1. Lorentz symmetries combine the energy density, the energy flux density, the momentum density, and the stress tensor into a symmetric Lorentz tensor $T^{\mu\nu} = +T^{\nu\mu}$ called the *stress-energy tensor*. For the electromagnetic fields in the vacuum, the stress-energy tensor can be written in covariant form as

$$T^{\mu\nu} = \frac{1}{4\pi} F^{\mu\alpha} g_{\alpha\beta} F^{\beta\nu} + \frac{1}{16\pi} g^{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} \tag{1}$$

(Gauss units).

(a) Spell out the components of the stress-energy tensor in 3D terms and explain the physical meaning of the components.

(b) Use Maxwell equations (in the covariant form) to show that

$$\partial_\mu T^{\mu\nu} = \frac{1}{c} J_\mu F^{\mu\nu}. \tag{2}$$

(c) Spell out eq. (2) for $\nu = 0$ and $\nu = j = 1, 2, 3$ in 3D terms and explain the physical meaning of these equations.

2. Compton scattering is elastic scattering of a high energy photon off an electron. In the lab frame (in which the electron is at rest before the scattering event) electron's recoil results in lowering of the energy — and hence the frequency — of the scattered photon compared to the incident photon.

Find out the energy $\hbar\omega'$ of the scattered photon as a function of of the incident photon's energy $\hbar\omega$ and the angle of scattering θ as measured in the lab frame.

Hint: use conservation laws for the energy-momentum p'^μ of the recoiling electron and demand $p'^\mu p'_\mu = m_e^2 c^2$.

3. Consider photo-production of pions via the following process:

$$p + \gamma \to \Delta^+ \to n + \pi^+ \tag{3}$$

— a proton absorbs a photon and becomes a Δ^+ resonance, which then decays into a neutron and a positive pion.

For your information, the rest masses of particles involved in this process are as follows: Proton, $M_p \approx 938$ MeV/c^2; pion, $M_{\pi^+} \approx 139$ MeV/c^2; Δ^+ resonance, $M_{\Delta^+} \approx 1226$ MeV/c^2; neutron, $M_n \approx 939$ MeV/c^2 (you may approximate $M_n \approx M_p$); and photon, exactly zero.

(a) In the center-of-mass frame — which is also the frame in which the Δ^+ resonance is at rest — what are the energies of the neutron and the pion?

(b) In the lab frame — the frame in which the initial proton is at rest — what should the photon's energy be in order to make the Δ^+ resonance?

(c) For the lab frame, derive the relation between the energy of the pion and the direction of its velocity. Give numerical values for the maximal and minimal values of pion's energy and find out whether the pion can move backwards (relative to the incident photon)?

(d) Now consider the case when the target proton is not at rest. If a photon collides with the proton "head on", it takes a lower photon energy to make a Δ^+ resonance. For an extremely high energy cosmic ray proton, even a photon from the cosmic microwave background can make a Δ^+. This is known as the GZK (Greisen?-Zatsepin?-Kuzmin) effect which makes it difficult for the ultra high energy protons to fly more than a few megaparsecs through the intergalactic space.

While the energy spectrum of the microwave background peaks about $\frac{2}{3} \times 10^{-3}$ eV, the photons with 3 times higher energy $\hbar\omega = 2 \cdot 10^{-3}$ eV are numerous enough to act as GZK obstacles for the ultra high energy protons. How much energy does a proton needs to collide with such a photon and make a Δ^+ resonance?