MAGNETIC ENERGY

Before I get to the magnetic energy, let me remind you of the Faraday's Law of Induction. Take any closed loop of coil of wire and place it in presence of magnetic fields; let Φ be the net magnetic flux through this loop or coil. *If the flux Φ changes for any reason whosoever, this induces an electromotive force (EMF) in the loop/coil according to*

$$\mathcal{E} = -\frac{d\Phi}{dt}. \tag{1}$$

The minus sign in this formula reflects the *Lenz rule:* the EMF (1) leads to a current in the loop whose magnetic field has opposite direction to the $\Delta\Phi$.

Now consider an inductor coil. For simplicity, let's assume that the coil either does not have a ferromagnetic core or else the ferromagnetic material of the core is linear, which means inside the core $\mathbf{B} = \mu\mu_0\mathbf{H}$. Consequently, when we run a current I through the coil, the magnetic fields $\mathbf{H}(\mathbf{x})$ and $\mathbf{B}(\mathbf{x})$ are proportional to the current and hence the magnetic flux Φ through the coil is also proportional to the current. Thus

$$\Phi = L \times I \tag{2}$$

for some constant coefficient L called the self-inductance of the coil. When the current through the coil changes for any reason, the flux also changes according to eq. (2), and according to the Faraday's law (1) this induces EMF in the coil,

$$\mathcal{E} = -L \times \frac{dI}{dt}. \tag{3}$$

Given these preliminaries, we may turn to the magnetic energy, and let's start with the magnetic energy stored in the inductor coil. Suppose we try to increase the current in the coil by an infinitesimal amount δI. Changing the current induces EMF in the coil, and to compensate for this negative EMF the power source providing the current must also provide voltage

$$V = -\mathcal{E} + \text{ a bit extra for the ohmic losses in the coil,} \tag{4}$$

and hence power $P = I \times V$. Some of this power is dissipated by the ohmic losses, but that would happen even for a time-independent current and hence $\mathcal{E} = 0$. So let's focus on the extra power due to $-\mathcal{E} \times I$ and hence the extra work done by the power supply while the current increases by δI in time δt:

$$\delta W_{\text{extra}} \;=\; -\mathcal{E} \times I \times \delta t \;=\; +L \times \frac{\delta I}{\delta t} \times I \times \delta t \;=\; LI \times \delta I \;=\; \delta\bigl(\tfrac{1}{2}LI^2\bigr). \tag{5}$$

This extra work — which is independent on the time δt it takes to raise the current — goes to the *magnetic energy of the coil*

$$U \;=\; \tfrac{1}{2}LI^2 \;=\; \tfrac{1}{2}\Phi I \;=\; \frac{\Phi^2}{2L}\,. \tag{6}$$

Now let's express this magnetic energy in terms of the magnetic field $\mathbf{B}(\mathbf{x})$ inside the coil. The magnetic flux though the coil can be expressed in terms of the vector potential as

$$\Phi \;=\; \oint\limits_{\text{coil}} d\mathbf{x} \cdot \mathbf{A}(\mathbf{x}), \tag{7}$$

hence the magnetic energy

$$U \;=\; \tfrac{1}{2}I\Phi \;=\; \frac{1}{2} \oint\limits_{\text{coil}} I\, d\mathbf{x} \cdot \mathbf{A}(\mathbf{x}). \tag{8}$$

This formula assumes a coil made of thin wires; if we replace them with thicker conductors carrying some volume currents $\mathbf{J}(\mathbf{x})$, then in the integral (8) we replace $I\,d\mathbf{x} \;\to\; d^3\mathbf{x}\,\mathbf{J}(\mathbf{x})$, hence

$$U \;=\; \frac{1}{2} \iiint d^3\mathbf{x}\,\mathbf{J}(\mathbf{x}) \cdot \mathbf{A}(\mathbf{x}). \tag{9}$$

The volume integral here is over the conductor's volume, whatever that is, but we may just as well extend it to the integral over the whole space since $\mathbf{J} = 0$ outside the conductor. This extension allows us to integrate by parts without worrying about the surface terms using the

Ampere's law

$$\nabla\times H \;=\; \mathbf{J}. \tag{10}$$

Indeed, for any two vector fields $\mathbf{f}$ and $\mathbf{g}$,

$$(\nabla\times\mathbf{f})\cdot\mathbf{g} \;=\; \epsilon_{ijk}(\nabla_i f_j)g_k \;=\; \epsilon_{ijk}\big(\nabla_i(f_j g_k) - f_j(\nabla_i g_k)\big) \;=\; \nabla\cdot(\mathbf{f}\times\mathbf{g}) \,+\, \mathbf{f}\cdot(\nabla\times\mathbf{g}), \tag{11}$$

hence

$$\mathbf{J}\cdot\mathbf{A} \;=\; (\nabla\times\mathbf{H})\cdot\mathbf{A} \;=\; \nabla\cdot(\mathbf{H}\times\mathbf{A}) \,+\, \mathbf{H}\cdot(\nabla\times\mathbf{A}) \;=\; \nabla\cdot(\mathbf{H}\times\mathbf{A}) \,+\, \mathbf{H}\cdot\mathbf{B}, \tag{12}$$

and therefore

$$\iiint\limits_{\mathcal{V}} d^3\mathbf{x}\,\mathbf{J}\cdot\mathbf{A} \;=\; \oiint\limits_{\substack{\text{surface}\\ \text{of }\mathcal{V}}} (\mathbf{H}\times\mathbf{A})\cdot d^2\mathbf{a} \,+\, \iiint\limits_{\mathcal{V}} d^3\mathbf{x}\,\mathbf{H}\cdot\mathbf{B}. \tag{13}$$

When the integration volume $\mathcal{V}$ expands to the whole space, the surface integral term goes away, so only the volume-integral term on the RHS remains. Thus, the magnetic energy (9) becomes

$$U \;=\; \frac{1}{2}\iiint\limits_{\substack{\text{whole}\\ \text{space}}} d^3\mathbf{x}\,\mathbf{H}\cdot\mathbf{B}. \tag{14}$$

Eq. (14) applies to magnetic energy in linear materials — the vacuum, diamagnetics, paramagnetics, and very soft ferromagnetics in weak fields, but we need a different approach to non-linear media. So let's go back to the inductor coil, but allow for a non-linear dependence between the current I and the magnetic flux Φ. Increasing the current in the coil — and hence the magnetic flux Φ — requires extra work by the power supply

$$\delta W_{\text{extra}} \;=\; -\mathcal{E}\times I\times\delta t \;=\; +\frac{\delta\Phi}{\delta t}\times I\times\delta t \;=\; \delta\Phi\times I, \tag{15}$$

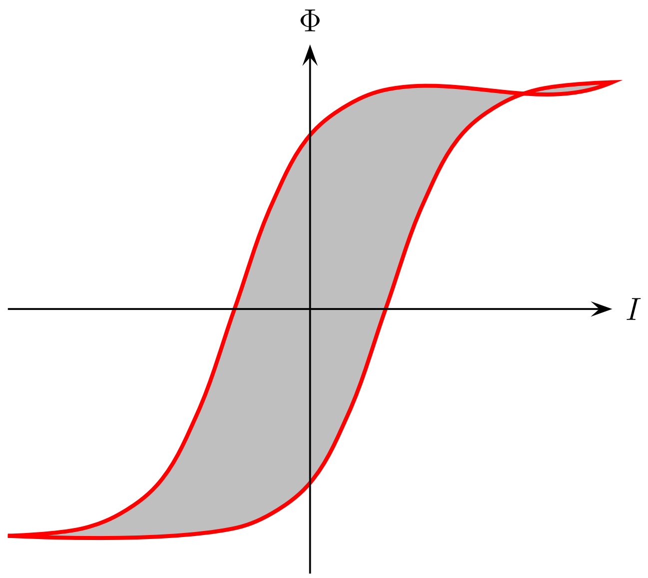
but this time we cannot re-express this work as δU without knowing the dependence of the flux Φ on the current I. If the relation between the current and the flux is non-linear but single-valued, then integrating eq. (13) yields magnetic energy

$$U(I) = \int_0^I I' \times d\Phi(I'), \tag{16}$$

but if there is hysteresis and Φ depends on the past history of the current, then the magnetic work

$$W = \int I \, d\Phi \tag{17}$$

is irreversible and cannot be accounted by any magnetic energy as a function of the current. Indeed, if we change the current back and force and come back to its original value, then the net work (17) is the area of the hysteresis loop in the (I, Φ) plot,



and this net work is dissipated as heat rather than stored as magnetic energy.

Similar formulae obtains in terms of the magnetic fields **H** and **B**:

$$\begin{aligned} \delta W \;&=\; I \times \delta\Phi \;=\; \oint\limits_{\text{coil}} I\, d\mathbf{x} \cdot \delta\mathbf{A}(\mathbf{x}) \\ &\longrightarrow \iiint d^3\mathbf{x}\, \mathbf{J}(\mathbf{x}) \cdot \delta\mathbf{A}(\mathbf{x}) \\ &= \iiint\limits_{\substack{\text{whole}\\ \text{space}}} s^3\mathbf{x}\, \mathbf{H}(\mathbf{x}) \cdot \delta\mathbf{B}(\mathbf{x}). \end{aligned} \tag{18}$$

If the relation between the **H** and the **B** fields is linear, then this magnetic work is accounted by the magnetic energy (14). If the relation is non-linear but single-valued, then the magnetic energy becomes

$$U \;=\; \iiint d^3\mathbf{x}\, F(\mathbf{B}) \quad \text{where} \quad F(\mathbf{B}) \;=\; \int\limits_0^{\mathbf{B}} \mathbf{H}(\mathbf{B}') \cdot d\mathbf{B}'. \tag{19}$$

But if there is hysteresis, then the magnetic work is irreversible and cannot be wholly accounted by the energy of the fields in the ferromagnetic involved.

Now consider what happens when we insert a piece of magnetic material inside a coil kept at constant current I. For simplicity, let's assume that the inserted magnetic material is linear, so its effect can be accounted by changing the self-inductance of the coil by δL: a piece of diamagnetic would reduce L little bit, a piece of paramagnetic would increase L a little but, and a piece of ferromagnetic would increase L quite a lot. By in any case, this change of L while the current I is held constant changes the magnetic energy of the coil by

$$\delta U \;=\; \delta(\tfrac{1}{2}I^2 L) \;=\; \tfrac{1}{2}I^2 \times \delta L. \tag{20}$$

This change of magnetic energy comes from 2 sources: mechanical work of moving the magnetic material, and the electric work of the power supply keeping the current constant. Indeed, changing the self-inductance while the current is constant changes the flux through the coil by $\delta\Phi = I \times \delta L$, which induces EMF

$$\mathcal{E} \;=\; -\frac{I\delta L}{\delta t}\,, \tag{21}$$

which calls for voltage spike from the power supply and hence extra work

$$W_{\rm el} \;=\; -\mathcal{E} \times I \times \delta t \;=\; +\frac{I\delta L}{\delta t} \times I \times \delta t \;=\; I^2 \times \delta L. \tag{22}$$

Consequently, the mechanical work of moving the magnetic material into or out from the coil is

$$\delta W_{\rm mech} \;=\; \delta U \;-\; \delta W_{\rm el} \;=\; +\tfrac{1}{2}I^2 \times \delta L \;-\; I^2 \times \delta L \;=\; -\tfrac{1}{2}I^2 \times \delta L. \tag{23}$$

In terms of the force by the coil on the piece of magnetic material, this means

$$F \;=\; \frac{I^2}{2} \times \frac{dL}{dx}\,, \tag{24}$$

with the positive direction of F being further into the coil. Thus:

- the ferromagnetic materials are strongly pulled into the coil;
- the paramagnetic material are weakly pulled into the coil;
- the diamagnetic material are weakly pushed out from the coil.

Similar situation applies when we set up some constant volume currents $\mathbf{x}(\mathbf{x})$ and then add a piece of linear magnetic material to the system while keeping the currents constant. In this case, the magnetic energy of the system changes by

$$\begin{aligned} \Delta U \;=\; U \;-\; U_0 \;&=\; \frac{1}{2}\iiint \big(\mathbf{J}\cdot\mathbf{A} \;-\; \mathbf{J}_0\cdot\mathbf{A}_0\big) d^3\mathbf{x} \\ &\langle\!\langle\, \text{using } \mathbf{J}(\mathbf{x}) = \mathbf{J}_0(\mathbf{x})\,\rangle\!\rangle \\ &=\; \frac{1}{2}\iiint \mathbf{J}_0\cdot(\mathbf{A}-\mathbf{A}_0) d^3\mathbf{x} \\ &\langle\!\langle\, \text{integrating by parts}\,\rangle\!\rangle \\ &=\; \frac{1}{2}\iiint \mathbf{H}_0\cdot(\mathbf{B}-\mathbf{B}_0) d^3\mathbf{x} \;=\; +\frac{\mu_0}{2}\iiint \mathbf{H}_0\cdot\mathbf{M}\, d^3\mathbf{x}. \end{aligned} \tag{25}$$

However, to keep the currents constant despite changing the magnetic fluxes, the power supply have to provide extra work which generalizes the

$$W_{\text{el}} \;=\; I \times \Delta\Phi \;=\; \oint_{\text{coil}} I\, d\mathbf{x} \cdot \mathbf{A}(\mathbf{x}) \tag{26}$$

for a coil, namely

$$\Delta W_{\text{el}} \;=\; \iiint d^3\mathbf{x}\, \mathbf{J}(\mathbf{x}) \cdot \Delta\mathbf{A}(\mathbf{x}) \;=\; \iiint d^3\mathbf{x}\, \mathbf{H}_0 \cdot \delta\mathbf{B} \;=\; +\mu_0 \iiint d^3\mathbf{x}\, \mathbf{H}_0 \cdot \mathbf{M}. \tag{27}$$

Consequently, the mechanical work of moving the magnetic material into the system is

$$W_{\text{mech}} \;=\; \Delta U \;-\; W_{\text{el}} \;=\; -\frac{\mu_0}{2} \iiint d^3\mathbf{x}\, \mathbf{H}_0 \cdot \mathbf{M}. \tag{28}$$

This formula explains while the ferromagnetic and paramagnetic materials are attracted to regions where the magnetic field is strongest while the diamagnetic materials are repelled by them. Indeed, take a small ball of magnetic material, so small that over its size we mat approximate $\mathbf{H}_0(\mathbf{x}) \approx \text{const} = \mathbf{H}_0(\text{center})$. Then proceeding similarly to the dielectric case we find that the $\mathbf{H}$ field inside the ball is is

$$\mathbf{H}_{\text{inside}} \;=\; \frac{3}{\mu+2}\, \mathbf{H}_0\,, \tag{29}$$

hence

$$\mathbf{M} \;=\; \frac{3(\mu-1)}{\mu+2}\, \mathbf{H}_0$$

and therefore

$$W_{\text{mech}} \;=\; -\frac{\mu_0}{2} \times \frac{4\pi R^3}{3} \times \frac{3(\mu-1)}{\mu+2} \times \mathbf{H}_0^2 \;=\; -\frac{2\pi R^3 \mu_0}{\mu+2} \times (\mu-1) \times \mathbf{H}_0^2. \tag{30}$$

The $\mathbf{H}_0^2$ in this formula should be evaluated at the ball's location, and the work (30) is the work of bringing the ball to that location from infinitely far away. Consequently, this mechanical work as a function of the ball's location acts as its potential energy

$$U_{\text{potential}}(\mathbf{x}) \;=\; -\frac{2\pi R^3\mu_0}{\mu+2}\times(\mu-1)\times\mathbf{H}_0^2(\mathbf{x}). \tag{31}$$

This potential energy governs the magnetic force on the ball according to

$$\mathbf{F} \;=\; -\nabla U_{\text{potential}} \;=\; (\mu-1)*\begin{pmatrix}\text{positive}\\ \text{factor}\end{pmatrix}*\nabla(\mathbf{H}_0^2), \tag{32}$$

so the direction of this force on diamagnetic material is opposite from the force on paramagnetic or ferromagnetic materials. Specifically:

- the ferromagnetic materials with $\mu \gg 1$ are strongly pulled towards the locations where $\mathbf{H}_0^2$ is strongest such as magnet's poles;
- the paramagnetic materials with $\mu = 1 + \text{small}$ are pulled in the same direction but with a weaker force;
- the diamagnetic materials with $\mu = 1 - \text{small}$ are pushed in the opposite direction, away from the magnet.

Thus, the diamagnetic materials — such as water — can be levitated above a strong magnet. Here is the famous levitating frog demo illustrating this effect.