HIGHLIGHTS OF MAGNETOSTATICS

Magnetostatics is based on two Laws: the Biot–Savart–Laplace Law for the magnetic field of *steady* currents in wires (discovered in 1820)

$$\mathbf{B}(\mathbf{x}) = \frac{\mu_0}{4\pi} \int_{\text{wire}} I\, d\mathbf{y} \times \frac{(\mathbf{x}-\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^3}, \tag{1}$$

and the Ampere's Force Law for the magnetic force on a wire (discovered in 1823)

$$\mathbf{F} = \int_{\text{wire}} I\, d\mathbf{x} \times \mathbf{B}(\mathbf{x}). \tag{2}$$

Both formulae easily generalize from currents in thin wires to volume currents in thick conductors,

$$\mathbf{B}(\mathbf{x}) = \frac{\mu_0}{4\pi} \iiint d^3\mathbf{y}\, \mathbf{J}(\mathbf{y}) \times \frac{(\mathbf{x}-\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^3}, \tag{3}$$

and to forces on such currents,

$$\mathbf{F} = \iiint d^3\mathbf{x}\, \mathbf{J}(\mathbf{x}) \times \mathbf{B}(\mathbf{x}). \tag{4}$$

Let me illustrate the Biot–Savart–Laplace Law and the Ampere's Force Law with the canonical example of two parallel wires. For the sake of definiteness, let the first wire run along the x_3 axis while the second wire runs parallel to it at $x_1 = a > 0$ and $x_2 = 0$. The magnetic field of the first wire at some point on the second wire is

$$\begin{aligned} \mathbf{B}_1(a,0,x_3) &= \frac{\mu_0}{4\pi} \int_{-\infty}^{+\infty} I_1\, dy_3 \times \frac{(0,0,1)\times(a,0,x_3-y_3)}{[a^2+(y_3-x_3)^2]^{3/2}} \\ &= \frac{\mu_0 I_1}{4\pi} \int_{-\infty}^{+\infty} \frac{(0,a,0)\,d\ell}{[a^2+\ell^2]^{3/2}} = \frac{\mu_0 I_1}{4\pi}\, \frac{2}{a^2}\,(0,a,0) \\ &= \frac{\mu_0 I_1}{2\pi a}\,(0,1,0). \end{aligned} \tag{5}$$

By the rotational symmetry of the fist wire — and hence of the field around it — this formula generalizes to give the magnetic field everywhere else in space: in cylindrical coordinates

(s,ϕ,z), a becomes s — the radial distance from the first wire, while the unit vector $(0,1,0)=\mathbf{n}_2$ becomes the unit vector $\mathbf{n}_\phi$ in the ϕ direction, *i.e.*, the circular direction around the wire, thus

$$\mathbf{B}_1 \;=\; \frac{\mu_0 I_1}{2\pi s}\,\mathbf{n}_\phi\,. \tag{6}$$

Going back to the second wire and plugging back the field (5) into the Ampere's Force Law, we have

$$I_2\,d\mathbf{x}\times\mathbf{B}_1 \;=\; \frac{\mu_0 I_1 I_2}{2\pi a}\bigl(\mathbf{n}_3\times\mathbf{n}_2\;=\;-\mathbf{n}_1\bigr) \tag{7}$$

and therefore

$$\frac{\text{Force}}{\text{Length}} \;=\; \frac{\mu_0 I_1 I_2}{2\pi a}\,(-\mathbf{n}_1); \tag{8}$$

the direction of this force is attractive if the two currents flow in the same direction ($I_1 I_2 > 0$) but repulsive if the two currents flow in opposite directions ($I_1 I_2 < 0$).

Third Law of Newton

Together, the Biot–Savart–Laplace Law and the Ampere's Force Law provide a rather complicated formula for the magnetic force between two electric circuits, so let's make sure the Third Law of Newton does work for the magnetic forces. For simplicity, let's start with a simple case of each circuit comprising a single closed loop of thin wire carrying a constant currents — respectively current I_1 in the loop L_1 and current I_2 in the loop L_2. Then, combining the BSL Law and the magnetic force law, we have

$$\mathbf{F}_{1\,\mathrm{on}\,2} \;=\; \oint\limits_{L_2} I_2\,d\mathbf{x}\times\mathbf{B}_{\mathrm{loop}\#1}(\mathbf{x}) \;=\; \oint\limits_{L_2} I_2\,d\mathbf{x}\times\frac{\mu_0}{4\pi}\oint\limits_{L_1} I_1\,d\mathbf{y}\times\frac{(\mathbf{x}-\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^3}\,. \tag{9}$$

Combining the two integrals into a single integral over 2 variables, we may rewrite this formula as

$$\mathbf{F}_{1\,\mathrm{on}\,2} \;=\; \frac{\mu_0 I_1 I_2}{4\pi}\oint\limits_{\mathbf{x}\in L_2}\oint\limits_{\mathbf{y}\in L_1} d\mathbf{x}\times\left(d\mathbf{y}\times\frac{(\mathbf{x}-\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^3}\right), \tag{10}$$

where the integrand can be rearranged as

$$d\mathbf{x}\times\left(d\mathbf{y}\times\frac{(\mathbf{x}-\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^3}\right)\;=\;-\big(d\mathbf{x}\cdot d\mathbf{y}\big)\,\frac{(\mathbf{x}-\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^3}\;+\;d\mathbf{y}\left(d\mathbf{x}\cdot\frac{(\mathbf{x}-\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^3}\right).\tag{11}$$

Separating the two terms into separate integrals, we arrive at

$$\mathbf{F}_{1\,\mathrm{on}\,2}\;=\;-\frac{\mu_0I_1I_2}{4\pi}\oint\limits_{\mathbf{x}\in L_2}\oint\limits_{\mathbf{y}\in L_1}\big(d\mathbf{x}\cdot d\mathbf{y}\big)\,\frac{(\mathbf{x}-\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^3}+\frac{\mu_0I_1I_2}{4\pi}\oint\limits_{\mathbf{x}\in L_2}\oint\limits_{\mathbf{y}\in L_1}d\mathbf{y}\left(d\mathbf{x}\cdot\frac{(\mathbf{x}-\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^3}\right),\tag{12}$$

where the second integral happens to vanish. To see how this works, note that in the double integral over $\mathbf{x}\in L_2$ and $\mathbf{y}\in L_1$, we may integrate over the two variables in whichever order we like, so let's first integrate over the $\mathbf{x}$ at a fixed $\mathbf{y}$ and only then integrate over the $\mathbf{y}$, thus

$$\text{(second term)}\;=\;\frac{\mu_0I_1I_2}{4\pi}\oint\limits_{\mathbf{y}\in L_1}d\mathbf{y}\left(\oint\limits_{\mathbf{x}\in L_2}d\mathbf{x}\cdot\frac{(\mathbf{x}-\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^3}\right).\tag{13}$$

Since the inner integral $d\mathbf{x}$ is taken at a fixed $\mathbf{y}$, we have

$$d\mathbf{x}\cdot\frac{(\mathbf{x}-\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^3}\;=\;d\mathbf{x}\cdot\nabla_{\mathbf{x}}\left(\frac{-1}{|\mathbf{x}-\mathbf{y}|}\right)\;=\;d\left(\frac{-1}{|\mathbf{x}-\mathbf{y}|}\right),\tag{14}$$

and an integral of a total differential over any *closed loop* such as L_2 always vanishes,

$$\oint\limits_{\mathbf{x}\in L_2}d\mathbf{x}\cdot\frac{(\mathbf{x}-\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^3}\;=\oint\limits_{L_2}d\left(\frac{-1}{|\mathbf{x}-\mathbf{y}|}\right)\;=\;\left[\frac{-1}{|\mathbf{x}-\mathbf{y}|}\right]_{\mathbf{x}=\text{start of }L_2}^{\mathbf{x}=\text{end of }L_2}\;=\;0.\tag{15}$$

Thus, the second term in eq. (12) for the force is zero.

The remaining first term in eq. (12) looks rather symmetric between the two loops:

$$\mathbf{F}_{1\,\mathrm{on}\,2}\;=\;-\frac{\mu_0I_1I_2}{4\pi}\oint\limits_{\mathbf{x}\in L_2}\oint\limits_{\mathbf{y}\in L_1}\big(d\mathbf{x}\cdot d\mathbf{y}\big)\,\frac{(\mathbf{x}-\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^3},\tag{16}$$

where the pre-integral factor is symmetric between the two loops, the double integral itself and the $(d\mathbf{x}\cdot d\mathbf{y})$ factor are also symmetric, while the remaining factor

$$\frac{(\mathbf{x}-\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^3}$$

is *antisymmetric* — it changes overall sign when we exchange toe two loos and hence $\mathbf{x}\leftrightarrow\mathbf{y}$. Consequently, the net force is antisymmetric WRT exchanging the two loops,

$$\mathbf{F}_{2\,\mathrm{on}\,1} \;=\; -\mathbf{F}_{1\,\mathrm{on}\,2} \tag{17}$$

— in perfect agreement with the Third Law of Newton.

Now consider two general circuits, each one comprising many wires, as long as the current in each wire is steady. By the Kirchhoff Law of Currents, each such circuit can be viewed as a superposition of several *closed* current loops, each loop L_i having is own constant current I_i. And since both the Biot–Savart–Laplace formula and the magnetic force formula are linear, the net magnetic force between the two circuits is simply the sum of forces between the current loops comprising the two circuits,

$$\mathbf{F}^{\text{from circuit\#1}}_{\text{on circuit\#2}} \;=\; \sum_{L_i\in\text{circuit\#1}}\;\sum_{L_j\in\text{circuit\#2}} \mathbf{F}_{L_i\,\mathrm{on}\,L_j}\,. \tag{18}$$

The magnetic forces between the loops L_i and L_j in this sum are exactly as in eq. (9) and hence as in eq. (16). As we saw above, such forces are antisymmetric WRT exchanging the loops $L_i\leftrightarrow L_j$, so the net force (18) is also antisymmetric WRT exchanging the two whole circuits, thus

$$\mathbf{F}^{\text{from circuit\#1}}_{\text{on circuit\#2}} \;=\; -\mathbf{F}^{\text{from circuit\#2}}_{\text{on circuit\#1}}\,, \tag{19}$$

in perfect agreement with the Third Law of Newton. *Quod erat demonstrandum.*

Magnetic Field Equations

In electrostatics, we may start with the Coulomb Law and then use it to derive the field equations for the electric field such as the Gauss Law $\nabla\cdot\mathbf{E}=\rho/\epsilon_0$ and the zero-curl law $\nabla\times\mathbf{E}=0$. But we may also start with these field equations and then derive the Coulomb Law as a solution. Likewise, in magnetostatics, the Biot–Savart–Laplace Law is mathematically equivalent to a pair of differential equations: the *magnetic Gauss law*

$$\nabla\cdot\mathbf{B}(\mathbf{x}) \;=0 \tag{20}$$

— which holds for any magnetic field, static or time-dependent, — and the *Ampere's Circuital Law* (or simply the *Ampere's Law*

$$\nabla\times\mathbf{B}(\mathbf{x}) \;=\; \mu_0\mathbf{J}(\mathbf{x}) \tag{21}$$

— which holds only for the static fields of steady currents.

In the integral form, the magnetic Gauss law says that *the magnetic flux through any closed surface is zero,*

$$\oint\!\!\!\!\oint_{\text{any closed surface}} \mathbf{B}\cdot d^2\mathbf{a} \;=\; 0, \tag{22}$$

or by analogy with the electric Gauss Law, *there are no magnetic charges.* As to the integral form of the Ampere's Circuital Law — which was named by J. C. Maxwell by analogy with similar formulae in hydrodynamics — it says that for any closed loop $\mathcal{L}$ in space

$$\oint_{\mathcal{L}}\mathbf{B}\cdot d\ell \;=\; \mu_0\, I^{\text{net}}[\text{through the loop }\mathcal{L}]. \tag{23}$$

For the currents in wires, the RHS here obtains by simply counting the wires which go inside the loop $\mathcal{L}$ and adding up the current they carry. For the volume current $\mathbf{J}(\mathbf{x})$, we need to pick a surface $\mathcal{S}$ spanning the loop $\mathcal{L}$ and then integrate

$$I^{\text{net}}[\text{through the loop }\mathcal{L}] \;=\; \iint_{\mathcal{S}}\mathbf{J}(\mathbf{x})\cdot d^2\mathbf{a}. \tag{24}$$

Since a *steady* current $\mathbf{J}(\mathbf{x})$ must have zero divergence, $\nabla\cdot\mathbf{J}=0$, we may integrate over any surface $\mathcal{S}$ spanning the loop $\mathcal{L}$, the integral would be the same. (And for non-steady currents with $\nabla\cdot\mathbf{J}\neq 0$, the Ampere's Circuital Law does not work anyway, and we would need the Ampere–Maxwell Law instead.)

I can formally verify that the magnetic field given by the Biot–Savart–Laplace formula (1) or (3) obeys the magnetic Gauss law and the Ampere's circuital law, but in these notes I would like to do it the other way around: I am going to formally solve the differential equations (20) and (21) by means of the *magnetic vector potential*, and then we shall see that the solution is precisely the Biot–Savart–Laplace formula (1) or (3).

Vector Potential for the Magnetic Field

Let me start with two theorems of Vector Calculus:

Theorem 1: If a vector field has zero curl *everywhere in space,* then that field is a gradient of some scalar field.

Theorem 2: If a vector field has zero divergence *everywhere in space,* then that field is a curl of some other vector field.

The first theorem allows us to introduce the scalar potential for the static electric field,

$$\nabla\times\mathbf{E}(\mathbf{x}) \;=\; 0 \quad \forall\mathbf{x} \quad\Longrightarrow\quad \mathbf{E}(\mathbf{x}) \;=\; -\nabla\Phi(\mathbf{x}) \text{ for some } \Phi(\mathbf{x}), \tag{25}$$

while the second theorem allows us to introduce the vector potential for the magnetic field,

$$\nabla\cdot\mathbf{B}(\mathbf{x}) \;=\; 0 \quad \forall\mathbf{x} \quad\Longrightarrow\quad \mathbf{B}(\mathbf{x}) \;=\; \nabla\times\mathbf{A}(\mathbf{x}) \text{ for some } \mathbf{A}(\mathbf{x}). \tag{26}$$

The potentials (25) and (26) have many uses. In particular, they are needed for the Lagrangian or Hamiltonian description of a charged particle's motion in classical mechanics,

$$L(\mathbf{x},\mathbf{v}) \;=\; \frac{m}{2}\mathbf{v}^2 \;-\; q\Phi(\mathbf{x}) \;+\; q\mathbf{v}\cdot\mathbf{A}(\mathbf{x}), \tag{27}$$

$$H(\mathbf{x},\mathbf{p}) \;=\; \frac{1}{2m}\big(\mathbf{p} \;-\; q\mathbf{A}(\mathbf{x})\big)^2 \;+\; q\Phi(\mathbf{x}), \tag{28}$$

or in quantum mechanics,

$$\hat{H} \;=\; \frac{1}{2m}\big(\hat{\mathbf{p}} \;-\; \mathbf{A}(\hat{\mathbf{x}})\big)^2 \;+\; q\Phi(\hat{\mathbf{x}}). \tag{29}$$

I shall explain these issues — as well as the Aharonov–Bohm effect and the Dirac's magnetic monopoles — in a couple of extra lectures this and next week (1/31 and 2/7); see also my notes "Dynamics of a Charged Particle and Gauge Transforms" and "Aharonov–Bohm Effect and Dirac Monopoles". But for the current set of notes, I would like to focus on using the vector potential $\mathbf{A}(\mathbf{x})$ to calculate the magnetic field.

Let me start with some general properties of the vector potential. While the electrostatic field $\mathbf{E}(\mathbf{x})$ determines the scalar potential $\Phi(\mathbf{x})$ up to an overall constant term, the magnetic field $\mathbf{B}(\mathbf{x})$ determines the vector potential $\mathbf{A}(\mathbf{x})$ only up to a gradient of an arbitrary scalar field $\Lambda(\mathbf{x})$. Indeed, the vector potentials $\mathbf{A}(\mathbf{x})$ and

$$\mathbf{A}'(\mathbf{x}) = \mathbf{A}(\mathbf{x}) + \nabla\Lambda(\mathbf{x}) \tag{30}$$

have the same curl everywhere, so they correspond to the same magnetic field,

$$\mathbf{B}'(\mathbf{x}) = \nabla\times\mathbf{A}'(\mathbf{x}) = \nabla\times\mathbf{A}(\mathbf{x}) + \nabla\times\nabla\Lambda(\mathbf{x}) = \mathbf{B}(\mathbf{x}) + 0. \tag{31}$$

The relations (30) between different vector potentials for the same magnetic field are called the *gauge transforms.*

Despite ambiguity of the vector potential itself, some of its properties are *gauge invariant, i.e.*, the same for all potentials related by gauge transforms. For example, for any closed loop $\mathcal{L}$, the integral

$$\oint\limits_{\mathcal{L}} \mathbf{A}\cdot d\vec{\ell} \tag{32}$$

is gauge invariant; indeed,

$$\oint\limits_{\mathcal{L}} \mathbf{A}'(\mathbf{x})\cdot d\mathbf{x} - \oint\limits_{\mathcal{L}} \mathbf{A}(\mathbf{x})\cdot d\mathbf{x} = \oint\limits_{\mathcal{L}} \nabla\Lambda(\mathbf{x})\cdot d\mathbf{x} = \oint\limits_{\mathcal{L}} d\Lambda(\mathbf{x}) = \Lambda(\text{end of }\mathcal{L}) - \Lambda(\text{start of }\mathcal{L}) = 0. \tag{33}$$

Physically, the integral (32) is the magnetic flux through the loop $\mathcal{L}$. Indeed, take any surface $\mathcal{S}$ spanning the loop $\mathcal{L}$; by the Stokes' theorem,

$$\Phi_B[\text{through }\mathcal{S}] = \iint\limits_{\mathcal{S}} \mathbf{B}\cdot d^2\mathbf{a} = \iint\limits_{\mathcal{S}} (\nabla\times\mathbf{A})\cdot d^2\mathbf{a} = \oint\limits_{\mathcal{L}} \mathbf{A}\cdot d\vec{\ell}. \tag{34}$$

We may use eq. (34) to easily find the vector potential for a magnetic field which have some symmetries. For example, consider the uniform magnetic field $\mathbf{B} = (0, 0, B)$ inside a long solenoid. By the rotational and translational symmetries of the solenoid, we expect that in cylindrical coordinates (s, ϕ, z) the vector potential has form

$$\mathbf{A}(s, \phi, z) \;=\; A(s)\mathbf{n}_\phi, \tag{35}$$

while the magnitude $A(s)$ follows from eq. (34): Take a circle of radius $s < R_{\text{solenoid}}$, then

$$\oint\limits_{\text{circle}} \mathbf{A} \cdot d\vec{\ell} \;=\; A(s) \times 2\pi s, \tag{36}$$

while the magnetic flux through that circle is

$$\Phi_B[\text{circle}] \;=\; B \times \pi s^2, \tag{37}$$

hence

$$A(s) \;=\; \frac{B \times \pi s^2}{2\pi s} \;=\; \tfrac{1}{2} B s. \tag{38}$$

In Cartesian coordinates, this vector potential becomes

$$\mathbf{A} \;=\; \frac{B}{2}\, s\mathbf{n}_\phi \;=\; \frac{B}{2}\, (x_2, -x_1, 0), \tag{39}$$

which makes it easy to verify $\nabla \times \mathbf{A} = (0, 0, B) = \mathbf{B}$.

Eq. (39) gives the vector potential inside the long solenoid. Outside the solenoid, the magnetic field is negligible, but the flux through a circle of radius $s > R_{\text{solenoid}}$ is non-zero due to the flux inside the solenoid. Thus,

$$\Phi_B[\text{circle}] \;=\; B \times \pi R^2 \tag{40}$$

and hence

$$2\pi s \times A(s) \;=\; \Phi_B \;=\; \pi R^2 B \quad\Longrightarrow\quad A(s) \;=\; \frac{BR^2}{2s}\,. \tag{41}$$

In vector notations,

$$\mathbf{A} \;=\; \frac{BR^2}{2}\,\frac{\mathbf{n}_\phi}{s} \;=\; \frac{BR^2}{2}\,\frac{(x_2, -x_1, 0)}{x^2+y^2} \;=\; \frac{BR^2}{2}\,\nabla\phi, \tag{42}$$

which agrees with zero magnetic field outside the solenoid,

$$\mathbf{B} \;=\; \nabla\times\mathbf{A} \;=\; \frac{BR^2}{2}\,\nabla\times\nabla\phi \;=\; 0. \tag{43}$$

Equations for the Vector Potential

A static magnetic field of steady currents obeys equations

$$\nabla\cdot\mathbf{B} \;=\; 0, \tag{44}$$

$$\nabla\times\mathbf{B} \;=\; \mu_0\mathbf{J}. \tag{45}$$

In terms of the vector potential $\mathbf{A}(\mathbf{x})$, the zero-divergence equation (44) is automatic: any $\mathbf{B} = \nabla\times\mathbf{A}$ has zero divergence. On the other hand, the Ampere Circuital Law (45) becomes a second-order differential equation

$$\mu_0\mathbf{J} \;=\; \nabla\times(\nabla\times\mathbf{A}) \;=\; \nabla(\nabla\cdot\mathbf{A}) \;-\; \nabla^2\mathbf{A}. \tag{46}$$

Moreover, for any solution $\mathbf{A}(\mathbf{x})$ of this equation for any given current density $\mathbf{J}(\mathbf{x})$, there is a whole family of other solutions related to each other by the gauge transforms

$$\mathbf{A}'(\mathbf{x}) \;=\; \mathbf{A}(\mathbf{x}) \;+\; \nabla\Lambda(\mathbf{x}), \quad \text{any } \Lambda(\mathbf{x}). \tag{30}$$

To avoid this redundancy, it is often convenient to impose an extra *gauge-fixing condition* on the vector potential besides $\nabla\times\mathbf{A} = \mathbf{B}$. In magnetostatics, the most commonly used condition is the *transverse gauge* $\nabla\cdot\mathbf{A}=0$. Note that any vector potential can be gauge-transformed to a potential which obeys the transversality condition. Indeed, suppose $\nabla\cdot\mathbf{A}_0\neq 0$, then for

$$\Lambda(\mathbf{x}) \;=\; \iiint \frac{(\nabla\cdot\mathbf{A}_0)(\mathbf{y})}{4\pi\,|\mathbf{x}-\mathbf{y}|}\,d^3\mathbf{y} \tag{47}$$

we have

$$\nabla^2\Lambda(\mathbf{x}) \;=\; -\nabla\cdot\mathbf{A}_0(\mathbf{x}) \tag{48}$$

and therefore $\mathbf{A}=\mathbf{A}_0+\nabla\Lambda$ — which is gauge-equivalent to the $\mathbf{A}_0$ — has zero divergence,

$$\nabla\cdot\mathbf{A} \;=\; \nabla\cdot\mathbf{A}_0 \;+\; \nabla^2\Lambda \;=\; 0. \tag{49}$$

In the transverse gauge, $\nabla\times\mathbf{B}$ becomes simply the (minus) Laplacian of the vector potential,

$$\nabla\times\mathbf{B} \;=\; \nabla\times(\nabla\times\mathbf{A}) \;=\; \nabla(\nabla\cdot\mathbf{A}) \;-\; \nabla^2\mathbf{A} \;\longrightarrow\; -\nabla^2\mathbf{A}, \tag{50}$$

so the Ampere Law equation (46) becomes the *Poisson equation for the vector potential*,

$$\nabla^2\,\mathbf{A}(\mathbf{x}) \;=\; -\mu_0\,\mathbf{J}(\mathbf{x}). \tag{51}$$

Component by component, it looks exactly like the Poisson equation for the scalar potential of the electrostatics,

$$\nabla^2\Phi(\mathbf{x}) \;=\; -\frac{1}{\epsilon_0}\,\rho(\mathbf{x}), \tag{52}$$

so its solution has a similar Coulomb-like form

$$\mathbf{A}(\mathbf{x}) \;=\; \frac{\mu_0}{4\pi}\iiint \frac{\mathbf{J}(\mathbf{y})\,d^3\mathbf{y}}{|\mathbf{x}-\mathbf{y}|}\,. \tag{53}$$

As written, this formula is for the volume current $\mathbf{J}(\mathbf{x})$ in a thick conductor; for a current in a thin wire it becomes

$$\mathbf{A}(\mathbf{x}) \;=\; \frac{\mu_0}{4\pi}\int\limits_{\text{wire}} \frac{I\,d\mathbf{y}}{|\mathbf{x}-\mathbf{y}|}\,. \tag{54}$$

The Biot–Savart–Laplace formulae (1) and (3) follows from these formulae by simply taking the curl of the vector potential. For example, for the volume current $\mathbf{J}(\mathbf{y})$,

$$\begin{aligned}
\mathbf{B}(\mathbf{x}) \;&=\; \nabla\times\mathbf{A}[\text{from eq. (53)}] \;=\; \nabla\times\left(\frac{\mu_0}{4\pi}\iiint\frac{\mathbf{J}(\mathbf{y})\,d^3\mathbf{y}}{|\mathbf{x}-\mathbf{y}|}\right) \\
&=\; \frac{\mu_0}{4\pi}\iiint\nabla_{\mathbf{x}}\times\left(\frac{\mathbf{J}(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|}\right)d^3\mathbf{y} \;=\; \frac{\mu_0}{4\pi}\iiint\left(\nabla_{\mathbf{x}}\frac{1}{|\mathbf{x}-\mathbf{y}|}\right)\times\mathbf{J}(\mathbf{y})\,d^3\mathbf{y} \qquad (55)\\
&=\; \frac{\mu_0}{4\pi}\iiint\frac{-(\mathbf{x}-\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^3}\times\mathbf{J}(\mathbf{y})\,d^3\mathbf{y} = \frac{\mu_0}{4\pi}\iiint d^3\mathbf{y}\,\mathbf{J}(\mathbf{y})\times\frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|^3}\,.
\end{aligned}$$

Quod erat demonstrandum.

EXAMPLES OF CALCULATING THE MAGNETIC FIELD:

For the practical calculation of the magnetic field of some wire circuits, sometimes it's more convenient to use the Biot–Savart–Laplace equation directly, while sometimes it's more convenient to first calculate the vector potential using eqs. (54) or or (53), and then take the curl.

As an example of direct usage of the Biot–Savart–Laplace formula, consider a circular wire loop of radius R carrying current I, and let's calculate the magnetic field along the axis through the loop's center and $\perp$ to the plane of the loop. In coordinates,

$$\mathbf{y} \;=\; (R\cos\phi, R\sin\phi, 0) \quad \text{for } 0\le\phi\le 2\pi, \tag{56}$$

while we focus on the magnetic field at $\mathbf{x}=(0,0,x_3)$ only. This makes the problem axially symmetric, with a constant denominator in eq. (1):

$$\frac{1}{|\mathbf{x}-\mathbf{y}|^3} \;=\; \frac{1}{[R^2+x_3^2]^{3/2}} \;=\; \text{const}, \tag{57}$$

while in the numerator

$$\begin{aligned} I\,d\mathbf{y} \times (\mathbf{x} - \mathbf{y}) &= IR(-\sin\phi, +\cos\phi, 0)\,d\phi \times (-R\cos\phi, -R\sin\phi, +x_3) \\ &= IR(x_3\cos\phi, x_3\sin\phi, R)\,d\phi. \end{aligned} \tag{58}$$

Also,

$$\int_0^{2\pi} (x_3\cos\phi, x_3\sin\phi, R)\,d\phi = (0, 0, 2\pi R) = 2\pi R\,\mathbf{n}_3 \tag{59}$$

where $\mathbf{n}_3 = (0, 0, 1)$ is the unit vector along the axis. Altogether, the magnetic field at point $(0, 0, x_3)$ on the axis is

$$\begin{aligned} \mathbf{B}(0,0,x_3) &= \frac{\mu_0}{4\pi} \int\limits_{\text{wire}} \frac{I\,d\mathbf{y} \times (\mathbf{x} - \mathbf{y}) = IR(x_3\cos\phi, x_3\sin\phi, R)\,d\phi}{|\mathbf{x} - \mathbf{y}|^3 = [R^2 + x_3^2]^{3/2}} \\ &= \frac{\mu_0}{4\pi} \frac{IR}{[R^2 + x_3^2]^{3/2}} \int_0^{2\pi} (x_3\cos\phi, x_3\sin\phi, R)\,d\phi \\ &= \frac{\mu_0 I R^2}{2[R^2 + x_3^2]^{3/2}}\,\mathbf{n}_3\,. \end{aligned} \tag{60}$$

In particular, at the center of the ring the field is $\mathbf{B}(0) = (\mu_0 I/2R)\mathbf{n}_3$.

In this case, using the BSL equation directly is easier because along the x_3 symmetry axis the integral for the magnetic field is drastically simplified by the constant denominator. Away from the axis, we would have ended up with a horrible elliptic integral. But had we tried to calculate the vector potential first and then take its curl, we would need to find the $\mathbf{A}$ not just along the axis but also in its infinitesimal vicinity, and for an off-axis $\mathbf{x}$, the integral

$$\mathbf{A}(x) = \frac{\mu_0}{4\pi} \int_0^{2\pi} \frac{IR(-\sin\phi, \cos\phi, 0)}{|\mathbf{x} - \mathbf{y}|} \tag{61}$$

is a messy elliptic function that's not easy to work with.

In another example — an infinitely long straight wire — both ways of calculating the magnetic field are easy, so let's do it using the vector potential. A quick look at eq. (54) — here it is again,

$$\mathbf{A}(\mathbf{x}) \;=\; \frac{\mu_0}{4\pi}\int\limits_{\text{wire}} \frac{I\,d\mathbf{y}}{|\mathbf{x}-\mathbf{y}|}\,, \tag{54}$$

— tells us that the vector potential $\mathbf{A}(\mathbf{x})$ has the same direction $\mathbf{n}_z = (0,0,1)$ as the current in the wire, while its magnitude

$$A(\mathbf{x}) \;=\; \frac{\mu_0}{4\pi}\int\limits_{\text{wire}} \frac{I\,dy}{|\mathbf{x}-\mathbf{y}|} \tag{62}$$

has the same form as the scalar potential of a uniformly charged straight wire, thus

$$\mathbf{A}(\mathbf{x}) \;=\; \frac{\mu_0 I}{2\pi}\Big(-\log(s)\,+\,\text{const}\Big)\mathbf{n}_3\,, \tag{63}$$

where s is the distance from the wire. Taking the curl of this vector potential in cylindrical coordinates, we immediately obtain

$$\mathbf{B}(\mathbf{x}) \;=\; \frac{\mu_0 I}{2\pi s}\,\mathbf{n}_\phi \tag{64}$$

where $\mathbf{n}_\phi$ is the unit vector in the circular direction around the wire.

For another example, consider a flat current sheet in the (x_1,x_2) plane with uniform current density $\mathbf{K}$ in the x_2 direction. In terms of the 3D current density,

$$\mathbf{J}(\mathbf{x}) \;=\; K\delta(x_3)\,(0,1,0). \tag{65}$$

Consequently, the Poisson equation for the vector potential of the current sheet is

$$\nabla^2\mathbf{A} \;=\; -\mu_0 K\delta(x_3)\,(0,1,0). \tag{66}$$

Thanks to the symmetries of this equation, we may look for a solution of the form

$$\mathbf{A}(\mathbf{x}) \;=\; A(x_3\text{ only})\,(0,1,0) \tag{67}$$

where $A(x_3)$ obey the 1D Poisson equation

$$\frac{d^2 A}{dx_3^2} \;=\; -\mu_0 K \delta(x_3). \tag{68}$$

Despite the delta function on the RHS, the solution of this differential equation is continuous at $x_3 = 0$, namely

$$A(x_3) \;=\; -\tfrac{1}{2}\mu_0 K \times |x_3|\,, \tag{69*}$$

although its derivative has a discontinuity,

$$\mathrm{disc}\left(\frac{dA}{dx_3}\right) \;=\; -\mu_0 K. \tag{70}$$

In terms of the magnetic field $\mathbf{B}$, the vector potential

$$\mathbf{A} \;=\; -\tfrac{1}{2}\mu_0 K\,|x_3|\,(0,1,0) \tag{71}$$

means

$$\mathbf{B} \;=\; \begin{cases} +\tfrac{1}{2}\mu_0 K\,(1,0,0) & \text{above the sheet } (x_3 > 0), \\ -\tfrac{1}{2}\mu_0 K\,(1,0,0) & \text{below the sheet } (x_3 < 0). \end{cases} \tag{72}$$

This example illustrates general behavior of the vector potential for all kinds of 2D current sheets, flat or curved, with uniform or non-uniform 2D currents: *The vector potential is continuous across the current sheet, but its normal derivative has a discontinuity,*

$$\mathrm{disc}\left(\frac{\partial \mathbf{A}}{\partial x_{\mathrm{normal}}}\right) \;=\; -\mu_0 \mathbf{K}. \tag{73}$$

Consequently, the magnetic field has a discontinuity

$$\mathrm{disc}(\mathbf{B}) \;=\; \mu_0 \mathbf{K} \times \mathbf{n} \tag{74}$$

where $\mathbf{n}$ is the unit vector $\perp$ to the current sheet.

$\star$ A general solution of eq. (68) is $A(x_3) = -\frac{1}{2}\mu_0 K \times |x_3| + \alpha x_3 + \beta$ for arbitrary constants α and β, but the upside-down symmetry $x_3 \to -x_3$ of the current sheet requires $\alpha = 0$, while β is physically irrelevant.

Multipole Expansion for the Vector Potential

Suppose electric current I flows through a closed wire loop of some complicated shape, and we want to find its magnetic field far away from the wire. Let's work through the vector potential according to the Coulomb-like formula

$$\mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi} \oint\limits_{\text{wire}} \frac{I\,d\mathbf{y}}{|\mathbf{x}-\mathbf{y}|}\,. \tag{75}$$

Far away from the wire, we may expand the denominator here into a power series in $(|y|/|x|)$, thus

$$\frac{1}{|\mathbf{x}-\mathbf{y}|} = \sum_{\ell=0}^{\infty} \frac{|\mathbf{y}|^{\ell}}{|\mathbf{x}|^{\ell+1}} \times P_\ell(\mathbf{n}_x \cdot \mathbf{n}_y) \tag{76}$$

where $\mathbf{n}_x \cdot \mathbf{n}_y = \cos(\text{angle between } \mathbf{x} \text{ and} \mathbf{y})$. Plugging the expansion (76) into eq. (75) for the vector potential, we obtain

$$\mathbf{A}(\mathbf{x}) = \frac{\mu_0 I}{4\pi} \sum_{\ell=0}^{\infty} \frac{1}{|\mathbf{x}|^{\ell+1}} \oint\limits_{\text{wire}} |\mathbf{y}|^{\ell} P_\ell(\mathbf{n}_x \cdot \mathbf{n}_y)\,d\mathbf{y} \tag{77}$$

— the *expansion of the vector potential into magnetic multipole terms.* Let me write down more explicit formulae for the three leading terms,

$$\mathbf{A}(\mathbf{x}) = \frac{\mu_0 I}{4\pi} \left[\begin{array}{ll} \displaystyle \frac{1}{r} \oint d\mathbf{y} & \langle\!\langle\,\text{monopole}\,\rangle\!\rangle \\ \displaystyle +\ \frac{1}{r^2} \oint (\mathbf{n}_x \cdot \mathbf{y})\,d\mathbf{y} & \langle\!\langle\,\text{dipole}\,\rangle\!\rangle \\ \displaystyle +\ \frac{1}{r^3} \oint \left(\tfrac{3}{2}(\mathbf{n}_x \cdot \mathbf{y})^2 - \tfrac{1}{2}\mathbf{y}^2\right) d\mathbf{y} & \langle\!\langle\,\text{quadrupole}\,\rangle\!\rangle \\ +\ \cdots & \langle\!\langle\,\text{higher multipoles}\,\rangle\!\rangle \end{array} \right]. \tag{78}$$

Naively, the leading term in this expansion is the monopole term for $\ell = 0$ (the top line in eq. (78)), but it vanishes for any closed current loop,

$$\oint d\mathbf{y} = 0 \tag{79}$$

Thus, *the magnetic multipole expansion starts with the dipole term* — which dominates the magnetic field at large distances from the wire loop. (Except when the dipole moment happens to vanish.)

Let's simplify the dipole term in (78) using a bit of vector calculus, Let $\mathbf{c}$ be some constant vector, then

$$\mathbf{c}\cdot\oint(\mathbf{n}\cdot\mathbf{y})\,d\mathbf{y} \;=\; \oint(\mathbf{n}\cdot\mathbf{y})\,\mathbf{c}\cdot d\mathbf{y} \;=\; \langle\!\langle\,\text{by Stokes' theorem}\,\rangle\!\rangle \;=\; \iint\Big(\nabla_y\times\big((\mathbf{n}\cdot\mathbf{y})\,\mathbf{c}\big)\Big)\cdot d^2\mathbf{a} \quad (80)$$

where

$$\nabla_y\times\big((\mathbf{n}\cdot\mathbf{y})\,\mathbf{c}\big) \;=\; \big(\nabla_y(\mathbf{n}\cdot\mathbf{y})\big)\times\mathbf{c} \;=\; \mathbf{n}\times\mathbf{c}, \tag{81}$$

and hence

$$\begin{aligned}
\mathbf{c}\cdot\oint(\mathbf{n}\cdot\mathbf{y})\,d\mathbf{y} \;&=\; \iint(\mathbf{n}\times\mathbf{c})\cdot d^2\mathbf{a} \;=\; (\mathbf{n}\times\mathbf{c})\cdot\iint d^2\mathbf{a}\\
&=\; (\mathbf{n}\times\mathbf{c})\cdot\mathbf{a} \quad \langle\!\langle\,\text{where } \mathbf{a} \text{ is net the vector area of the loop}\,\rangle\!\rangle\\
&=\; (\mathbf{a}\times\mathbf{n})\cdot\mathbf{c}.
\end{aligned} \tag{82}$$

Since $\mathbf{c}$ here can be *any* constant vector, it follows that

$$\oint(\mathbf{n}\cdot\mathbf{y})\,d\mathbf{y} \;=\; \mathbf{a}\times\mathbf{n}. \tag{83}$$

Finally, plugging this integral into the dipole term in the expansion (78), we arrive at

$$\mathbf{A}_{\text{dipole}}(\mathbf{x}) \;=\; \frac{\mu_0}{4\pi}\,\frac{I\mathbf{a}\times\mathbf{n}}{r^2} \;=\; \frac{\mu_0}{4\pi}\,\frac{\mathbf{m}\times\mathbf{n}}{r^2} \tag{84}$$

where $\mathbf{m} = I\mathbf{a}$ is the *magnetic dipole moment* of the current loop.

I am going to skip over the higher multipoles in these notes. Instead, let me consider replacing a single wire loop with a circuit of several connected wires. In this case, we may use the Kirchhoff Law to express the whole circuit as several overlapping loops with independent currents; if a wire belongs to several loops, the current in that wire is the algebraic sum of the appropriate loop currents. By the superposition principle, the vector potential of the whole circuit is the sum of vector potentials of the individual loops, and as long as the whole circuit occupies small volume of size $\ll r$, we may expand each loop's $\mathbf{A}$ into multipoles, exactly as we did it for a single loop. In general, the leading contribution is the *net dipole term*,

$$\mathbf{A}_{\text{dipole}}(\mathbf{n}) \;=\; \sum_i^{\text{loops}} \frac{\mu_0}{4\pi}\,\frac{\mathbf{m}_i\times\mathbf{n}}{r^2} \;=\; \frac{\mu_0}{4\pi}\,\frac{\mathbf{m}_{\text{net}}\times\mathbf{n}}{r^2} \tag{85}$$

where

$$\mathbf{m}_{\text{net}} \;=\; \sum_i^{\text{loops}} \mathbf{m}_i \;=\; \sum_i^{\text{loops}} I_i\,\mathbf{a}_i \tag{86}$$

is the net dipole moment of the whole circuit.

Now suppose instead of a circuit of thin wires we have some current density $\mathbf{J}(\mathbf{y})$ flowing through the volume of some thick conductor. However, the conductor's size is much smaller than the distance r to where we want to calculate the vector potential and the magnetic field. In this case, we may use the multipole expansion, but the algebra is a bit different from what we had for a thin wire:

$$\mathbf{A}(\mathbf{y}) \;=\; \frac{\mu_0}{4\pi}\iiint\frac{\mathbf{J}(\mathbf{y})\,d^3\mathbf{y}}{|\mathbf{x}-\mathbf{y}|} \;=\; \frac{\mu_0}{4\pi}\sum_{\ell=0}^{\infty}\frac{1}{|\mathbf{x}|^{\ell+1}}\iiint|\mathbf{y}|^{\ell}P_\ell(\mathbf{n}_x\cdot\mathbf{n}_y)\,\mathbf{J}(\mathbf{y})\,d^3\mathbf{y}, \tag{87}$$

or in a more explicit form

$$\mathbf{A}(\mathbf{x}) \;=\; \frac{\mu_0 I}{4\pi}\left[\begin{array}{ll} \frac{1}{r}\iiint\mathbf{J}(\mathbf{y})\,d^3\mathbf{y} & \langle\!\langle\,\text{monopole}\,\rangle\!\rangle \\ +\;\frac{1}{r^2}\iiint(\mathbf{n}\cdot\mathbf{y})\,\mathbf{J}(\mathbf{y})\,d^3\mathbf{y} & \langle\!\langle\,\text{dipole}\,\rangle\!\rangle \\ +\;\frac{1}{r^3}\iiint(\tfrac{3}{2}(\mathbf{n}\cdot\mathbf{y})^2-\tfrac{1}{2}\mathbf{y}^2)\,\mathbf{J}(\mathbf{y})\,d^3\mathbf{y} & \langle\!\langle\,\text{quadrupole}\,\rangle\!\rangle \\ +\;\cdots & \langle\!\langle\,\text{higher multipoles}\,\rangle\!\rangle \end{array}\right]. \tag{88}$$

The monopole term here vanishes just as it did for the wire loop, albeit in a less obvious way. To see how this works, pick a constant vector $\mathbf{c}$ and take the divergence

$$\nabla_y\cdot\big((\mathbf{c}\cdot\mathbf{y})\mathbf{J}(\mathbf{y})\big) \;=\; \mathbf{c}\cdot\mathbf{J} \;+\; (\mathbf{c}\cdot\mathbf{y})\,(\nabla\cdot\mathbf{J}), \tag{89}$$

where the second term on the RHS vanishes for a steady — and hence divergence-less —

current. Consequently,

$$\begin{aligned}
\mathbf{c}\cdot\iiint\limits_{\mathcal{V}}\mathbf{J}(\mathbf{y})\,d^3\mathbf{y} &= \iiint\limits_{\mathcal{V}}(\mathbf{c}\cdot\mathbf{J}(\mathbf{y}))\,d^3\mathbf{y} \\
&\quad\langle\!\langle\text{ by eq. (89) }\rangle\!\rangle \\
&= \iiint\limits_{\mathcal{V}}\nabla_y\cdot\big((\mathbf{c}\cdot\mathbf{y})\mathbf{J}(\mathbf{y})\big)\,d^3\mathbf{y} \\
&\quad\langle\!\langle\text{ by Gauss theorem }\rangle\!\rangle \\
&= \iint\limits_{\mathcal{S}}\big((\mathbf{c}\cdot\mathbf{y})\mathbf{J}(\mathbf{y})\big)\cdot d^2\mathbf{a}(\mathbf{y})
\end{aligned} \tag{90}$$

where $\mathcal{S}$ is the surface of the volume $\mathcal{V}$. That volume must include the whole conductor, but we may also make it a bit bigger, which would put the surface $\mathcal{S}$ outside the conductor. But then there would be no current along or across $\mathcal{S}$, so the integral on the bottom line of (90) must vanish. Consequently, the top line of eq. (90) must vanish too, and since $\mathbf{c}$ is an arbitrary constant vector, this means zero monopole moment,

$$\iiint\limits_{\mathcal{V}}\mathbf{J}(\mathbf{y})\,d^3\mathbf{y} = 0. \tag{91}$$

Next, consider the dipole term in (88) and try to rewrite it in the form (84) for some dipole moment vector $\mathbf{m}$. This time, the algebra is a bit more complicated. For an arbitrary but constant vector $\mathbf{c}$, we have

$$\mathbf{c}\cdot\big(\mathbf{n}\times(\mathbf{J}\times\mathbf{y})\big) = (\mathbf{c}\cdot\mathbf{J})\,(\mathbf{n}\cdot\mathbf{y}) - (\mathbf{c}\cdot\mathbf{y})\,(\mathbf{n}\cdot\mathbf{J}), \tag{92}$$

$$\nabla_{\mathbf{y}}\cdot\big((\mathbf{c}\cdot\mathbf{y})\,(\mathbf{n}\cdot\mathbf{y})\,\mathbf{J}(\mathbf{y})\big) = (\mathbf{c}\cdot\mathbf{J})\,(\mathbf{n}\cdot\mathbf{y}) + (\mathbf{c}\cdot\mathbf{y})\,(\mathbf{n}\cdot\mathbf{J}) + (\mathbf{c}\cdot\mathbf{y})\,(\mathbf{n}\cdot\mathbf{y})\,\cancel{(\nabla\cdot\mathbf{J})}, \tag{93}$$

$\langle\!\langle$ where the last term vanishes for a steady current. $\rangle\!\rangle$

$\langle\!\langle$ which has $\nabla\cdot\mathbf{J}=0$ $\rangle\!\rangle$

and hence

$$(\mathbf{c}\cdot\mathbf{J})\,(\mathbf{n}\cdot\mathbf{y}) = \tfrac{1}{2}\mathbf{c}\cdot\big(\mathbf{n}\times(\mathbf{J}\times\mathbf{y})\big) + \tfrac{1}{2}\nabla_y\cdot\big((\mathbf{c}\cdot\mathbf{y})\,(\mathbf{n}\cdot\mathbf{y})\,\mathbf{J}(\mathbf{y})\big). \tag{94}$$

Consequently, dotting $\mathbf{c}$ with the dipole integral, we obtain

$$\begin{aligned}
\mathbf{c}\cdot\iiint\limits_{\mathcal{V}}(\mathbf{n}\cdot\mathbf{y})\,\mathbf{J}(\mathbf{y})\,d^3\mathbf{y} &= \iiint\limits_{\mathcal{V}}(\mathbf{c}\cdot\mathbf{J})\,(\mathbf{n}\cdot\mathbf{y})\,d^3\mathbf{y} \\
&= \frac{1}{2}\iiint\limits_{\mathcal{V}}\Big(\mathbf{c}\cdot\big(\mathbf{n}\times(\mathbf{J}\times\mathbf{y})\big)\Big)d^3\mathbf{y} \\
&\quad + \frac{1}{2}\iiint\limits_{\mathcal{V}}\Big(\nabla_y\cdot\big((\mathbf{c}\cdot\mathbf{y})\,(\mathbf{n}\cdot\mathbf{y})\,\mathbf{J}(\mathbf{y})\big)\Big)d^3\mathbf{y} \\
&= \frac{1}{2}\,\mathbf{c}\cdot\left(\mathbf{n}\times\iiint\limits_{\mathcal{V}}(\mathbf{J}(\mathbf{y})\times\mathbf{y})\,d^3\mathbf{y}\right) \\
&\quad + \frac{1}{2}\iint\limits_{\mathcal{S}}(\mathbf{c}\cdot\mathbf{y})\,(\mathbf{n}\cdot\mathbf{y})\,\mathbf{J}(\mathbf{y})\cdot d^2\mathbf{a}'
\end{aligned} \tag{95}$$

Similar to what we did for the monopole term, let's take the integration volume $\mathcal{V}$ a bit larger that the whole conductor, so its surface $\mathcal{S}$ is completely outside the conductor. Then on the last line of eq. (95) the current $\mathbf{J}$ vanishes everywhere on the surface, which kills the surface integral. This leaves us with

$$\mathbf{c}\cdot\iiint\limits_{\text{conductor}+}(\mathbf{n}\cdot\mathbf{y})\,\mathbf{J}(\mathbf{y})\,d^3\mathbf{y} = \frac{1}{2}\,\mathbf{c}\cdot\left(\mathbf{n}\times\iiint\limits_{\text{conductor}+}(\mathbf{J}(\mathbf{y})\times\mathbf{y})\,d^3\mathbf{y}\right), \tag{96}$$

and since $\mathbf{c}$ is an arbitrary constant vector,

$$\begin{aligned}
\iiint\limits_{\text{conductor}+}(\mathbf{n}\cdot\mathbf{y})\,\mathbf{J}(\mathbf{y})\,d^3\mathbf{y} &= \frac{\mathbf{n}}{2}\times\iiint\limits_{\text{conductor}+}\mathbf{J}(\mathbf{y})\times\mathbf{y}\,d^3\mathbf{y} \\
&= \left(\frac{1}{2}\iiint\limits_{\text{conductor}+}\mathbf{y}\times\mathbf{J}(\mathbf{y})\,d^3\mathbf{y}\right)\times\mathbf{n}.
\end{aligned} \tag{97}$$

Plugging this formula into the dipole term in the vector potential (88), we arrive at

$$\mathbf{A}_{\text{dipole}}(\mathbf{n}) = \frac{\mu_0}{4\pi r^2}\iiint\limits_{\text{conductor}+}(\mathbf{n}\cdot\mathbf{y})\,\mathbf{J}(\mathbf{y})\,d^3\mathbf{y} = \frac{\mu_0}{4\pi}\,\frac{\mathbf{m}\times\mathbf{n}}{r^2} \tag{98}$$

— exactly as in eq. (84) for the current loop — for the *magnetic dipole moment*

$$\mathbf{m} = \frac{1}{2} \iiint\limits_{\text{conductor}+} \mathbf{y} \times \mathbf{J}(\mathbf{y})\, d^3\mathbf{y}. \tag{99}$$

As a useful cross-check of this formula, suppose the conductor in question is a thin wire loop. Then the current element $\mathbf{J}(y)\, d^3\mathbf{y}$ reduces to $I\, d\mathbf{y}$ and the volume integral becomes a line integral along the wire, thus

$$\mathbf{m} = \frac{1}{2} \oint\limits_{\text{wire}} \mathbf{y} \times I\, d\mathbf{y} = I\, \mathbf{a} \tag{100}$$

where

$$\mathbf{a} = \frac{1}{2} \oint \mathbf{y} \times d\mathbf{y} \tag{101}$$

is the vector area of any surface spanned by the wire loop. Thus, when the current happens to flow around a loop of thin wire, eq. (99) for its magnetic dipole moment agrees with the simper formula $\mathbf{m} = I\, \mathbf{a}$.

EXAMPLE:

Consider a rotating body made of material with a uniform electric charge density to mass density ratio,

$$\frac{dQ/d\text{volume}}{dM/d\text{volume}} = \text{const} = \frac{Q}{M}. \tag{102}$$

Regardless of the body's shape or size or of its angular velocity, the magnetic moment of the rotating body and its angular momentum $\mathbf{L}$ point in the same directions (modulo sign of Q) and their ratio is fixed at $Q/2M$,

$$\mathbf{M} = \frac{Q}{2M} \mathbf{L}. \tag{103}$$

Proof: consider an infinitesimal part of the body of charge dQ located at $\mathbf{x}$ and moving at velocity $\mathbf{v}$. Then this part carries current

$$\mathbf{J}\, d\text{volume} \;=\; dQ\,\mathbf{v}, \tag{104}$$

so its contribution to the magnetic dipole moment (99) is

$$d\mathbf{m} \;=\; \tfrac{1}{2}\,\mathbf{x}\times(dQ\,\mathbf{v}), \tag{105}$$

and hence the net magnetic moment of the body is

$$\mathbf{m} \;=\; \frac{1}{2}\int(\mathbf{x}\times\mathbf{v})\,dQ \tag{106}$$

At the same time, the net angular momentum of the body is

$$\mathbf{L} \;=\; \int(\mathbf{x}\times\mathbf{v})\,dM, \tag{107}$$

so for the fixed dQ/dM ratio,

$$\mathbf{m} \;=\; \frac{Q}{2M}\,\mathbf{L}. \tag{108}$$

Quod erat demonstrandum.

Note that this argument does not care if the rotating body keeps a rigid shape or if its different part rotate at different rates, or even move along non-circular paths. We may even replace a single rotating body with a system of particles moving independently from each other, but as long as each moving particle has the same charge-to-mass ratio, the net magnetic moment and the net angular momentum of the system are related according to the *gyromagnetic relation* (103).

For example, *classically* the net magnetic moment and the net angular momentum of an atom are related as

$$\mathbf{m} \;=\; \frac{-e}{2m_e}\,\mathbf{L}. \tag{109}$$

In quantum mechanics, we would have a similar relation for charged particles without spin, but the electrons have spin, which complicates the situation. The spin — the intrinsic angular momentum of an electron, independent on its orbital motion — also comes with a magnetic moment, but in a different ratio,

$$\mathbf{m} \;=\; \frac{-e}{2m_e}\,g_e\,\mathbf{S} \tag{110}$$

where $g_e \approx 2$ is the *gyromagnetic factor of the electron.* According to the Dirac equation, g_e should be exactly 2, but in quantum field theory there are small corrections to this value due to interactions between the electrons and the virtual photons. In fact, g_e has been calculated theoretically to an incredible precision of 13 significant figures, and it has also been measured experimentally to a similar precision,

$$g_e \;=\; 2.002\,319\,304\,361\,8\,(5) \tag{111}$$

The net magnetic moment of an atom is the sum of individual electron's magnetic moments due to both spin and orbital angular momentum, thus

$$\mathbf{m}_{\text{net}} \;=\; \frac{-e}{2m_e}\sum_i(\mathbf{L}_i + g_e\mathbf{S}_i) \;=\; \frac{-e}{2m_e}\big(\mathbf{L}_{\text{net}} \,+\, g_e\mathbf{S}_{\text{net}}\big). \tag{112}$$

Dipole Fields

Given the vector potential

$$\mathbf{A}_{\text{dipole}}(\mathbf{x}) \;=\; \frac{\mu_0}{4\pi}\,\frac{\mathbf{m}\times\mathbf{n}}{r^2} \tag{84}$$

of a magnetic dipole, let's calculate its magnetic field $\mathbf{B}(\mathbf{x})$. For comparison, let me also calculate the electric field $\mathbf{E}(\mathbf{x})$ of an electric dipole $\mathbf{p}$.

Let me start with a bit of vector calculus. Naively, the second derivative tensor of $1/r$ is

$$\nabla_i\nabla_j\left(\frac{1}{r}\right) \;=\; \nabla_i\left(\frac{-n_j}{r^2} \;=\; \frac{-x_j}{r^3}\right) \;=\; \frac{-\delta_{ij}}{r^3} \;+\; \frac{3x_jx_i}{r^5} \;=\; \frac{3n_in_j-\delta_{ij}}{r^3}\,, \tag{113}$$

but then the trace of this tensor would give us $\nabla^2(1/r) = 0$, which is right for $\mathbf{x} \neq 0$ but misses the delta-function at $\mathbf{x} = 0$. So we need to add this delta-function to the second derivative tensor, and to make sure the extra term is spherically symmetric, its index dependence should be δ_{ij}, thus

$$\nabla_i\nabla_j\left(\frac{1}{r}\right) \;=\; \frac{3n_in_j-\delta_{ij}}{r^3} \;-\; \frac{4\pi}{3}\,\delta_{ij}\delta^{(3)}(\mathbf{x}). \tag{114}$$

Now consider the electric dipole potential

$$\Phi_{\rm dipole}(\mathbf{x}) \;=\; \frac{1}{4\pi\epsilon_0}\,\frac{\mathbf{p}\cdot\mathbf{n}}{r^2} \;=\; \frac{\mathbf{p}}{4\pi\epsilon_0}\cdot\nabla\left(\frac{-1}{r}\right). \tag{115}$$

Consequently, the electric field of this dipole is

$$E_i(\mathbf{x}) \;=\; -\nabla_i\Phi(\mathbf{x}) \;=\; \frac{p_j}{4\pi\epsilon_0}\,\nabla_i\nabla_j\left(\frac{-1}{r}\right) \;=\; \frac{p_j}{4\pi\epsilon_0}\left(\frac{3n_in_j-\delta_{ij}}{r^3} \;-\; \frac{4\pi}{3}\,\delta_{ij}\delta^{(3)}(\mathbf{x})\right), \tag{116}$$

or in vector notations,

$$\mathbf{E}_{\rm dipole}(\mathbf{x}) \;=\; \frac{1}{4\pi\epsilon_0}\left(\frac{\mathbf{3n(n\cdot p)\;-\;p}}{r^3} \;-\; \frac{4\pi}{3}\,\mathbf{p}\,\delta^{(3)}(\mathbf{x})\right). \tag{117}$$

By comparison, the magnetic monopole field obtains as a curl of the vector potential (84), which we may rewire as

$$\mathbf{A}(\mathbf{x}) \;=\; \frac{\mu_0\mathbf{m}}{4\pi}\times\left(\frac{\mathbf{n}}{r^2}\;=\;\nabla\left(\frac{-1}{r}\right)\right) \;=\; +\nabla\times\left(\frac{\mu_0\mathbf{m}}{4\pi\,r}\right).$$

Consequently,

$$\mathbf{B}(\mathbf{x}) \;=\; \nabla\times\mathbf{A}(\mathbf{x}) \;=\; \nabla\times\nabla\times\left(\frac{\mu_0\mathbf{m}}{4\pi\,r}\right) \;=\; \nabla\left(\nabla\cdot\left(\frac{\mu_0\mathbf{m}}{4\pi\,r}\right)\right) \;-\; \nabla^2\left(\frac{\mu_0\mathbf{m}}{4\pi\,r}\right), \tag{118}$$

where by analogy with the electric dipole calculation above

$$\nabla\left(\nabla\cdot\left(\frac{\mathbf{m}}{r}\right)\right) \;=\; \frac{\mathbf{3n(n\cdot m)\;-\;m}}{r^3} \;-\; \frac{4\pi}{3}\,\mathbf{m}\,\delta^{(3)}(\mathbf{x}), \tag{119}$$

while

$$\nabla^2\left(\frac{\mathbf{m}}{r}\right) \;=\; -4\pi\,\mathbf{m}\,\delta^{(3)}(\mathbf{x}). \tag{120}$$

Altogether, we have

$$\mathbf{B}_{\rm dipole}(\mathbf{x}) \;=\; \frac{\mu_0}{4\pi}\left(\frac{\mathbf{3n(n\cdot m)\;-\;m}}{r^3} \;+\; \frac{8\pi}{3}\,\mathbf{m}\,\delta^{(3)}(\mathbf{x})\right). \tag{121}$$

Remarkably, the electric and the magnetic dipole fields have exactly similar forms at

$\mathbf{x} \neq 0$, although the delta-function terms at $\mathbf{x} = 0$ are different. These delta-function terms are often ignored, but sometimes they become important. For example, in the hyperfine structure of atomic energy levels — which arises from electrons integrating with the nuclear magnetic moments — the delta-function term is important for the s–wave electrons.

Force and Torque on a Magnetic Dipole

Besides producing similar-looking fields, the electric and the magnetic dipoles also feel similar forces and torques in external fields:

$$\text{Electric:} \quad \mathrm{F} = \nabla(\mathbf{p}\cdot\mathbf{E}), \quad \boldsymbol{\tau} = \mathbf{p}\times\mathbf{E}; \tag{122}$$

$$\text{Magnetic:} \quad \mathrm{F} = \nabla(\mathbf{m}\cdot\mathbf{B}), \quad \boldsymbol{\tau} = \mathbf{m}\times\mathbf{B}. \tag{123}$$

You should have derived the electric force and torque equations (122) in your homework (set#1, problem 3),[⋆] so let's derive their magnetic analogies (123).

Let's start by showing that any closed current-carrying wire loop — whatever its shape or size — placed in a *uniform* external magnetic field $\mathbf{B} = \text{const}$ feels zero net force but a non-zero net torque,

$$\text{for } \mathbf{B} = \text{const}, \quad \mathbf{F}_{\text{net}} = 0, \quad \boldsymbol{\tau}_{\text{net}} = \mathbf{m}\times\mathbf{B}. \tag{124}$$

For the net force, this is trivial,

$$\mathbf{F}_{\text{net}} = \oint I d\vec{\ell}\times\mathbf{B} = I\left(\oint d\vec{\ell} = 0\right)\times\mathbf{B} = 0, \tag{125}$$

but to calculate the net torque

$$\boldsymbol{\tau}_{\text{net}} = \oint \mathbf{x}\times d(\text{on } \mathbf{x}) = \oint \mathbf{x}\times(I\,\mathbf{x}\times\mathbf{B}), \tag{126}$$

we need to use some algebraic tricks. Note that for a uniform magnetic field $\mathbf{B}$,

$$d\big(\mathbf{x}\times(\mathbf{x}\times\mathbf{B})\big) = d\mathbf{x}\times(\mathbf{x}\times\mathbf{B}) + \mathbf{x}\times(d\mathbf{x}\times\mathbf{B}), \tag{127}$$

⋆ In the homework, the dipole term in the electric force appears as $\mathbf{F} = (\mathbf{p}\cdot\nabla)\mathbf{E}$ rather than $\mathbf{F} = \nabla(\mathbf{p}\cdot\mathbf{E})$ as in eq. (122). However, for an electrostatic field obeying $\nabla\times\mathbf{E} = 0$, the two formulae are equivalent. Indeed,

$$\nabla(\mathbf{p}\cdot\mathbf{E}) - (\mathbf{p}\cdot\nabla)\mathbf{E} = \mathbf{p}\times(\nabla\times\mathbf{E}) = 0.$$

$$\begin{aligned}
\mathbf{B}\times(d\mathbf{x}\times\mathbf{x}) &= -d\mathbf{x}\times(\mathbf{x}\times\mathbf{B}) \;-\; \mathbf{x}\times(\mathbf{B}\times d\mathbf{x}) \quad \langle\!\langle\,\text{by the Jacobi identity}\,\rangle\!\rangle \\
&= -d\mathbf{x}\times(\mathbf{x}\times\mathbf{B}) \;+\; \mathbf{x}\times(d\mathbf{x}\times\mathbf{B}), \qquad (128)\\
\text{hence } \mathbf{x}\times(d\mathbf{x}\times\mathbf{B}) &= \tfrac{1}{2}\,d\big(\mathbf{x}\times(\mathbf{x}\times\mathbf{B})\big) \;+\; \tfrac{1}{2}\,\mathbf{B}\times(d\mathbf{x}\times\mathbf{x}). \qquad (129)
\end{aligned}$$

Consequently,

$$\begin{aligned}
\boldsymbol{\tau}_{\text{net}} &= \oint \mathbf{x}\times(I\,d\mathbf{x}\times\mathbf{B}) \\
&= \frac{I}{2}\oint d\big(\mathbf{x}\times(\mathbf{x}\times\mathbf{B})\big) \;+\; \frac{I}{2}\oint \mathbf{B}\times(d\mathbf{x}\times\mathbf{x}) \\
&\quad \langle\!\langle\,\text{where the first } \textstyle\oint \text{ of a total differential is zero}\,\rangle\!\rangle \qquad (130)\\
&= 0 \;+\; \mathbf{B}\times\frac{I}{2}\oint d\mathbf{x}\times\mathbf{x} \\
&= \left(\frac{I}{2}\oint \mathbf{x}\times d\mathbf{x}\right)\times\mathbf{B}.
\end{aligned}$$

But for any closed loop, the vector area of any surface spanning that loop may be calculated as

$$\mathbf{a} \;=\; \frac{1}{2}\oint \mathbf{x}\times d\mathbf{x}, \tag{131}$$

hence on the bottom line of eq. (130),

$$\frac{I}{2}\oint \mathbf{x}\times d\mathbf{x} \;=\; I\mathbf{a} \;=\; \mathbf{m} \tag{132}$$

— the magnetic moment of the loop. Altogether, the net torque is indeed

$$\boldsymbol{\tau}_{\text{net}} \;=\; \mathbf{m}\times\mathbf{B}. \tag{133}$$

When the external magnetic field is not uniform but varies on the distance scales much larger than the loop's size, we may expand $\mathbf{B}(\mathbf{x}) = \mathbf{B}(0) + (\mathbf{x}\cdot\nabla)\mathbf{B}(0) + \cdots$, and then calculate the force and the torque on the loop for each term. For the torque, the leading term comes from the $\mathbf{B}(0)$ itself, and it evaluates to $\boldsymbol{\tau} = \mathbf{m} \times \mathbf{B}(0)$. The first subleading term $(\mathbf{x} \cdot \nabla)\mathbf{B}(0)$ produces a small correction to the net torque,

$$\Delta\boldsymbol{\tau} \;=\; \oint \mathbf{x} \times \Big(I\, d\mathbf{x} \times \big((\mathbf{x} \cdot \nabla)\mathbf{B}(0) \big) \Big), \tag{134}$$

which can be related to the magnetic quadrupole moment of the loop, but let me skip those details. Altogether, in a non-uniform but slowly varying magnetic field, the net torque on the loop is

$$\boldsymbol{\tau}_{\rm net} \;=\; \mathbf{m} \times \mathbf{B}(0) \;+\; \text{corrections due to higher multipoles.} \tag{135}$$

For an ideal magnetic dipole, this formula reduces to the first term, $\boldsymbol{\tau} = \mathbf{m} \times \mathbf{B}$.

As to the net force on the loop, the leading contribution comes from the first derivative of the magnetic field, thus

$$\mathbf{F}_{\rm net} \;=\; \oint I\, d\mathbf{x} \times \big((\mathbf{x} \cdot \nabla)\mathbf{B}(0) \big) \;+\; \text{higher order terms,} \tag{136}$$

or in index notations,

$$F_i^{\rm leading} \;=\; \oint \epsilon_{ijk} (I\, dx_i)\, x_\ell \nabla_\ell B_k(0) \;=\; I \times \epsilon_{ijk} \nabla_\ell B_k(0) \times \oint dx_j\, x_\ell . \tag{137}$$

To evaluate the integral here, we use

$$dx_j\, x_\ell \;+\; dx_\ell\, x_j \;=\; d(x_j x_\ell), \tag{138}$$

$$dx_j\, x_\ell \;-\; dx_\ell\, x_j \;=\; \epsilon_{j\ell p} (d\mathbf{x} \times \mathbf{x})_p\,, \tag{139}$$

$$\text{hence} \quad dx_j\, x_\ell \;=\; \tfrac{1}{2} d(x_i x_\ell) \;+\; \tfrac{1}{2} \epsilon_{j\ell p} (d\mathbf{x} \times \mathbf{x})_p\,, \tag{140}$$

and therefore

$$\oint dx_j\, x_\ell \;=\; \frac{1}{2} \oint d(x_j x_\ell) \;+\; \frac{1}{2}\, \epsilon_{j\ell p} \times \oint (d\mathbf{x} \times \mathbf{x})_p \;=\; 0 \;+\; \frac{1}{2}\, \epsilon_{j\ell p} \times (-2\mathbf{a})_p \;=\; \epsilon_{\ell jp} (\mathbf{a})_p \tag{141}$$

where $\mathbf{a}$ is the vector area of the loop. Plugging this integral into eq. (137) for the force on the loop, we obtain

$$F_i^{\text{leading}} \;=\; I\times\epsilon_{ijk}\nabla_\ell B_k(0)\times\epsilon_{\ell jp}(\mathbf{a})_p \;=\; (Ia_p)\times\epsilon_{ijk}\epsilon_{\ell jp}\nabla_\ell B_k(0). \tag{142}$$

In this formula,

$$\epsilon_{ijk}\epsilon_{\ell jp} \;=\; \epsilon_{kij}\epsilon_{jp\ell} \;=\; \delta_{kp}\delta_{i\ell} \;-\; \delta_{k\ell}\delta_{ip} \tag{143}$$

and hence

$$\epsilon_{ijk}\epsilon_{\ell jp}\nabla_\ell B_k(0) \;=\; (\delta_{kp}\delta_{i\ell}-\delta_{k\ell}\delta_{ip})\nabla_\ell B_k(0) \;=\; \nabla_i B_p(0) \;-\; \delta_{ip}\,\nabla_\ell B_\ell(0), \tag{144}$$

where in the second term $\nabla_\ell B_\ell = \nabla\cdot\mathbf{B} = 0$. This leaves us with

$$F_i^{\text{leading}} \;=\; (Ia_p \;=\; m_p)\times\nabla_i B_p(0) \;=\; \nabla_i(\mathbf{m}\cdot\mathbf{B}(0)), \tag{145}$$

where the derivative ∇_i acts on the magnetic field but not on the dipole moment $\mathbf{m}$ (which we treat as fixed).

Altogether, we have the net force on the loop as

$$\mathbf{F}^{\text{net}} \;=\; \nabla(\mathbf{m}\cdot\mathbf{B}) \;+\; \text{corrections due to higher multipoles}. \tag{146}$$

For a pure magnetic dipole, this formula reduces to its first term $\mathbf{F} = \nabla(\mathbf{m}\cdot\mathbf{B})$.

For atoms and molecules, the magnetic dipole moment is fixed by the quantum effects. Consequently, the magnetic force (9) on an atom or a molecules acts as a potential force, with a potential energy

$$U(\mathbf{x}) \;=\; -\mathbf{m}\cdot\mathbf{B}(\mathbf{x}). \tag{147}$$

The same potential energy — or rather its variation when the magnetic moment $\mathbf{m}$ changes its direction — is also responsible for the magnetic torque $\boldsymbol{\tau} = \mathbf{m}\times\mathbf{B}$.

In quantum mechanics, this potential energy becomes the Hamiltonian term for particles in an external magnetic field. As we saw earlier, the net magnetic moment of an atom is

$$\mathbf{m}_{\rm net} \;=\; \frac{-e}{2m_e}\big(\mathbf{L}_{\rm net}\,+\,g_e\mathbf{S}_{\rm net}\big) \tag{148}$$

where $g_e \approx 2$, so interaction of the atom with an external magnetic field is governed by the Hamiltonian term

$$\Delta\hat{H} \;=\; +\frac{e}{2m_e}\,\mathbf{B}\cdot\big(\hat{\mathbf{L}}_{\rm net}\,+\,g_e\hat{\mathbf{S}}_{\rm net}\big). \tag{149}$$

It is this term in the net Hamiltonian of the atom which is responsible for the Zeemann effect — the splitting of the otherwise-degenerate energy level by an external magnetic field.

The hyperfine structure is a similar effect due to magnetic field of the atomic nucleus, in case it happens to have a non-zero magnetic moment $\mathbf{m}_N$. Since the nucleus is has very small size compared to the electron's orbits, we may approximate its magnetic field as pure dipole (121), thus

$$\Delta\hat{H}_{HF} \;=\; \frac{\mu_0 e}{8\pi m_e}\left[\begin{array}{l}\dfrac{8\pi g_e}{3}\,(\hat{\mathbf{m}}_N\cdot\hat{\mathbf{S}})\,\delta^{(3)}(\hat{\mathbf{x}}) \\ \qquad\qquad -\dfrac{(\hat{\mathbf{m}}_N\cdot\hat{\mathbf{L}})}{\hat{r}^3}\;+\;g_e\,\dfrac{3(\hat{\mathbf{m}}_N\cdot\hat{\mathbf{n}})(\hat{\mathbf{S}}\cdot\hat{\mathbf{n}})\,-\,(\hat{\mathbf{m}}_N\cdot\hat{\mathbf{S}})}{\hat{r}^3}\end{array}\right]. \tag{150}$$

The delta-function term on the first line inside $[\cdots]$ affects the s–wave electrons, while the terms on the second line affect the other electrons with $\ell \neq 0$.