# Electric Quadrupole radiation

As I explained last week, and oscillating electric quadrupole radiates EM waves which in the far zone become

$$\begin{aligned} \mathbf{H}(\mathbf{x}) &= \frac{i\omega k^2}{12\pi}\,\frac{e^{ikr}}{r}\,\big(\mathbf{n}\times(\mathcal{Q}\cdot\mathbf{n})\big),\\ \mathbf{E}(\mathbf{x}) &= \frac{i\omega k^2}{12\pi}\,Z_0\,\frac{e^{ikr}}{r}\,\big((\mathcal{Q}.\mathbf{n})\,-\,\mathbf{n}(\mathbf{n}\cdot\mathcal{Q}\cdot\mathbf{n})\big). \end{aligned} \tag{1}$$

In my notations here, $r$ and $\mathbf{n}$ are the radius and the direction of the position $\mathbf{x}$, and $\mathcal{Q}$ is the amplitude of the quadrupole moment tensor in matrix form. The product $\mathcal{Q}\cdot\mathbf{n}$ is the matrix product of a matrix and a vector — its a vector with components $(\mathcal{Q}\cdot\mathbf{n})_i = Q_{ij}n_j$.

The power carried by the EM waves (1) in the direction $\mathbf{n}$ is given by

$$\begin{aligned} \frac{dP}{d\Omega} &= \frac{Z_0\omega^2k^4}{288\pi^2}\,\Big\|\mathbf{n}\times(\mathcal{Q}\cdot\mathbf{n})\Big\|^2\\ &= \frac{\omega^6}{288\pi^2\epsilon_0c^5}\,\Big((\mathcal{Q}^*\cdot\mathbf{n})\cdot(\mathcal{Q}\cdot\mathbf{n})\,-\,\big|\mathbf{n}\cdot\mathcal{Q}\cdot\mathbf{n}\big|^2\Big). \end{aligned} \tag{2}$$

To calculate the net radiated power, we need to integrate this formula over the $4\pi$ solid angle. In components,

$$\oiint d^2\Omega\Big((\mathcal{Q}^*\cdot\mathbf{n})\cdot(\mathcal{Q}\cdot\mathbf{n}) - \big|\mathbf{n}\cdot\mathcal{Q}\cdot\mathbf{n}\big|^2\Big) \;=\; Q^*_{ij}Q_{ik}\oiint d^2\Omega\,n_jn_k - Q^*_{ij}Q_{k\ell}\oiint d^2\Omega\,n_in_jn_kn_\ell \tag{3}$$

where the remaining integrals on the RHS must be rotationaly invariant and also totally symmetric in the indices of all the $\mathbf{n}$ vectors. Thus

$$\oiint d^2\Omega\,n_jn_k \;=\; A_2\delta_{jk}\,,\quad \oiint d^2\Omega\,n_in_jn_kn_\ell \;=\; A_4\big(\delta_{ij}\delta_{k\ell}+\delta_{ik}\delta_{j\ell}+\delta_{i\ell}\delta_{jk}\big) \tag{4}$$

for some overall coefficients $A_2$ and $A_4$ which obtain by setting all the indices to 3 (*i.e.*, $z$):

$$A_2 \;=\oiint d^2\Omega\,\cos^2\theta \;=\; \frac{4\pi}{3}\,,\quad 3A_4 \;=\oiint d^2\Omega\,\cos^4\theta \;=\; \frac{4\pi}{5}\,. \tag{5}$$

Consequently,

$$\oiint d^2\Omega\left((\mathcal{Q}^*\cdot\mathbf{n})\cdot(\mathcal{Q}\cdot\mathbf{n})\;-\;\left|\mathbf{n}\cdot\mathcal{Q}\cdot\mathbf{n}\right|^2\right)\;=$$

$$\begin{aligned}
&=\;Q^*_{ij}Q_{ik}\times\frac{4\pi}{3}\,\delta_{jk}\;-\;Q^*_{ij}Q_{k\ell}\times\frac{4\pi}{15}\big(\delta_{ij}\delta_{k\ell}\,+\,\delta_{ik}\delta_{j\ell}\,+\,\delta_{i\ell}\delta_{jk}\big)\\
&=\;\frac{4\pi}{3}\,Q^*_{ij}Q_{ij}\;-\;\frac{4\pi}{15}\Big(Q^*_{ii}Q_{kk}\,+\,Q^*_{ij}Q_{ji}\,+\,Q^*_{ij}Q_{ij}\Big)\\
&\qquad\langle\!\langle\text{ using symmetry and tracelessness of the quadrupole moment tensor }\rangle\!\rangle\\
&=\;\frac{4\pi}{3}\,Q^*_{ij}Q_{ij}\;-\;\frac{4\pi}{15}\Big(0\,+\,2Q^*_{ij}Q_{ij}\Big)\\
&=\;\frac{4\pi}{5}\,Q^*_{ij}Q_{ij}\;=\;\frac{4\pi}{5}\,\mathrm{tr}\big(\mathcal{Q}^\dagger\mathcal{Q}\big),
\end{aligned}\tag{6}$$

and therefore the net radiated power is

$$P_{\mathrm{net}}\;=\;\frac{\omega^6}{360\pi\epsilon_0c^5}\,\mathrm{tr}\big(\mathcal{Q}^\dagger\mathcal{Q}\big).\tag{7}$$

The angular distribution of the quadrupole radiation depends on the structure of the quadrupole moment tensor, which can range from a linear quadrupole (all charges arranged along a line) to planer quadrupole (all charges in the same plane) to complicated 3D setups where the charges move in different directions with different phases. For specific examples, let's consider the quadrupole moment tensors proportional to the spherical harmonics $Y_{\ell,m}$ with $\ell=2$, namely

$$\begin{aligned}
\mathcal{Q}^{(m=0)}\;&=\;\frac{Q}{\sqrt{3/2}}\begin{pmatrix}-\frac12&0&0\\0&-\frac12&0\\0&0&+1\end{pmatrix},\\
\mathcal{Q}^{(m=\pm1)}\;&=\;\frac{Q}{2}\begin{pmatrix}0&0&+1\\0&0&\pm i\\+1&\pm i&0\end{pmatrix},\\
\mathcal{Q}^{(m=\pm2)}\;&=\;\frac{Q}{2}\begin{pmatrix}+1&\pm i&0\\\pm i&-1&0\\0&0&0\end{pmatrix}.
\end{aligned}\tag{8}$$

- The $m=0$ quadrupole mode included all linear quadrupoles as well as other configurations with similar symmetries. (An axial symmetry, or at least a symmetry of 90°

rotations around the $z$ axis.) For this mode

$$\mathcal{Q}\cdot\mathbf{n} \;=\; \frac{Q}{\sqrt{3/2}}\begin{pmatrix}-\frac{1}{2}n_x\\ -\frac{1}{2}n_y\\ +n_z\end{pmatrix} \quad\Longrightarrow\quad \begin{aligned}\|\mathcal{Q}\cdot\mathbf{n}\|^2 \;&=\; \frac{Q^2}{3/2}\left(\tfrac{1}{4}n_x^2+\tfrac{1}{4}n_y^2+n_z^2\right)\\ &=\; \frac{Q^2}{6}\Big(1+3n_z^2 \;=\; 1+3\cos^2\theta\Big)\end{aligned} \tag{9}$$

while

$$\mathbf{n}\cdot\mathcal{Q}\cdot\mathbf{n} \;=\; \frac{Q}{\sqrt{3/2}}\Big(-\tfrac{1}{2}n_x^2-\tfrac{1}{2}n_y^2+n_z^2 \;=\; \tfrac{3}{2}n_z^2-\tfrac{1}{2} \;=\; \tfrac{3}{2}\cos^2\theta-\tfrac{1}{2}\Big). \tag{10}$$

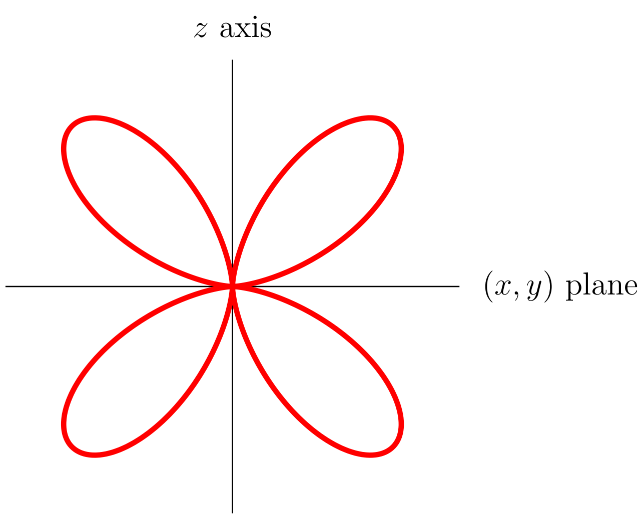
Consequently,

$$\begin{aligned}\|\mathcal{Q}\cdot\mathbf{n}\|^2 \;-\; |\mathbf{n}\cdot\mathcal{Q}\cdot\mathbf{n}|^2 \;&=\; \frac{Q^2}{6}\left(1+3\cos^2\theta\right) \;-\; \frac{Q^2}{3/2}\left(\tfrac{3}{2}\cos^2\theta-\tfrac{1}{2}\right)^2\\ &=\; \frac{Q^2}{6}\Big(1+3\cos^2\theta-1+6\cos^2\theta-9\cos^4\theta\Big)\\ &=\; \frac{Q^2}{6}\left(9\cos^2\theta-9\cos^4\theta\right)\\ &=\; \frac{3Q^2}{2}\times\cos^2\theta\sin^2\theta\end{aligned} \tag{11}$$

and therefore the angular distribution of the radiated power is

$$\frac{dP}{d\Omega} \;\propto\; \cos^2\theta\sin^2\theta. \tag{12}$$

Graphically,



- Next, consider the $m = \pm 1$ quadrupole modes, for which

$$\mathcal{Q}\cdot\mathbf{n} \;=\; \frac{Q}{2}\begin{pmatrix} +n_z \\ \pm i n_z \\ n_x \pm i n_y \end{pmatrix} \quad\Longrightarrow\quad \begin{aligned} \|\mathcal{Q}\cdot\mathbf{n}\|^2 \;&=\; \frac{Q^2}{4}\left(2n_z^2+n_x^2+n_y^2\right) \\ &=\; \frac{Q^2}{4}\,(1+\cos^2\theta) \end{aligned} \tag{13}$$

while

$$\mathbf{n}\cdot\mathcal{Q}\cdot\mathbf{n} \;=\; \frac{Q}{2}\times 2n_z(n_x\pm i n_y) \;=\; Q\times\cos\theta\sin\theta\,e^{\pm i\phi}. \tag{14}$$

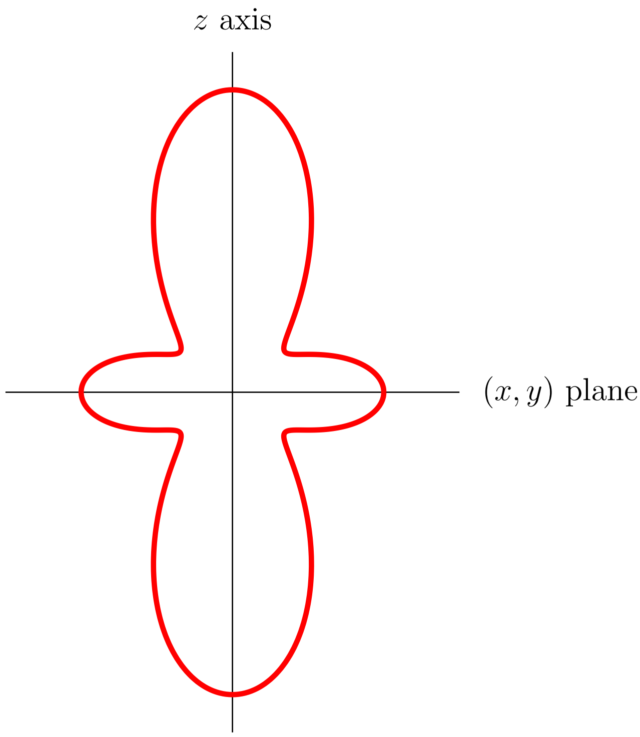
Consequently,

$$\begin{aligned} \|\mathcal{Q}\cdot\mathbf{n}\|^2 \;-\; |\mathbf{n}\cdot\mathcal{Q}\cdot\mathbf{n}|^2 \;&=\; \frac{Q^2}{4}\,(1+\cos^2\theta) \;-\; Q^2\cos^2\theta\sin^2\theta \\ &=\; \frac{Q^2}{4}\times\left(1-3\cos^2\theta+4\cos^4\theta\right) \end{aligned} \tag{15}$$

and therefore

$$\frac{dP}{d\Omega} \;\propto\; 1 \;-\; 3\cos^2\theta \;+\; 4\cos^4\theta. \tag{16}$$

Graphically,



- Finally, the $m = \pm 2$ quadrupole modes, which include the planar quadrupoles. For these modes

$$\mathcal{Q}\cdot\mathbf{n} \;=\; \frac{Q}{2}\begin{pmatrix} n_x \pm i n_y \\ \pm i n_x - n_y \\ 0 \end{pmatrix} \qquad\Longrightarrow\qquad \begin{aligned} \|\mathcal{Q}\cdot\mathbf{n}\|^2 \;&=\; \frac{Q^2}{4}\times 2|n_x \pm i n_y|^2 \\ &=\; \frac{Q^2}{2}\,\sin^2\theta \end{aligned} \tag{17}$$

while

$$\mathbf{n}\cdot\mathcal{Q}\cdot\mathbf{n} \;=\; \frac{Q}{2}\times(n_x \pm i n_y)^2 = \frac{Q}{2}\times\sin^2\theta\, e^{\pm 2i\phi}. \tag{18}$$

Consequently,

$$\begin{aligned} \|\mathcal{Q}\cdot\mathbf{n}\|^2 \;-\; |\mathbf{n}\cdot\mathcal{Q}\cdot\mathbf{n}|^2 \;&=\; \frac{Q^2}{2}\times\sin^2\theta \;-\; \frac{Q^2}{4}\times\sin^4\theta \\ &=\; \frac{Q^2}{4}\times(\sin^2 \;=\; 1-\cos^2\theta)\theta(2-\sin^2\theta \;=\; 1+\cos^2\theta) \\ &=\; \frac{Q^2}{4}\times(1-\cos^4\theta) \end{aligned} \tag{19}$$

and therefore

$$\frac{dP}{d\Omega} \;\propto\; 1-\cos^4\theta. \tag{20}$$

Graphically,