Bose–Einstein Condensation and Superfluidity

In these notes I explain the approximate ground state of a Bose–Einstein condensate, the spectrum of quasi-particle excitations of that ground state, and then explain why the BEC can flow without resistance. For simplicity, I assume zero temperature these notes, but if I have time in class I shall explain what happens at finite temperatures.

Let me start with the Landau–Ginzburg theory with Hamiltonian

$$\hat{H} - \mu\hat{N} = \int d^3\mathbf{x} \left(\frac{1}{2m} \nabla\hat{\psi}^\dagger \cdot \nabla\hat{\psi} - \mu\hat{\psi}^\dagger\hat{\psi} + \frac{\lambda}{2} \hat{\psi}^\dagger\hat{\psi}^\dagger\hat{\psi}\hat{\psi} \right) \tag{1}$$

where $\hat{\psi}(\mathbf{x})$ is the atom-annihilation quantum field while $\hat{\psi}^\dagger(\mathbf{x})$ is the atom-creation field. The Landau–Ginzburg theory with local $\hat{\psi}^\dagger\hat{\psi}^\dagger\hat{\psi}\hat{\psi}$ interactions — which correspond to zero-distance two-body repulsion $V_2(\mathbf{x} - \mathbf{y}) = \lambda\delta^{(3)}(\mathbf{x} - \mathbf{y})$ between the atoms — is a good approximation to reality for a low-density ultra-cold BEC of heavy atoms, and it's a good starting point for qualitative understanding of the superfluid liquid helium. Later in these notes we shall get to a better description of superfluid helium, but for the moment let's stick with the Landau–Ginzburg theory.

In the undergraduate StatMech textbooks, Bose–Einstein condensation is often described as all the atoms having exactly zero momenta, or in terms of occupation numbers $n_\mathbf{k}$ for momentum modes k, the BEC has $n_0 = N$ while all the other $n_\mathbf{k} = 0$,

$$|\text{naive BEC}\rangle = \frac{\left(\hat{a}_0^\dagger\right)^N}{\sqrt{N!}} |\text{vac}\rangle . \tag{2}$$

However, this naive ground state has unphysical long-distance correlations between the creation and annihilation fields at distant points in space,

$$G(\mathbf{x}-\mathbf{y}) = \left\langle \hat{\psi}^\dagger(\mathbf{x})\hat{\psi}(\mathbf{y}) \right\rangle - \left\langle \hat{\psi}^\dagger(\mathbf{x}) \right\rangle \left\langle \hat{\psi}(\mathbf{y}) \right\rangle = n - 0 \times 0 = n \quad \text{for any } \mathbf{x}-\mathbf{y}. \tag{3}$$

In real-life such correlations can never happen for any macroscopic number of atoms, so let's make a better starting approximation for the ground state, namely the coherent state of the

$\mathbf{k} = 0$ mode,

$$|\text{coherent}\rangle \;=\; e^{-N/2} \exp(\sqrt{N}\hat{a}_0^\dagger)\,|\text{vac}\rangle\,. \tag{4}$$

In QFT terms, this coherent state obeys

$$\hat{\psi}(\mathbf{x})\,|\text{coherent}\rangle \;=\; \phi(\mathbf{x})\,|\text{coherent}\rangle \quad \text{for } \phi(x) \;=\; \text{const}, \tag{5}$$

where the specific value of the $\phi(\mathbf{x}) = \left\langle \hat{\psi}(\mathbf{x}) \right\rangle$ obtains from the classical field theory of the Landau–Ginzburg theory:

$$H[\phi,\phi^*] \;=\; \int d^3\mathbf{x} \left(\frac{1}{2m}\,|\nabla\phi|^2 \;+\; \frac{\lambda}{2}\,|\phi|^4 \;-\; \mu|\phi|^2 \right). \tag{6}$$

For a negative chemical potential μ the lowest-energy state of this classical Hamiltonian is $\phi(\mathbf{x}) \equiv 0$ — which corresponds to the vacuum state of the quantum field theory, — while for positive chemical potential $\mu > 0$ the lowest-energy state has a non-zero value of the scalar field ϕ, namely

$$|\phi|^2 \;=\; \bar{n}_s \;=\; \frac{\mu}{\lambda}\,. \tag{7}$$

The phase of ϕ is arbitrary, as long as it is the same at all $\mathbf{x}$ (in the non-moving BEC), so without loss of generality we assume $\phi_{\text{ground}} = \sqrt{\bar{n}_s}$. In the quantum theory, this corresponds to a constant non-zero ground-state expectation value

$$\left\langle \hat{\psi}(\mathbf{x}) \right\rangle \;=\; \sqrt{\bar{n}_s} \;=\; \text{const}. \tag{8}$$

The simplest quantum state with this expectation value $\hat{\psi}$ is the coherent state (5). However, interactions of this expectation value with the fluctuations of the quantum fields around this coherent state changes the ground states of the $\mathbf{k} \neq 0$ modes and they no longer correspond to $n_{\mathbf{k}} = 0$. To see how this works, consider the *shifted quantum fields*

$$\begin{aligned} \delta\hat{\psi}(\mathbf{x}) \;&=\; \hat{\psi}(\mathbf{x}) \;-\; \left\langle \hat{\psi} \right\rangle \;=\; L^{-3/2} \sum_{\mathbf{k}\neq 0} e^{i\mathbf{k}\cdot\mathbf{x}} \hat{a}_{\mathbf{k}} \\ \text{and } \delta\hat{\psi}^\dagger(\mathbf{x}) \;&=\; \hat{\psi}^\dagger(\mathbf{x}) \;-\; \left\langle \hat{\psi} \right\rangle^* \;=\; L^{-3/2} \sum_{\mathbf{k}\neq 0} e^{i\mathbf{k}\cdot\mathbf{x}} \hat{a}_{\mathbf{k}}^\dagger\,, \end{aligned} \tag{9}$$

and let's rewrite the Hamiltonian (1) in terms of these shifted fields. Term-by-term in the

Hamiltonian density, we have

$$
\begin{aligned}
\nabla\hat{\psi}^\dagger\cdot\nabla\hat{\psi} &= \nabla\delta\hat{\psi}^\dagger\cdot\nabla\delta\hat{\psi}, & (10)\\
\hat{\psi}^\dagger\hat{\psi} &= (\sqrt{\bar{n}_s}+\delta\hat{\psi}^\dagger)(\sqrt{\bar{n}_s}+\delta\hat{\psi}) \;=\; \bar{n}_s \;+\; \sqrt{\bar{n}_s}\big(\delta\hat{\psi}^\dagger+\delta\hat{\psi}\big) \;+\; (\delta\hat{\psi}^\dagger)(\delta\hat{\psi}), & (11)\\
\hat{\psi}^\dagger\hat{\psi}^\dagger\hat{\psi}\hat{\psi} &= (\sqrt{\bar{n}_s}+\delta\hat{\psi}^\dagger)^2(\sqrt{\bar{n}_s}+\delta\hat{\psi})^2\\
&= \bar{n}_s^2 \;+\; 2\bar{n}_s^{3/2}\big(\delta\hat{\psi}^\dagger+\delta\hat{\psi}\big) \;+\; \bar{n}_s\big((\delta\hat{\psi}^\dagger)^2 \;+\; 4(\delta\hat{\psi}^\dagger)(\delta\hat{\psi}) \;+\; (\delta\hat{\psi})^2\big)\\
&\quad +\; 2\sqrt{\bar{n}_s}(\delta\hat{\psi}^\dagger)(\delta\hat{\psi}^\dagger+\delta\hat{\psi})(\delta\hat{\psi}) \;+\; (\delta\hat{\psi}^\dagger)^2(\delta\hat{\psi})^2, & (12)
\end{aligned}
$$

hence for $\lambda\bar{n}_s=\mu$,

$$
\begin{aligned}
\frac{\lambda}{2}(\hat{\psi}^\dagger)^2(\hat{\psi})^2 \;-\; \mu\hat{\psi}^\dagger\hat{\psi} \;=\; &-\frac{\lambda n_s^2}{2} \;+\; 0\times(\delta\hat{\psi}^\dagger+\delta\hat{\psi})\\
&+\; \lambda\bar{n}_s\big(\tfrac{1}{2}(\delta\hat{\psi}^\dagger)^2 \;+\; (\delta\hat{\psi}^\dagger)(\delta\hat{\psi}) \;+\; \tfrac{1}{2}(\delta\hat{\psi})^2\big)\\
&+\; \lambda\sqrt{\bar{n}_s}(\delta\hat{\psi}^\dagger)(\delta\hat{\psi}^\dagger+\delta\hat{\psi})(\delta\hat{\psi}) \;+\; \tfrac{1}{2}\lambda(\delta\hat{\psi}^\dagger)^2(\delta\hat{\psi})^2.
\end{aligned}
\qquad (13)
$$

Note the organization of the RHS here according to net powers of the *shifted* fields $\delta\hat{\psi}^\dagger$ and $\delta\hat{\psi}$. Reorganizing the whole Landau–Ginzburg Hamiltonian along the similar lines, we get

$$
\hat{H} \;-\; \mu\hat{N} \;=\; \text{const} \;+\; \hat{H}_{\text{free}} \;+\; \hat{H}_{\text{int}} \qquad (14)
$$

where

$$
\begin{aligned}
\hat{H}_{\text{free}} &= \int d^3\mathbf{x}\left(\frac{1}{2m}\nabla\delta\hat{\psi}^\dagger\cdot\nabla\delta\hat{\psi} \;+\; \lambda\bar{n}_s\big((\delta\hat{\psi}^\dagger)(\delta\hat{\psi}) \;+\; \tfrac{1}{2}(\delta\hat{\psi})^2 \;+\; \tfrac{1}{2}(\delta\hat{\psi}^\dagger)^2\big)\right)\\
&= \sum_{\mathbf{k}\neq0}\left(\left(\frac{\mathbf{k}^2}{2m} \;+\; \lambda\bar{n}_s\right)\hat{a}^\dagger_{\mathbf{k}}\hat{a}_{\mathbf{k}} \;+\; \tfrac{1}{2}\lambda\bar{n}_s\big(\hat{a}_{\mathbf{k}}\hat{a}_{-\mathbf{k}} \;+\; \hat{a}^\dagger_{\mathbf{k}}\hat{a}^\dagger_{-\mathbf{k}}\big)\right)
\end{aligned}
\qquad (15)
$$

comprises the quadratic (and bilinear) terms in the shifted fields, while

$$
\hat{H}_{\text{int}} \;=\; \int d^3\mathbf{x}\left(\lambda\sqrt{\bar{n}_s}(\delta\hat{\psi}^\dagger)(\delta\hat{\psi}^\dagger+\delta\hat{\psi})(\delta\hat{\psi}) \;+\; \tfrac{1}{2}\lambda(\delta\hat{\psi}^\dagger)^2(\delta\hat{\psi})^2\right) \qquad (16)
$$

comprises the cubic and the quartic terms. Physically, the $\hat{H}_{\text{free}}$ describes the free quanta of the shifted fields — *i.e.*, of the quantum fields' fluctuations around their ground-state expectation values, — while the $\hat{H}_{\text{int}}$ describes the interactions between such quanta.

Our next task is to diagonalize the $\hat{H}_{\text{free}}$; this should give us the leading approximation to the excitation spectrum as well as the next approximation to the ground state (next after the coherent state). The better approximations after that would obtain by perturbation theory in $\hat{H}_{\text{int}}$, but I won't do it in these notes. Instead, diagonalizing just the free Hamiltonian for the fluctuations would be interesting enough.

In terms of atomic creation and annihilation operators, the Hamiltonian (15) has form

$$\hat{H} \;=\; \sum_{\mathbf{k}} \Big(A_{\mathbf{k}} \hat{a}^\dagger_{\mathbf{k}} \hat{a}_{\mathbf{k}} \;+\; \tfrac{1}{2} B_{\mathbf{k}} \big(\hat{a}_{\mathbf{k}} \hat{a}_{-\mathbf{k}} \,+\, \hat{a}^\dagger_{\mathbf{k}} \hat{a}^\dagger_{-\mathbf{k}} \big) \Big), \tag{17}$$

for some real $A_{\mathbf{k}} = A_{-\mathbf{k}}$ and $B_{\mathbf{k}} = B_{-\mathbf{k}}$, so it may be diagonalized via a **Bogolyubov transform** *of the creation and annihilation operators:*

$$\begin{aligned} \hat{b}_{\mathbf{k}} \;&=\; \cosh(t_{\mathbf{k}}) \times \hat{a}_{\mathbf{k}} \;+\; \sinh(t_{\mathbf{k}}) \times \hat{a}^\dagger_{-\mathbf{k}}\,, \\ \hat{b}^\dagger_{\mathbf{k}} \;&=\; \cosh(t_{\mathbf{k}}) \times \hat{a}^\dagger_{\mathbf{k}} \;+\; \sinh(t_{\mathbf{k}}) \times \hat{a}_{-\mathbf{k}}\,. \end{aligned} \tag{18}$$

for appropriate real parameters $t_{\mathbf{k}} = t_{-\mathbf{k}}$.

Lemma 1: For any real $t_{\mathbf{k}} = t_{-\mathbf{k}}$, the $\hat{b}_{\mathbf{k}}$ and $\hat{b}^\dagger_{\mathbf{k}}$ operators obey the same bosonic commutation relations as the $\hat{a}_{\mathbf{k}}$ and $\hat{a}^\dagger_{\mathbf{k}}$ operators,

$$\big[\hat{b}_{\mathbf{k}}, \hat{b}_{\mathbf{k}'}\big] \;=\; 0, \quad \big[\hat{b}^\dagger_{\mathbf{k}}, \hat{b}^\dagger_{\mathbf{k}'}\big] \;=\; 0, \quad \big[\hat{b}_{\mathbf{k}}, \hat{b}^\dagger_{\mathbf{k}'}\big] \;=\; \delta_{\mathbf{k},\mathbf{k}'}\,. \tag{19}$$

Lemma 2: For any Hamiltonian of the form (17) with real $A_{\mathbf{k}} = A_{-\mathbf{k}}$, real $B_{\mathbf{k}} = B_{-\mathbf{k}}$ and $|B_{\mathbf{k}}| < A_{\mathbf{k}}$, there is a Bogolyubov transform (18) with

$$t_{\mathbf{k}} \;=\; \frac{1}{2}\,\mathrm{artanh}\,\frac{B_{\mathbf{k}}}{A_{\mathbf{k}}}\,, \tag{20}$$

which leads to

$$\hat{H} \;=\; \sum_{\mathbf{k}} \omega(\mathbf{k})\, \hat{b}^\dagger_{\mathbf{k}} \hat{b}_{\mathbf{k}} \;+\; \text{constant} \tag{21}$$

$$\text{for } \omega(\mathbf{k}) \;=\; \sqrt{A^2_{\mathbf{k}} - B^2_{\mathbf{k}}}\,. \tag{22}$$

Clearly, the ground state of the Hamiltonian (21) is the state annihilated by all the $\hat{b}_{\mathbf{k}}$ operators,

$$\forall\mathbf{k}, \quad \hat{b}_{\mathbf{k}}\,|\text{ground}\rangle \;=\; 0, \tag{23}$$

while the excited states obtain by acting with the $\hat{b}^\dagger_{\mathbf{k}}$ operators on the ground state,

$$|\text{excited}\rangle \;=\; \hat{b}^\dagger_{\mathbf{k}_1}\cdots\hat{b}^\dagger_{\mathbf{k}_n}\,|\text{ground}\rangle\,, \quad E_{\text{excited}} \;-\; E_{\text{ground}} \;=\; \omega(\mathbf{k}_1) \;+\; \cdots \;+\; \omega(\mathbf{k}_n). \tag{24}$$

Physically, we may interpret such excitations as containing n *quasiparticles* of respective energies $\omega(\mathbf{k}_1),\ldots\omega(\mathbf{k}_n)$. Thus, the operators $\hat{b}^\dagger_{\mathbf{k}}$ create quasiparticles, the operators $\hat{b}_{\mathbf{k}}$ annihilate those quasiparticles, and the ground state defined by eq. (23) is the *quasiparticle vacuum.*

Lemma 3: The quasiparticle creation and annihilation operators $\hat{b}^\dagger_{\mathbf{k}}$ and $\hat{b}_{\mathbf{k}}$ are related to the atomic creation and annihilation operators $\hat{a}^\dagger_{\mathbf{k}}$ and $\hat{a}_{\mathbf{k}}$ by a unitary operator transform,

$$\hat{b}^\dagger_{\mathbf{k}} \;=\; e^{+\hat{F}}\times\hat{a}^\dagger_{\mathbf{k}}\times e^{-\hat{F}}, \qquad \hat{b}_{\mathbf{k}} \;=\; e^{+\hat{F}}\times\hat{a}_{\mathbf{k}}\times e^{-\hat{F}}, \tag{25}$$

for the antihermitian operator

$$\hat{F} \;=\; \frac{1}{2}\sum_{\mathbf{k}} t_{\mathbf{k}}\big(\hat{a}_{\mathbf{k}}\hat{a}_{-\mathbf{k}} \;-\; \hat{a}^\dagger_{\mathbf{k}}\hat{a}^\dagger_{-\mathbf{k}}\big). \tag{26}$$

Consequently, the state

$$|\text{ground}\rangle \;=\; e^{+\hat{F}}\,|\text{coherent}\rangle \tag{27}$$

is annihilated by all the quasiparticle annihilation operators $\hat{b}_{\mathbf{k}}$, so it's the ground state of the Hamiltonian (21).

Unlike the $|\text{coherent}\rangle$ state of the BEC which has all the atoms in the $\mathbf{k}=0$ mode, the ground state (27) also has a lot of atoms paired in $(+\mathbf{k},-\mathbf{k})$ modes. In fact, experiments with the Bose–Einstein condensates of ultra-cold atoms show more atoms in such $\pm\mathbf{k}$ pairs than the atoms in the $\mathbf{k}=0$ mode itself.

Lemma 4: For the state (27), the net number of atoms in $\mathbf{k}\neq 0$ modes is

$$N_{\mathbf{k}\neq 0} \;=\; \langle\text{ground}|\,\hat{N}_{\mathbf{k}\neq 0}\,|\text{ground}\rangle \;=\; \sum_{\mathbf{k}\neq 0}\sinh^2(t_{\mathbf{k}}). \tag{28}$$

Lemma 5: The quasiparticle vacuum state (23) has zero net mechanical momentum, while the quasiparticles have definite momenta $\mathbf{k}$, thus

$$\hat{\mathbf{P}}_{\text{net}} \;=\; \sum_{\mathbf{k}}\mathbf{k}\hat{a}^\dagger_{\mathbf{k}}\hat{a}_{\mathbf{k}} \;=\; \sum_{\mathbf{k}}\mathbf{k}\hat{b}^\dagger_{\mathbf{k}}\hat{b}_{\mathbf{k}}\,. \tag{29}$$

I shall prove the Lemmas 1–5 later in these notes, but first let me put them in the specific context of the Bose–Einstein condensate. Let's go back to the Landau–Ginzburg theory and the Hamiltonian (14) for the fluctuation fields. — or rather the free part of that Hamiltonian. In terms of the atomic creation and annihilation operators, the free part (15) of that Hamiltonian is

$$\hat{H}_{\text{free}} \;=\; \sum_{\mathbf{k}\neq 0}\left(\left(\frac{\mathbf{k}^2}{2m}\,+\,\lambda\bar{n}_s\right)\hat{a}^\dagger_{\mathbf{k}}\hat{a}_{\mathbf{k}} \;+\; \tfrac{1}{2}\lambda\bar{n}_s\big(\hat{a}_{\mathbf{k}}\hat{a}_{-\mathbf{k}}\,+\,\hat{a}^\dagger_{\mathbf{k}}\hat{a}^\dagger_{-\mathbf{k}}\big)\right), \tag{30}$$

which is a special case of (17) with

$$A_{\mathbf{k}} \;=\; \frac{\mathbf{k}^2}{2m}\,+\,\lambda\bar{n}_s\,, \qquad B_{\mathbf{k}} \;=\; \lambda\bar{n}_s, \tag{31}$$

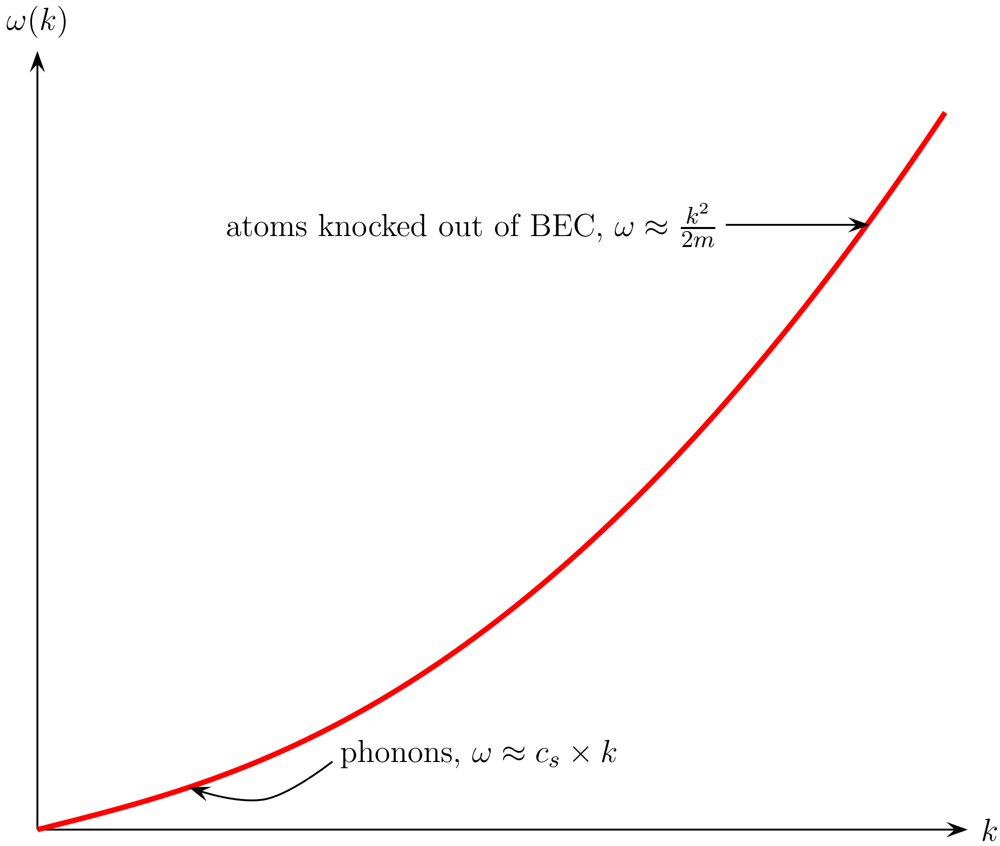
hence

$$t_{\mathbf{k}} \;=\; \frac{1}{2}\,\text{artanh}\,\frac{2\lambda\bar{n}_s m}{2\lambda\bar{n}_2 m\,+\,k^2} \;\longrightarrow\; \begin{cases}\infty & \text{for small } k,\\ 0 & \text{for large } k.\end{cases} \tag{32}$$

while

$$\omega(\mathbf{k}) \;=\; \sqrt{\left(\frac{\mathbf{k}^2}{2m}+\lambda\bar{n}_s\right)^2-(\lambda\bar{n}_s)^2} \;=\; k\times\sqrt{\frac{k^2}{4m^2}\,+\,\frac{\lambda\bar{n}_s}{m}}\,. \tag{33}$$

Graphically,



(34)

- Note that at high quasiparticle momenta k we have $\omega(k) \approx k^2/2m$ while $t(k) \ll 1$ and hence $\hat{b}_{\mathbf{k}} \approx \hat{a}_{\mathbf{k}}$ and $\hat{b}^\dagger_{\mathbf{k}} \approx \hat{a}^\dagger_{\mathbf{k}}$. In other words, the quasiparticle created by the $\hat{b}^\dagger_{\mathbf{k}}$ and annihilated by the $\hat{b}_{\mathbf{k}}$ is approximately a free atom, or rather *an atom kicked out of the BEC condensate* and given a high momentum $\mathbf{k}$.

- On the other hand, for low (but non-zero) quasiparticles momenta

$$\omega(k) \;\approx\; k \times \sqrt{\lambda \bar{n}_s/m} \;\equiv\; k \times c_s \tag{35}$$

  while $t(k)$ is large, hence $\hat{b}^\dagger_{\mathbf{k}} \propto (\hat{a}^\dagger_{\mathbf{k}} + \hat{a}_{-\mathbf{k}})$ which means that it creates a quantum of $\delta\psi^*(\mathbf{x}) + \delta\phi(\mathbf{x}) \propto \delta n_s(\mathbf{x})$, a quantum of the condensate density wave. In other words, the quasiparticle created by the $\hat{b}^\dagger_{\mathbf{k}}$ (and annihilated by the $\hat{b}_{\mathbf{k}}$) is *a phonon*; indeed, the quasiparticle's velocity $c_s = \sqrt{\lambda \bar{n}_s/m}$ is the speed of sound in the BEC.

- Finally, at the intermediate momenta k, the quasiparticles *interpolate between the phonons and the atoms kicked out of the BEC.*

Thus far, we have ignored the interactions between the fluctuation fields $\delta\hat{\psi}(\mathbf{x})$ and $\delta\hat{\psi}^\dagger(\mathbf{x})$ and hence between the quasiparticles. In quasiparticle terms, the interaction Hamiltonian

$$\hat{H}_{\text{int}} \;=\; \int d^3\mathbf{x} \left(\lambda\sqrt{\bar{n}_s}(\delta\hat{\psi}^\dagger)(\delta\hat{\psi}^\dagger + \delta\hat{\psi})(\delta\hat{\psi}) \;+\; \tfrac{1}{2}\lambda(\delta\hat{\psi}^\dagger)^2(\delta\hat{\psi})^2\right) \tag{16}$$

comprises terms of the form

$$\hat{b}\hat{b}\hat{b}, \quad \hat{b}^\dagger\hat{b}\hat{b}, \quad \hat{b}^\dagger\hat{b}^\dagger\hat{b}, \quad \hat{b}^\dagger\hat{b}^\dagger\hat{b}^\dagger, \quad \text{and} \quad \hat{b}\hat{b}\hat{b}\hat{b}, \quad \hat{b}^\dagger\hat{b}\hat{b}\hat{b}, \quad \hat{b}^\dagger\hat{b}^\dagger\hat{b}\hat{b}, \quad \hat{b}^\dagger\hat{b}^\dagger\hat{b}^\dagger\hat{b}, \quad \hat{b}^\dagger\hat{b}^\dagger\hat{b}^\dagger\hat{b}^\dagger. \tag{36}$$

In particular, the $\hat{b}^\dagger\hat{b}^\dagger\hat{b}^\dagger$ and the $\hat{b}^\dagger\hat{b}^\dagger\hat{b}^\dagger\hat{b}^\dagger$ terms do not annihilate the $|\text{ground}\rangle$ state of the free Hamiltonian, so it suffers perturbative corrections. Fortunately, there is a way to recast all such corrections in terms of the unitary operator transform,

$$|\text{true ground state}\rangle \;=\; e^{\hat{F}}\,|\text{coherent}\rangle \tag{37}$$

for

$$\begin{aligned}\hat{F} \;=\; &\tfrac{1}{2}\sum_{\mathbf{k}\neq 0} t_{\mathbf{k}}(\hat{a}_{\mathbf{k}}\hat{a}_{-\mathbf{k}} - \hat{a}^\dagger_{\mathbf{k}}\hat{a}^\dagger_{-\mathbf{k}})\\ &+\;\text{perturbative corrections involving terms of the form}\\ &\qquad(\hat{a}\hat{a}\hat{a} - \hat{a}^\dagger\hat{a}^\dagger\hat{a}^\dagger), \quad (\hat{a}\hat{a}\hat{a}\hat{a} - \hat{a}^\dagger\hat{a}^\dagger\hat{a}^\dagger\hat{a}^\dagger), \quad \ldots\end{aligned} \tag{38}$$

Consequently, we may redefine the quasiparticle creation and annihilation operators as

$$\hat{b}^\dagger_{\mathbf{k}} \;=\; e^{+\hat{F}} \times \hat{a}^\dagger_{\mathbf{k}} \times e^{-\hat{F}}, \qquad \hat{b}_{\mathbf{k}} \;=\; e^{+\hat{F}} \times \hat{a}_{\mathbf{k}} \times e^{-\hat{F}}, \tag{25}$$

in terms of the perturbatively corrected unitary transform $e^{\hat{F}}$, so that the $\hat{b}_{\mathbf{k}}$ and the $\hat{b}^\dagger_{\mathbf{k}}$ obey the bosonic commutation relations and all the $\hat{b}_{\mathbf{k}}$ annihilate the true ground state. Thus, we still have the quasiparticle picture of the excited states of the complete Hamiltonian, although the relation $\omega(k)$ between quasiparticles energy and momentum suffers perturbative corrections. Nevertheless, *qualitatively* the $\omega(k)$ function remains as on the diagram (34), so the low-momentum quasiparticles are phonons, the high-momentum quasiparticles are atoms kicked out of the BEC, and the intermediate-momentum quasiparticles interpolate between the two.

Superfluid Helium

Thus far we have used the Landau–Ginzburg theory which is good for the BEC of ultra-cold atoms but becomes inaccurate for the superfluid liquid helium. The problem stems from higher average momenta of the helium atoms and hence shorter De Broglie wavelength which becomes comparable to the range of the interatomic forces. Consequently, we may longer approximate the two-body potential as $V_2(\mathbf{x}-\mathbf{y}) = \lambda\delta^{(3)}(\mathbf{x}-\mathbf{y})$, so instead of the Landau–Ginzburg Hamiltonian we should use

$$\hat{H} - \mu\hat{N} \;=\; \int\! d^3\mathbf{x} \left(\frac{1}{2m}\,\hat{\nabla}\hat{\psi}^\dagger\cdot\nabla\hat{\psi} \;-\; \mu\hat{\psi}^\dagger\hat{\psi}\right) + \frac{1}{2}\int\! d^3\mathbf{x}\int\! d^4\mathbf{y}\, V_2(\mathbf{x}-\mathbf{y})\times\hat{\psi}^\dagger(\mathbf{x})\hat{\psi}^\dagger(\mathbf{y})\hat{\psi}(\mathbf{y})\hat{\psi}(\mathbf{x}). \tag{39}$$

To find the ground state of this Hamiltonian, we start with the classical field limit and minimize the classical Hamiltonian

$$H[\phi,\phi^*] \;=\; \int\! d^3\mathbf{x}\left(\frac{1}{2m}\,|\nabla\phi|^2 \;-\; \mu|\phi|^2\right) + \frac{1}{2}\int\! d^3\mathbf{x}\int\! d^4\mathbf{y}\, V_2(\mathbf{x}-\mathbf{y})\times|\phi(\mathbf{x})|^2\times|\phi(\mathbf{y})|^2. \tag{40}$$

Again, the minimum obtains for $\phi(\mathbf{x}) = \text{const}$, specifically

$$|\phi|^2 \;=\; \bar{n}_s \;=\; \frac{\mu}{\lambda}, \quad \text{any constant phase of } \phi, \tag{41}$$

where

$$\lambda \;\stackrel{\text{def}}{=}\; \int\! d^3\mathbf{x}\, V_2(\mathbf{x}) \;>\; 0, \tag{42}$$

or in terms of the Fourier transform of the two-atom potential

$$W(\mathbf{k}) \;=\; \int\! d^3\mathbf{x}\, V_2(\mathbf{x})e^{-i\mathbf{k}\cdot\mathbf{x}}, \tag{43}$$

$$\lambda \;=\; W(0). \tag{44}$$

Given the classical ground-state expectation value of the condensate field ϕ, we go back to the quantum field theory and shift the quantum field just as we did before,

$$\begin{aligned} \delta\hat{\psi}(\mathbf{x}) \;&=\; \hat{\psi}(\mathbf{x}) \;-\; \left\langle\hat{\psi}\right\rangle \;=\; L^{-3/2}\sum_{\mathbf{k}\neq 0} e^{i\mathbf{k}\cdot\mathbf{x}}\hat{a}_{\mathbf{k}} \\ \text{and } \delta\hat{\psi}^\dagger(\mathbf{x}) \;&=\; \hat{\psi}^\dagger(\mathbf{x}) \;-\; \left\langle\hat{\psi}\right\rangle^* \;=\; L^{-3/2}\sum_{\mathbf{k}\neq 0} e^{i\mathbf{k}\cdot\mathbf{x}}\hat{a}^\dagger_{\mathbf{k}}, \end{aligned} \tag{9}$$

Next, we re-express the Hamiltonian (39) in terms of the shifted quantum fields and rearrange the terms according to the net power of the shifted fields. Just as we had earlier for the Landau–Ginzburg theory, we end up with

$$\hat{H} - \mu\hat{N} = \text{const} + \hat{H}_{\text{free}} + \hat{H}_{\text{int}} \tag{14}$$

where $\hat{H}_{\text{free}}$ comprises the quadratic (and bilinear) terms while $\hat{H}_{\text{int}}$ comprises the cubic and the quartic terms which we treat as perturbations.

Lemma 6:

$$\begin{aligned}
\hat{H}_{\text{free}} &= \frac{1}{2m}\int d^3\mathbf{x}\,\nabla\delta\hat{\phi}^\dagger\cdot\nabla\delta\hat{\psi} \\
&\quad + \int d^3\mathbf{x}\int d^3\mathbf{y}\,V_2(\mathbf{x}-\mathbf{y})\times\Big(\hat{\psi}^\dagger(\mathbf{x})\hat{\psi}(\mathbf{y}) + \tfrac{1}{2}\hat{\psi}(\mathbf{x})\hat{\psi}(\mathbf{y}) + \tfrac{1}{2}\hat{\psi}^\dagger(\mathbf{x})\hat{\psi}^\dagger(\mathbf{y})\Big) \\
&= \sum_{\mathbf{k}\neq 0}\left(\left(\frac{\mathbf{k}^2}{2m} + \bar{n}_s W(\mathbf{k})\right)\hat{a}^\dagger_{\mathbf{k}}\hat{a}_{\mathbf{k}} + \tfrac{1}{2}\bar{n}_s W(\mathbf{k})\big(\hat{a}_{\mathbf{k}}\hat{a}_{-\mathbf{k}} + \hat{a}^\dagger_{\mathbf{k}}\hat{a}^\dagger_{-\mathbf{k}}\big)\right).
\end{aligned} \tag{45}$$

Again, this Hamiltonian can be diagonalized by a Bogolyubov transform, exactly as we did it for the LG theory in lemmas 1–5, except for the new values of

$$A_{\mathbf{k}} = \frac{k^2}{2m} + \bar{n}_s W(\mathbf{k}) \quad \text{and} \quad B_k = \bar{n}_s W(\mathbf{k}). \tag{46}$$

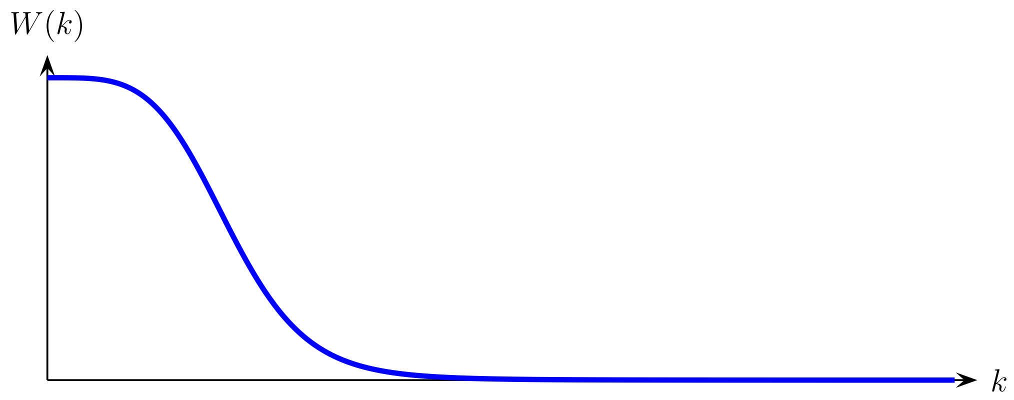
Consequently, we end up with

$$\hat{H}_{\text{free}} = \sum_{\mathbf{k}\neq 0}\omega(\mathbf{k})\hat{b}^\dagger_{\mathbf{k}}\hat{b}_{\mathbf{k}} + \text{const} \tag{47}$$

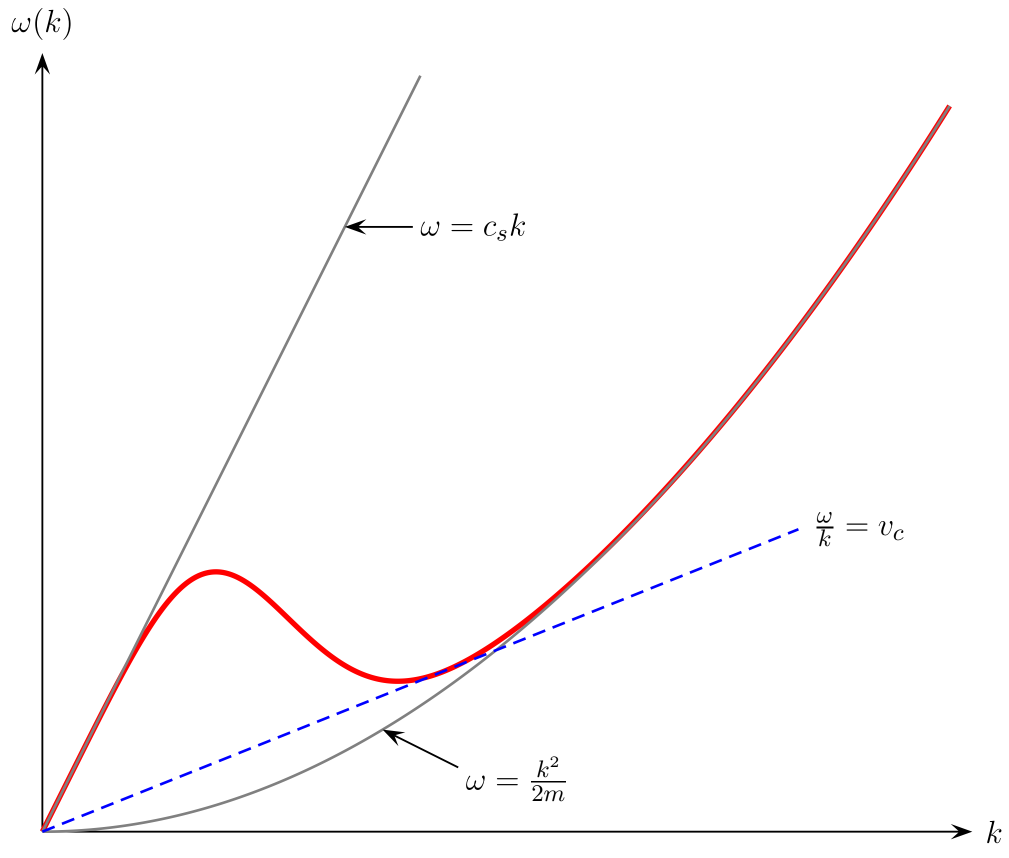
so the ground state is the quasiparticle vacuum, while the quasiparticles have definite momenta $\mathbf{k}$ and energies

$$\omega(\mathbf{k}) = \sqrt{\left(\frac{\mathbf{k}^2}{2m} + \bar{n}_s W(\mathbf{k})\right)^2 - (\lambda\bar{n}_s)^2} = k\times\sqrt{\frac{k^2}{4m^2} + \frac{\bar{n}_s}{m}W(k)}\,. \tag{48}$$

For the helium atoms, the $W(\mathbf{k})$ drops off at large momenta,



hence the energy-momentum relation $\omega(k)$ for the quasiparticles — or equivalently, the wavenumber-frequency *dispersion relation* for the waves of small fluctuations — has a dip:



(49)

Again, this curve shows that the low-momenta quasiparticles are phonons while the high momenta quasiparticles are helium atoms knocked out from the BEC. But now we also have unexpectedly-low energy quasiparticles at intermediate momenta; they are called the *rotons* for historical reason. The rotons have much larger phase space than the phonons, so at temperatures $T \sim 1$ K the rotons dominate the quasiparticle gas, which is the normal-fluid component of the finite-temperature liquid Helium II.

The most important feature of the $\omega(k)$ curve (49) is the positive lower bound on the energy-to-momentum ratio,

$$\forall \mathbf{k}: \quad \omega(\mathbf{k}) > v_c \times |\mathbf{k}| \quad \text{for a positive } v_c. \tag{50}$$

We shall see momentarily that it is this lower bound which gives rise to the superfluidity.

Superfluidity

Consider a flowing superfluid; for simplicity, let it flow with a uniform velocity $\mathbf{v}$. Classically, this flow is described by

$$\phi(\mathbf{x}) = \sqrt{\bar{n}_s} \times \exp(im\mathbf{v}\cdot\mathbf{x}) \tag{51}$$

while quantum mechanically, we have a coherent pile up of atoms into the $\mathbf{k} = m\mathbf{v}$ mode, while other atoms form pairs with momenta $\mathbf{k}_{1,2} = m\mathbf{v} \pm \mathbf{k}_{\text{red}}$. Altogether, we have a quantum state very much like the state of the superfluid at rest, except all atom's momenta are shifted by $m\mathbf{v}$. In other words, the state of the flowing superfluid obtains from the ground state of the superfluid at rest via *Galilean boost* of velocity $\mathbf{v}$.

The excitation spectrum of the moving superfluid also obtains via Galilean boost of the Hamiltonian,

$$\hat{H}' = \hat{H} + \mathbf{v}\cdot\hat{P}_{\text{net}} + \tfrac{1}{2}v^2 M_{\text{net}}\,. \tag{52}$$

For simplicity, let's ignore the interactions between the quasiparticles and focus on their free Hamiltonian. In quasiparticle terms,

$$\hat{H}_{\text{free}} = \text{const} + \sum_{\mathbf{k}} \omega(\mathbf{k})\,\hat{b}^\dagger_{\mathbf{k}}\hat{b}_{\mathbf{k}}\,, \tag{53}$$

$$\hat{P}_{\text{net}} = \sum_{\mathbf{k}} \mathbf{k}\,\hat{b}^\dagger_{\mathbf{k}}\hat{b}_{\mathbf{k}}\,, \tag{54}$$

hence for a moving superfluid

$$\hat{H}'(\mathbf{v}) \;=\; \text{const}' \;+\; \sum_{\mathbf{k}} \big(\omega(\mathbf{k}) + \mathbf{v}\cdot\mathbf{k}\big) \hat{b}^\dagger_{\mathbf{k}} \hat{b}_{\mathbf{k}}\,. \tag{55}$$

(In this formula, the quasiparticle momenta $\mathbf{k}$ are in the rest frame of the superfluid rather than the lab frame. In the lab frame, they have $\mathbf{p} = \mathbf{k} + m\mathbf{v}$.)

Before we apply this formula to the liquid Helium II, consider the ideal gas. For the ideal gas at rest, the Hamiltonian has form (53) where the 'quasiparticles' are simple the atoms and $\omega(k) = k^2/2m$. Consequently, the uniformly flowing gas has

$$\hat{H}'(\mathbf{v}) \;=\; \text{const} \;+\; \sum_{\mathbf{k}} \left(\frac{\mathbf{k}^2}{2m} + \mathbf{v}\cdot\mathbf{k} \right) \hat{b}^\dagger_{\mathbf{k}} \hat{b}_{\mathbf{k}} \tag{56}$$

where the frequencies

$$\omega'(\mathbf{k}) \;=\; \frac{\mathbf{k}^2}{2m} \;+\; \mathbf{v}\cdot\mathbf{k} \tag{57}$$

are positive for some modes $\mathbf{k}$ and negative for other modes. In particular, for $\mathbf{k}$ in opposite direction from the gas flow $\mathbf{v}$ and of magnitude $k < 2mv$, the frequency $\omega'(\mathbf{k})$ is negative.

Now, while a harmonic oscillator with a positive frequency ω has a unique ground state, the oscillator with a negative frequency has energy spectrum unlimited from below. Which means that any perturbation — however small it might be — would cause transitions building up the number of quanta while lowering the energy. For the ideal gas, this means spontaneous build up of atoms with $\omega'(\mathbf{k}) < 0$ — *i.e.*, with lab-frame velocities

$$\left| \mathbf{v}_{\mathbf{k}} \;=\; \mathbf{v} \;+\; \frac{\mathbf{k}}{m} \right| \;<\; |\mathbf{v}|, \tag{58}$$

by taking them out of the coherent motion with the gas. In other words, any interactions with the outside world (for example, the walls of the pipe the gas flows through) would spontaneously knock the atoms out of the coherent flow of the gas and slow them down. Such slowed-down atoms would dissipate the net energy and the net momentum of the flowing gas; it is this dissipation that we experimentally observe as *resistance to the flow.*

Now consider the superfluid Helium with $\omega(\mathbf{k})$ as on the diagram (49). Unlike the ideal gas with $\omega \propto \mathbf{k}^2$, the superfluid has $\omega \propto |\mathbf{k}|$ at low momenta, and for any quasiparticle momenta $\omega(\mathbf{k}) > v_c \times |\mathbf{k}|$ for some positive v_c. Consequently, as long as the Helium flows at speed v less than the critical speed v_c, we have

$$\omega'(\mathbf{k}) \;=\; \omega(\mathbf{k}) \;+\; \mathbf{v}\cdot\mathbf{k} \;>\; 0 \quad \text{for all } \mathbf{k}. \tag{59}$$

Indeed,

$$\omega(\mathbf{k}) \;+\; \mathbf{v}\cdot\mathbf{k} \;>\; \omega(k) \;-\; vk \;>\; \omega(k) \;-\; v_c k \;\geq\; 0. \tag{60}$$

Consequently, there are no negative-energy quasiparticles — like the slowed-down atoms — so there are no microscopic transitions lowering the superfluid's net energy. This means no energy dissipation, and that's why the superfluid flows without resistance, hence the name *superfluidity*.

Proofs of the Lemmas

Lemma 1: the bosonic commutation relations (19) for the quasiparticle creation and annihilation operators. Starting from the bosonic commutation relations

$$[\hat{a}_{\mathbf{k}}, \hat{a}_{\mathbf{k}'}] \;=\; 0, \quad [\hat{a}^\dagger_{\mathbf{k}}, \hat{a}^\dagger_{\mathbf{k}'}] \;=\; 0, \quad [\hat{a}_{\mathbf{k}}, \hat{a}^\dagger_{\mathbf{k}'}] \;=\; \delta_{\mathbf{k},\mathbf{k}'} \tag{61}$$

for the operators creating and annihilating the atoms, and treating eqs. (18) as the definitions of the $\hat{b}_{\mathbf{k}}$ and $\hat{b}^\dagger_{\mathbf{k}}$ operators, we immediately calculate

$$\begin{aligned}
[\hat{b}_{\mathbf{k}}, \hat{b}_{\mathbf{k}'}] \;&=\; \cosh(t_{\mathbf{k}})\sinh(t_{\mathbf{k}'}) \times \big([\hat{a}_{\mathbf{k}}, \hat{a}^\dagger_{-\mathbf{k}'}] = \delta_{\mathbf{k},-\mathbf{k}'}\big)\\
&\quad + \sinh(t_{\mathbf{k}})\cosh(t_{\mathbf{k}'}) \times \big([\hat{a}^\dagger_{-\mathbf{k}}, \hat{a}_{\mathbf{k}'}] = -\delta_{-\mathbf{k},\mathbf{k}'}\big)\\
&=\; \delta_{\mathbf{k}',-\mathbf{k}} \times \Big(\cosh(t_{\mathbf{k}})\sinh(t_{\mathbf{k}'}) \;-\; \sinh(t_{\mathbf{k}})\cosh(t_{\mathbf{k}'}) \;=\; \sinh(t_{\mathbf{k}'} - t_{\mathbf{k}})\Big)\\
&=\; 0 \quad \text{because } t'_{\mathbf{k}} = t_{\mathbf{k}} \text{ when } \mathbf{k}' = -\mathbf{k}.
\end{aligned} \tag{62}$$

In the same way, $[\hat{b}^\dagger_{\mathbf{k}}, \hat{b}^\dagger_{\mathbf{k}'}] = 0$.

Finally,

$$
\begin{aligned}
[\hat{b}_{\mathbf{k}}, \hat{b}^\dagger_{\mathbf{k}'}] \;&=\; \cosh(t_{\mathbf{k}})\cosh(t_{\mathbf{k}'}) \times \big([\hat{a}_{\mathbf{k}}, \hat{a}^\dagger_{\mathbf{k}'}] = \delta_{\mathbf{k},\mathbf{k}'}\big) \\
&\quad+\; \sinh(t_{-\mathbf{k}})\sinh(t_{-\mathbf{k}'}) \times \big([\hat{a}^\dagger_{-\mathbf{k}}, \hat{a}_{-\mathbf{k}'}] = \delta_{-\mathbf{k},-\mathbf{k}'}\big) \\
&=\; \delta_{\mathbf{k},\mathbf{k}'} \times \Big(\cosh^2(t_{\mathbf{k}}) - \sinh^2(t_{-\mathbf{k}}) \;= \cosh^2(t_{\mathbf{k}}) - \sinh^2(t_{\mathbf{k}}) \;=\; 1\Big) \\
&=\; \delta_{\mathbf{k},\mathbf{k}'}\,.
\end{aligned}
\tag{63}
$$

Quod erat demonstrandum.

Lemma 2: bringing the Hamiltonian (17) to the form (21). Let's start by expressing the product $\hat{b}^\dagger_{\mathbf{k}}\hat{b}_{\mathbf{k}}$ in terms of the $\hat{a}^\dagger$ and $\hat{a}$ operators. Applying both definitions (18), we immediately obtain

$$
\begin{aligned}
\hat{b}^\dagger_{\mathbf{k}}\hat{b}_{\mathbf{k}} \;&=\; \cosh^2(t_{\mathbf{k}})\,\hat{a}^\dagger_{\mathbf{k}}\hat{a}_{\mathbf{k}} \;+\; \cosh(t_{\mathbf{k}})\sinh(t_{\mathbf{k}})\,(\hat{a}^\dagger_{\mathbf{k}}\hat{a}^\dagger_{-\mathbf{k}} \;+\; \hat{a}_{-\mathbf{k}}\hat{a}_{\mathbf{k}}) \\
&\quad+\; \sinh^2(t_{\mathbf{k}})\,(\hat{a}_{-\mathbf{k}}\hat{a}^\dagger_{-\mathbf{k}} \;=\; \hat{a}^\dagger_{-\mathbf{k}}\hat{a}_{-\mathbf{k}} + 1).
\end{aligned}
\tag{64}
$$

Likewise,

$$
\begin{aligned}
\hat{b}^\dagger_{-\mathbf{k}}\hat{b}_{-\mathbf{k}} \;&=\; \cosh^2(t_{-\mathbf{k}})\,\hat{a}^\dagger_{-\mathbf{k}}\hat{a}_{-\mathbf{k}} \;+\; \cosh(t_{-\mathbf{k}})\sinh(t_{\mathbf{k}})\,(\hat{a}^\dagger_{-\mathbf{k}}\hat{a}^\dagger_{\mathbf{k}} \;+\; \hat{a}_{\mathbf{k}}\hat{a}_{-\mathbf{k}}) \\
&\quad+\; \sinh^2(t_{-\mathbf{k}})\,(\hat{a}_{\mathbf{k}}\hat{a}^\dagger_{\mathbf{k}} \;=\; \hat{a}^\dagger_{\mathbf{k}}\hat{a}_{\mathbf{k}} + 1).
\end{aligned}
\tag{65}
$$

Assuming $t_{-\mathbf{k}} = t_{\mathbf{k}}$, we may combine

$$
\begin{aligned}
\hat{b}^\dagger_{\mathbf{k}}\hat{b}_{\mathbf{k}} \;+\; \hat{b}^\dagger_{-\mathbf{k}}\hat{b}_{-\mathbf{k}} \;&=\; \Big(\cosh^2(t_{\mathbf{k}}) + \sinh^2(t_{\mathbf{k}}) = \cosh(2t_{\mathbf{k}})\Big) \times (\hat{a}^\dagger_{\mathbf{k}}\hat{a}_{\mathbf{k}} + \hat{a}^\dagger_{-\mathbf{k}}\hat{a}_{-\mathbf{k}}) \\
&\quad+\; \Big(2\cosh(t_{\mathbf{k}})\sinh(t_{\mathbf{k}}) = \sinh(2t_{\mathbf{k}})\Big) \times (\hat{a}^\dagger_{\mathbf{k}}\hat{a}^\dagger_{-\mathbf{k}} \;+\; \hat{a}_{-\mathbf{k}}\hat{a}_{\mathbf{k}}) \;+\; \text{const.}
\end{aligned}
\tag{66}
$$

Now let's plug this formula into a Hamiltonian of the form (21) for some $\omega_{\mathbf{k}}$ and require that the result matches the original Hamiltonian (17). Assuming $\omega_{-\mathbf{k}} \equiv \omega_{\mathbf{k}}$, we obtain

$$
\begin{aligned}
\hat{H} \;&=\; \sum_{\mathbf{k}} \omega_{\mathbf{k}}\hat{b}^\dagger_{\mathbf{k}}\hat{b}_{\mathbf{k}} \;=\; \frac{1}{2}\sum_{\mathbf{k}} \omega_{\mathbf{k}}(\hat{b}^\dagger_{\mathbf{k}}\hat{b}_{\mathbf{k}} + \hat{b}^\dagger_{-\mathbf{k}}\hat{b}_{-\mathbf{k}}) \\
&= \sum_{\mathbf{k}} \omega_{\mathbf{k}}\cosh(2t_{\mathbf{k}})\hat{a}^\dagger_{\mathbf{k}}\hat{a}_{\mathbf{k}} \;+\; \tfrac{1}{2}\sum_{\mathbf{k}} \omega_{\mathbf{k}}\sinh(2t_{\mathbf{k}})\Big(\hat{a}^\dagger_{\mathbf{k}}\hat{a}^\dagger_{-\mathbf{k}} \;+\; \hat{a}_{-\mathbf{k}}\hat{a}_{\mathbf{k}}\Big) \;+\; \text{const.}
\end{aligned}
\tag{67}
$$

This formula must match (up to a constant) the original Hamiltonian (17), so we need to choose the parameters $\omega_{\mathbf{k}} = \omega_{-\mathbf{k}}$ and $t_{\mathbf{k}} = t_{-\mathbf{k}}$ such that

$$\omega_{\mathbf{k}} \cosh(2t_{\mathbf{k}}) \;=\; A_{\mathbf{k}} \quad \text{and} \quad \omega_{\mathbf{k}} \sinh(2t_{\mathbf{k}}) \;=\; B_{\mathbf{k}}\,. \tag{68}$$

These equations are easy to solve, and the solution exists as long as $A_{\mathbf{k}} = A_{-\mathbf{k}}$, $B_{\mathbf{k}} = B_{-\mathbf{k}}$, and $A_{\mathbf{k}} > |B_{\mathbf{k}}|$, namely

$$t_{\mathbf{k}} \;=\; \frac{1}{2}\,\text{artanh}\,\frac{B_{\mathbf{k}}}{A_{\mathbf{k}}} \quad \text{and} \quad \omega_{\mathbf{k}} \;=\; \sqrt{A_{\mathbf{k}}^2 - B_{\mathbf{k}}^2}\,. \tag{69}$$

Quod erat demonstrandum.

Lemma 3: By the Hadamard Lemma,

$$e^{+\hat{F}} \times \hat{a}_{\mathbf{k}} \times e^{-\hat{F}} \;=\; \hat{a}_{\mathbf{k}} \;+\; [\hat{F}, \hat{a}_{\mathbf{k}}] \;+\; \tfrac{1}{2}[\hat{F}, [\hat{F}, \hat{a}_{\mathbf{k}}]] \;+\; \tfrac{1}{6}[\hat{F}, [\hat{F}, [\hat{F}, \hat{a}_{\mathbf{k}}]]] \;+\; \cdots \tag{70}$$

and likewise for the $\hat{a}^\dagger_{\mathbf{k}}$. Specifically, for

$$\hat{F} \;=\; \frac{1}{2}\sum_{\mathbf{p}} t_{\mathbf{p}} \big(\hat{a}_{\mathbf{p}}\hat{a}_{-\mathbf{p}} \;-\; \hat{a}^\dagger_{\mathbf{p}}\hat{a}^\dagger_{-\mathbf{p}}\big), \tag{26}$$

we have

$$[\hat{F}, \hat{a}_{\mathbf{k}}] \;=\; -t_{\mathbf{k}}[\hat{a}^\dagger_{\mathbf{k}}\hat{a}^\dagger_{-\mathbf{k}}, \hat{a}_{\mathbf{k}}] \;=\; +t_{\mathbf{k}}\hat{a}^\dagger_{-\mathbf{k}} \tag{71}$$

and

$$[\hat{F}, \hat{a}^\dagger_{-\mathbf{k}}] \;=\; +t_{\mathbf{k}}[\hat{a}_{\mathbf{k}}\hat{a}_{-\mathbf{k}}, \hat{a}^\dagger_{-\mathbf{k}}] \;=\; +t_{\mathbf{k}}\hat{a}_{\mathbf{k}}\,. \tag{72}$$

Consequently, multiple commutators yield

$$[\hat{F}, [\hat{F}, \ldots [\hat{F}, \hat{a}_{\mathbf{k}}] \cdots]]_{n\,\text{times}} \;=\; (t_{\mathbf{k}})^n \times \begin{cases} \hat{a}_{\mathbf{k}} & \text{for even } n, \\ \hat{a}^\dagger_{-\mathbf{k}} & \text{for odd } n. \end{cases}, \tag{73}$$

and therefore

$$\begin{aligned} e^{+\hat{F}} \times \hat{a}_{\mathbf{k}} \times e^{-\hat{F}} &= \sum_{\text{even}\,n} \frac{(t_{\mathbf{k}})^n}{n!} \times \hat{a}_{\mathbf{k}} \;+\; \sum_{\text{odd}\,n} \frac{(t_{\mathbf{k}})^n}{n!} \times \hat{a}^\dagger_{-\mathbf{k}} \\ &= \cosh(t_{\mathbf{k}}) \times \hat{a}_{\mathbf{k}} \;+\; \sinh(t_{\mathbf{k}}) \times \hat{a}^\dagger_{-\mathbf{k}} \\ &= \hat{b}_{\mathbf{k}}\,. \end{aligned} \tag{74}$$

In exactly the same way

$$[\hat{F},[\hat{F},\ldots[\hat{F},\hat{a}^\dagger_{\mathbf{k}}]\cdots]]_{n\,\text{times}} \;=\; (t_{\mathbf{k}})^n\times\begin{cases}\hat{a}^\dagger_{\mathbf{k}} & \text{for even } n,\\ \hat{a}_{-\mathbf{k}} & \text{for odd } n.\end{cases}, \tag{75}$$

and therefore

$$\begin{aligned}e^{+\hat{F}}\times\hat{a}^\dagger_{\mathbf{k}}\times e^{-\hat{F}} \;&=\; \sum_{\text{even}\,n}\frac{(t_{\mathbf{k}})^n}{n!}\times\hat{a}^\dagger_{\mathbf{k}} \;+\; \sum_{\text{odd}\,n}\frac{(t_{\mathbf{k}})^n}{n!}\times\hat{a}_{-\mathbf{k}}\\ &=\; \cosh(t_{\mathbf{k}})\times\hat{a}^\dagger_{\mathbf{k}} \;+\; \sinh(t_{\mathbf{k}})\times\hat{a}_{-\mathbf{k}}\\ &=\; \hat{b}^\dagger_{\mathbf{k}}\,.\end{aligned} \tag{76}$$

This establish the unitary transform (25) of the creation and annihilation operators.

The unitary transforms (25) automatically preserve the commutation relations between the operators, so instead of going through the algebra of proving the Lemma 1, we may simply use

$$[\hat{b}_{\mathbf{k}},\hat{b}^\dagger_{\mathbf{k}'}] \;=\; e^{\hat{F}}[\hat{a}_{\mathbf{k}},\hat{a}^\dagger_{\mathbf{k}'}]e^{-\hat{F}} \;=\; e^{\hat{F}}\delta_{\mathbf{k},\mathbf{k}'}e^{-\hat{F}} \;=\; \delta_{\mathbf{k},\mathbf{k}'} \tag{77}$$

and likewise for the $[\hat{b}_{\mathbf{k}},\hat{b}_{\mathbf{k}'}]$ and $[\hat{b}^\dagger_{\mathbf{k}},\hat{b}^\dagger_{\mathbf{k}'}]$.

In the same matter, $|\text{ground}\rangle = e^{\hat{F}}\,|\text{coherent}\rangle$ being the quasiparticle vacuum automatically follows from the unitary operator transform (25) and the fact that the $|\text{coherent}\rangle$ state is annihilated by all the $\hat{a}_{\mathbf{k}}$ operators with $\mathbf{k}\neq 0$. Indeed,

$$\hat{b}_{\mathbf{k}}\,|\text{ground}\rangle \;=\; e^{\hat{F}}\hat{a}_{\mathbf{k}}e^{-\hat{F}}\times e^{\hat{F}}\,|\text{coherent}\rangle \;=\; e^{\hat{F}}\hat{a}_{\mathbf{k}}\,|\text{coherent}\rangle \;=\; e^{\hat{F}}\times 0 \;=\; 0. \tag{78}$$

$\mathcal{Q}.\mathcal{E}.\mathcal{D}.$

Lemma 4: The operator $\hat{F}$ — and hence its exponential $e^{\hat{F}}$ — creates and annihilates the atoms in $\pm\mathbf{k}$ pairs independently from all the other pairs. Consequently, the BEC ground state $|\text{ground}\rangle = e^{\hat{F}}\,|\text{coherent}\rangle$ can be written as a direct product of independent states of

$\pm\mathbf{k}$ modes (and the coherent state for the $\mathbf{k}=0$ mode),

$$|\text{ground}\rangle \;=\; |\Psi_{\text{coherent}}(\mathbf{k}=0)\rangle \otimes \bigoplus_{\pm\mathbf{k}\,\text{pairs}} |\Psi(\pm\mathbf{k})\rangle\,, \tag{79}$$

where each

$$|\Psi\rangle\,(\pm\mathbf{k}) \;=\; \exp\bigl(t_{\mathbf{k}}(\hat{a}_{\mathbf{k}}\hat{a}_{-\mathbf{k}} - \hat{a}^\dagger_{\mathbf{k}}\hat{a}^\dagger_{-\mathbf{k}})\bigr)\,|0_{\mathbf{k}},0_{-\mathbf{k}}\rangle \;=\; \sum_{n=1}^{\infty} C_n(t_{\mathbf{k}})\,|n_{\mathbf{k}},n_{-\mathbf{k}}\rangle \tag{80}$$

for some coefficients $C_n(t_{\mathbf{k}})$. To calculate these coefficients, we use

$$\frac{d}{dt}e^{t\hat{F}}\,|0,0\rangle \;=\; \hat{F}e^{t\hat{F}}\,|0,0\rangle \tag{81}$$

hence

$$\begin{aligned}\frac{d}{dt}\sum_n C_n(t)\,|n,n\rangle \;&=\; (\hat{a}_{\mathbf{k}}\hat{a}_{-\mathbf{k}} - \hat{a}^\dagger_{\mathbf{k}}\hat{a}^\dagger_{-\mathbf{k}})\sum_n C_n(t)\,|n,n\rangle\\ &=\; \sum_n C_n(t)\Bigl(n\,|n-1,n-1\rangle \;-\; (n+1)\,|n+1,n+1\rangle\Bigr)\\ &=\; \sum_n |n,n\rangle \times \Bigl((n+1)C_{n+1}(t) \;-\; nC_{n-1}(t)\Bigr),\end{aligned} \tag{82}$$

which leads us to differential equations

$$\frac{d}{dt}C_n(t) \;=\; (n+1)C_{n+1}(t) \;-\; nC_{n-1}(t) \tag{83}$$

subject to initial conditions $C_n(0)=\delta_{n,0}$. Instead of going through the long song and dance of solving the equations (83), let me simply give you the solutions

$$C_n(t) \;=\; \frac{(\tanh(t))^n}{\cosh(t)} \tag{84}$$

and verify that they indeed solve the equations (83):

$$\begin{aligned}\frac{d}{dt}\frac{(\tanh(t))^n}{\cosh(t)} \;&=\; n\frac{(\tanh t)^{n-1}}{\cosh t}\times\frac{1}{\cosh^2 t} \;-\; (\tanh t)^n \times \frac{\sinh t}{\cosh^2 t}\\ &=\; \frac{1}{\cosh t}\Bigl(n(\tanh t)^{n-1}\times(1-\tanh^2 t) \;-\; (\tanh t)^{n+1}\Bigr)\\ &=\; n\,\frac{(\tanh t)^{n-1}}{\cosh t} \;-\; (n+1)\,\frac{(\tanh t)^{n+1}}{\cosh t},\end{aligned} \tag{85}$$

quod erat demonstrandum.

Now, given the quantum state (80) of the $\pm\mathbf{k}$ modes of the atoms, we may calculate the expectation value of the atom number in these two modes as

$$N_{\pm\mathbf{k}} = \sum_{n=0}^{\infty}(2n)\times C_n^2(t_{\mathbf{k}}). \tag{86}$$

Specifically, for the C_n coefficients as in eq. (84),

$$N_{\pm\mathbf{k}} = \frac{2}{\cosh^2 t_{\mathbf{k}}}\times\sum_n n(\tanh t_{\mathbf{k}})^{2n} = \frac{2}{\cosh^2 t_{\mathbf{k}}}\times\frac{\tanh^2(t_{\mathbf{k}})}{(1-\tanh^2(t_b k))^2} = 2\sinh^2(t_{\mathbf{k}}). \tag{87}$$

Finally, combining all such $\pm\mathbf{k}$ pairs of modes, we get the net (average) number of atoms in all the $\mathbf{k}\neq 0$ modes as

$$N_{\mathbf{k}\neq 0} = \frac{1}{2}\sum_{\mathbf{k}} N_{\pm\mathbf{k}} = \sum_{\mathbf{k}}\sinh^2(t_{\mathbf{k}}). \tag{28}$$

$\mathcal{Q.E.D.}$

Lemma 5: the net momentum operator is

$$\hat{\mathbf{P}}_{\text{net}} = \sum_{\mathbf{k}}\mathbf{k}\hat{a}^\dagger_{\mathbf{k}}\hat{a}_{\mathbf{k}}\,. \tag{88}$$

Using eqs. (64) and (65) from the proof of Lemma 2 and $t_{-\mathbf{k}} = t_{\mathbf{k}}$, we immediately see that

$$\hat{b}^\dagger_{\mathbf{k}}\hat{b}_{\mathbf{k}} - \hat{b}^\dagger_{-\mathbf{k}}\hat{b}_{-\mathbf{k}} = \Big(\cosh^2(t_{\mathbf{k}}) - \sinh^2(t_{\mathbf{k}}) = 1\Big)\times(\hat{a}^\dagger_{\mathbf{k}}\hat{a}_{\mathbf{k}} - \hat{a}^\dagger_{-\mathbf{k}}\hat{a}_{-\mathbf{k}}). \tag{89}$$

Consequently, for the momentum operator (88) we have

$$\begin{aligned}\hat{\mathbf{P}} &= \sum_{\mathbf{k}}\mathbf{k}\times\hat{a}^\dagger_{\mathbf{k}}\hat{a}_{\mathbf{k}} = \sum_{\mathbf{k}}(-\mathbf{k})\times\hat{a}^\dagger_{-\mathbf{k}}\hat{a}_{-\mathbf{k}}\\ &= \frac{1}{2}\sum_{\mathbf{k}}\mathbf{k}\times(\hat{a}^\dagger_{\mathbf{k}}\hat{a}_{\mathbf{k}} - \hat{a}^\dagger_{-\mathbf{k}}\hat{a}_{-\mathbf{k}})\\ &= \frac{1}{2}\sum_{\mathbf{k}}\mathbf{k}\times(\hat{b}^\dagger_{\mathbf{k}}\hat{b}_{\mathbf{k}} - \hat{b}^\dagger_{-\mathbf{k}}\hat{b}_{-\mathbf{k}})\\ &= \sum_{\mathbf{k}}\mathbf{k}\times\hat{b}^\dagger_{\mathbf{k}}\hat{b}_{\mathbf{k}}\,.\end{aligned} \tag{90}$$

Quod erat demonstrandum.

Alternatively, we may use the unitary operator transform of Lemma 3 and the fact that the $\hat{F}$ operator commutes with the net momentum $\hat{\mathbf{P}}$. Indeed, for every $\mathbf{k}$ mode

$$\hat{\mathbf{P}}\hat{a}_{\mathbf{k}}\hat{a}_{-\mathbf{k}} \;=\; \hat{a}_{\mathbf{k}}(\hat{\mathbf{P}}-\mathbf{k})\hat{a}_{-\mathbf{k}} \;=\; \hat{a}_{\mathbf{k}}\hat{a}_{-\mathbf{k}}\hat{\mathbf{P}} \tag{91}$$

and likewise

$$\hat{\mathbf{P}}\hat{a}^\dagger_{\mathbf{k}}\hat{a}^\dagger_{-\mathbf{k}} \;=\; \hat{a}^\dagger_{\mathbf{k}}(\hat{\mathbf{P}}+\mathbf{k})\hat{a}^\dagger_{-\mathbf{k}} \;=\; \hat{a}^\dagger_{\mathbf{k}}\hat{a}^\dagger_{-\mathbf{k}}\hat{\mathbf{P}}, \tag{92}$$

hence $\hat{\mathbf{P}}\hat{F} = \hat{F}\hat{\mathbf{P}}$. Consequently, the BEC ground state $|\text{ground}\rangle = e^{\hat{F}}\,|\text{coherent}\rangle$ has the same momentum as the $|\text{coherent}\rangle$ state, namely zero, and the quasiparticles created by the $\hat{b}^\dagger_{\mathbf{k}}$ and annihilated by the $\hat{b}_{\mathbf{k}}$ carry the same definite momenta $\mathbf{k}$ as the atoms created by the $\hat{a}^\dagger_{\mathbf{k}}$ and annihilated by the $\hat{a}_{\mathbf{k}}$.

Lemma 6: the finite-range potential $V_2(\mathbf{x}-\mathbf{y})$ for the helium atoms. Consider the net potential operator

$$\hat{V} \;=\; \frac{1}{2}\int\! d^3\mathbf{x}\int\! d^3\mathbf{y}\, V_2(\mathbf{x}-\mathbf{y}) \times \hat{\psi}^\dagger(\mathbf{x})\hat{\psi}^\dagger(\mathbf{y})\hat{\psi}(\mathbf{y})\hat{\psi}(\mathbf{x}). \tag{93}$$

In terms of the shifted fields $\delta\hat{\psi}(\mathbf{x}) = \hat{\psi}(\mathbf{x}) - \sqrt{\bar{n}_s}$ and $\delta\hat{\psi}^\dagger(\mathbf{x}) = \hat{\psi}^\dagger(\mathbf{x}) - \sqrt{\bar{n}_s}$, we have

$$\begin{aligned}\hat{\psi}^\dagger(\mathbf{x})\hat{\psi}^\dagger(\mathbf{y})\hat{\psi}(\mathbf{y})\hat{\psi}(\mathbf{x}) \;=\; & \bar{n}_s^2 \;+\; \bar{n}_s^{3/2}\Big(\delta\hat{\psi}^\dagger(\mathbf{x}) \;+\; \delta\hat{\psi}^\dagger(\mathbf{y}) \;+\; \delta\hat{\psi}(\mathbf{x}) \;+\; \delta\hat{\psi}(\mathbf{y})\Big)\\ & +\; \bar{n}_s\Big(\delta\hat{\psi}^\dagger(\mathbf{x})\delta\hat{\psi}(\mathbf{x}) \;+\; \delta\hat{\psi}^\dagger(\mathbf{y})\delta\hat{\psi}(\mathbf{y})\Big)\\ & +\; \bar{n}_s\Big(\delta\hat{\psi}^\dagger(\mathbf{x})\delta\hat{\psi}(\mathbf{y}) \;+\; \delta\hat{\psi}^\dagger(\mathbf{y})\delta\hat{\psi}(\mathbf{x})\Big)\\ & +\; \bar{n}_s\Big(\delta\hat{\psi}^\dagger(\mathbf{x})\delta\hat{\psi}^\dagger(\mathbf{y}) \;+\; \delta\hat{\psi}(\mathbf{y})\delta\hat{\psi}(\mathbf{x})\Big)\\ & +\; \text{cubic} \;+\; \text{quartic}.\end{aligned} \tag{94}$$

The terms on the first two lines here depend only on the $\mathbf{x}$ or only on the $\mathbf{y}$, so when we plug them into the potential operator (93), we may immediately integrate over the other space position to obtain

$$[\text{@any fixed }\mathbf{y}]\;\int\! d^3\mathbf{x}\, V_2(\mathbf{x}-\mathbf{y}) \;=\; [\text{@any fixed }\mathbf{x}]\;\int\! d^3\mathbf{y}\, V_2(\mathbf{x}-\mathbf{y}) \;=\; W(0). \tag{95}$$

Consequently, integrating over the expansion (94) in the context of the potential (93) and making use of the $\mathbf{x} \leftrightarrow \mathbf{y}$ symmetry, we obtain

$$\begin{aligned}\hat{V} \;=\; & \bar{n}_s \times W(0) \times \int\! d^3\mathbf{x} \Big(\tfrac{1}{2}\bar{n}_s \;+\; \sqrt{\bar{n}_s}\big(\delta\hat{\psi}^\dagger(\mathbf{x}) + \delta\hat{\psi}(\mathbf{x})\big) \;+\; \delta\hat{\psi}^\dagger(\mathbf{x})\delta\hat{\psi}(\mathbf{x}) \Big) \\ & +\; \frac{\bar{n}_2}{2} \times \int\! d^3\mathbf{x} \int\! d^3\mathbf{y}\, V_2(\mathbf{x}-\mathbf{y}) \times \Big(2\delta\hat{\psi}^\dagger(\mathbf{x})\delta\hat{\psi}(\mathbf{y}) \;+\; \delta\hat{\psi}^\dagger(\mathbf{x})\delta\hat{\psi}^\dagger(\mathbf{y}) \;+\; \delta\hat{\psi}(\mathbf{y})\delta\hat{\psi}(\mathbf{x}) \Big) \\ & +\; \text{cubic} \;+\; \text{quartic.}\end{aligned} \tag{96}$$

Now consider the other non-derivative term in the Helium's Hamiltonian

$$\hat{H}_{\text{net}} \;=\; \hat{K} \;+\; \hat{V} \;-\; \mu\hat{N}, \tag{97}$$

namely the chemical potential term,

$$\begin{aligned} -\mu\hat{N} \;&=\; -\mu \int\! d^3\mathbf{x}\, \hat{\psi}^\dagger(\mathbf{x})\hat{\psi}(\mathbf{x}) \\ &=\; -\mu \int\! d^3\mathbf{x} \Big(\bar{n}_s \;+\; \sqrt{\bar{n}_s}\big(\delta\hat{\psi}(\mathbf{x}) + \delta\hat{\psi}^\dagger(\mathbf{x})\big) \;+\; \delta\hat{\psi}^\dagger(\mathbf{x})\delta\hat{\psi}(\mathbf{x}) \Big) \end{aligned} \tag{98}$$

If we generalize the $\mu = \lambda\bar{n}_s$ formula of the Landau–Ginzburg theory to the

$$\mu \;=\; W(0) \times \bar{n}_s\,, \tag{99}$$

then the chemical potential term (98) cancels the top line of the two-body potential (96) (except for the constant part), hence

$$\begin{aligned}\hat{V} \;-\; \mu\hat{N} \;=\; & \frac{\bar{n}_2}{2} \int\! d^3\mathbf{x} \int\! d^3\mathbf{y}\, V_2(\mathbf{x}-\mathbf{y}) \times \Big(2\delta\hat{\psi}^\dagger(\mathbf{x})\delta\hat{\psi}(\mathbf{y}) \;+\; \delta\hat{\psi}^\dagger(\mathbf{x})\delta\hat{\psi}^\dagger(\mathbf{y}) \;+\; \delta\hat{\psi}(\mathbf{y})\delta\hat{\psi}(\mathbf{x}) \Big) \\ & +\; \text{constant} \;+ \text{cubic} \;+\; \text{quartic.}\end{aligned} \tag{100}$$

Thus altogether,

$$\hat{H} \;=\; \text{constant} \;+\; \hat{H}_{\text{free}} \;+\; \hat{H}_{\text{interactions}} \tag{101}$$

where

$$\begin{aligned}\hat{H}_{\text{free}} \;=\; & \frac{1}{2m} \int\! d^3\mathbf{x}\, \nabla\delta\hat{\psi}^\dagger(\mathbf{x}) \cdot \nabla\delta\hat{\psi}(\mathbf{x}) \\ & +\; \frac{\bar{n}_2}{2} \int\! d^3\mathbf{x} \int\! d^3\mathbf{y}\, V_2(\mathbf{x}-\mathbf{y}) \times \Big(2\delta\hat{\psi}^\dagger(\mathbf{x})\delta\hat{\psi}(\mathbf{y}) \;+\; \delta\hat{\psi}^\dagger(\mathbf{x})\delta\hat{\psi}^\dagger(\mathbf{y}) \;+\; \delta\hat{\psi}(\mathbf{y})\delta\hat{\psi}(\mathbf{x}) \Big).\end{aligned} \tag{102}$$

This completes the proof of the first part of the Lemma 6 — the top two lines of the eq. (45).

To prove the second part of the Lemma (the bottom line of eq. (45)) we simply Fourier transform from the shifted creation and annihilation fields to the creation and annihilation operators for modes $\mathbf{k} \neq 0$,

$$\delta\hat{\psi}^{\dagger}(\mathbf{x}) \;=\; L^{-3/2}\sum_{\mathbf{k}\neq 0} e^{+i\mathbf{k}\mathbf{x}}\hat{a}^{\dagger}_{\mathbf{k}}\,, \qquad \delta\hat{\psi}(\mathbf{x}) \;=\; L^{-3/2}\sum_{\mathbf{k}\neq 0} e^{-i\mathbf{k}\mathbf{x}}\hat{a}_{\mathbf{k}}\,. \tag{103}$$

Consequently,

$$\begin{aligned}
&\int\! d^3\mathbf{x}\int\! d^3\mathbf{y}\, V_2(\mathbf{x}-\mathbf{y}) \times \delta\hat{\psi}^{\dagger}(\mathbf{x})\delta\hat{\psi}(\mathbf{y}) \;= \\
&\qquad = \int\! d^3\mathbf{x}\int\! d^3\mathbf{y}\, V_2(\mathbf{x}-\mathbf{y}) \times L^{-3}\sum_{\mathbf{k},\mathbf{k}'} e^{i\mathbf{k}\mathbf{x}-i\mathbf{k}'\mathbf{y}} \times \hat{a}^{\dagger}_{\mathbf{k}}\hat{a}_{\mathbf{k}'} \\
&\qquad = \sum_{\mathbf{k},\mathbf{k}'} \hat{a}^{\dagger}_{\mathbf{k}}\hat{a}_{\mathbf{k}'} \times L^{-3}\int\! d^3\mathbf{x}\int\! d^3\mathbf{y}\, V_2(\mathbf{x}-\mathbf{y}) \times e^{i\mathbf{k}\mathbf{x}-i\mathbf{k}'\mathbf{y}}
\end{aligned} \tag{104}$$

where

$$\begin{aligned}
&L^{-3}\int\! d^3\mathbf{x}\int\! d^3\mathbf{y}\, V_2(\mathbf{x}-\mathbf{y}) \times e^{i\mathbf{k}\mathbf{x}-i\mathbf{k}'\mathbf{y}} \;= \\
&\qquad =\; L^{-3}\int\! d^3\mathbf{y}\int\! d^3(\mathbf{z}=\mathbf{x}-\mathbf{y})\, V_2(\mathbf{z}) \times e^{i\mathbf{k}(\mathbf{y}+\mathbf{z})-i\mathbf{k}'\mathbf{y}} \\
&\qquad = \int\! d^3\mathbf{z}\, V_2(\mathbf{z})e^{i\mathbf{k}\mathbf{z}} \times L^{-3}\int_{\text{box}} d^3\mathbf{y}\; e^{i\mathbf{k}\mathbf{y}-i\mathbf{k}'\mathbf{y}} \\
&\qquad =\; W(\mathbf{k}) \times \delta_{\mathbf{k},\mathbf{k}'}\,,
\end{aligned} \tag{105}$$

hence

$$\int\! d^3\mathbf{x}\int\! d^3\mathbf{y}\, V_2(\mathbf{x}-\mathbf{y}) \times \delta\hat{\psi}^{\dagger}(\mathbf{x})\delta\hat{\psi}(\mathbf{y}) \;=\; \sum_{\mathbf{k}} W(\mathbf{k}) \times \hat{a}^{\dagger}_{\mathbf{k}}\hat{a}_{\mathbf{k}}\,. \tag{106}$$

In the same way we obtain

$$\begin{aligned}
&\int\! d^3\mathbf{x}\int\! d^3\mathbf{y}\, V_2(\mathbf{x}-\mathbf{y}) \times \delta\hat{\psi}^{\dagger}(\mathbf{x})\delta\hat{\psi}^{\dagger}(\mathbf{y}) \;= \\
&\qquad = \int\! d^3\mathbf{x}\int\! d^3\mathbf{y}\, V_2(\mathbf{x}-\mathbf{y}) \times L^{-3}\sum_{\mathbf{k},\mathbf{k}'} e^{i\mathbf{k}\mathbf{x}-i\mathbf{k}'\mathbf{y}} \times \hat{a}^{\dagger}_{\mathbf{k}}\hat{a}^{\dagger}_{-\mathbf{k}} \\
&\qquad = \sum_{\mathbf{k}} W(\mathbf{k}) \times \hat{a}^{\dagger}_{\mathbf{k}}\hat{a}^{\dagger}_{-\mathbf{k}}
\end{aligned} \tag{107}$$

and likewise

$$\int d^3\mathbf{x} \int d^3\mathbf{y}\, V_2(\mathbf{x}-\mathbf{y}) \times \delta\hat{\psi}(\mathbf{x})\delta\hat{\psi}(\mathbf{y}) \;=\; \sum_{\mathbf{k}} W(\mathbf{k}) \times \hat{a}_{-\mathbf{k}}\hat{a}_{\mathbf{k}}\,. \tag{108}$$

Combining all these formulae with the gradient term in the Hamiltonian (102),

$$\hat{K} \;=\; \frac{1}{2m}\int d^3\mathbf{x}\, \nabla\psi^\dagger \cdot \nabla\psi \;=\; \frac{1}{2m}\int d^3\mathbf{x}\, \nabla\delta\psi^\dagger \cdot \nabla\delta\psi \;=\; \sum_{\mathbf{k}} \frac{\mathbf{k}^2}{2m}\, \hat{a}^\dagger_{\mathbf{k}}\hat{a}_{\mathbf{k}}\,, \tag{109}$$

we finally assemble all quadratic terms to

$$\hat{H}_{\text{free}} \;=\; \sum_{\mathbf{k}} \left(\left(\frac{\mathbf{k}^2}{2m} \,+\, W(\mathbf{k})\bar{n}_s \right) \hat{a}^\dagger_{\mathbf{k}}\hat{a}_{\mathbf{k}} \,+\, \tfrac{1}{2} W(\mathbf{k})\bar{n}_s \big(\hat{a}^\dagger_{\mathbf{k}}\hat{a}^\dagger_{-\mathbf{k}} \,+\, \hat{a}_{-\mathbf{k}}\hat{a}_{\mathbf{k}} \big) \right). \tag{110}$$

Quod erat demonstrandum.