Fermions of the ElectroWeak Theory

The Quarks, The Leptons, and their Masses.

This is my second set of notes on the Glashow–Weinberg–Salam theory of weak and electromagnetic interactions. The first set was about the bosonic fields of the theory — the gauge fields of the $SU(2)\times U(1)$ gauge theory and the Higgs fields that give mass to the $W^\pm_\mu$ and Z^0_μ vector particles. This set is about the fermionic fields — the quarks and the leptons.

From the fermionic point of view, the electroweak gauge symmetry $SU(2)_W\times U(1)_Y$ is *chiral* — the left-handed and the right-handed fermions form different types of multiplets — and consequently, the weak interactions do not respect the parity or the charge-conjugation symmetries. Specifically, all the left-handed quarks and leptons form doublets of the $SU(2)_W$ while the all right-handed quarks and leptons are singlets, so the charged weak currents are purely left-handed,

$$J^\mu_\pm \;=\; \tfrac{1}{2}\bigl(V^\mu\,-\,A^\mu\bigr) \;=\; \overline{\Psi}\gamma^\mu\frac{1-\gamma^5}{2}\Psi \;=\; \psi^\dagger_L\bar\sigma^\mu\psi_L \quad\text{without a } \psi_R \text{ term.} \tag{1}$$

The left-handed and the right-handed fermions also have different $U(1)$ hypercharges, which is needed to give them similar electric charges $Q=Y+T^3$. For example, the LH up and down quarks — which form an $SU(2)_W$ doublet — have $Y=+\frac{1}{6}$, while the RH quarks are $SU(2)$ singlets and have $Y_u=+\frac{2}{3}$ and $Y_d=-\frac{1}{3}$. Consequently, their electric charges come out to be

$$\begin{aligned}
&\left.\begin{aligned}
Q(u,L) \;&=\; Y(u,L)\,+\,T^3(u,L) \;=\; +\tfrac{1}{6}\,+\,\tfrac{1}{2} \;=\; +\tfrac{2}{3}\\
Q(u,R) \;&=\; Y(u,R)\,+\,T^3(u,R) \;=\; +\tfrac{2}{3}\,+\,0 \;=\; +\tfrac{2}{3}
\end{aligned}\right\}\quad\text{same,}\\
&\left.\begin{aligned}
Q(d,L) \;&=\; Y(d,L)\,+\,T^3(d,L) \;=\; +\tfrac{1}{6}\,-\,\tfrac{1}{2} \;=\; -\tfrac{1}{3}\\
Q(d,R) \;&=\; Y(d,R)\,+\,T^3(d,R) \;=\; -\tfrac{1}{3}\,+\,0 \;=\; -\tfrac{1}{3}
\end{aligned}\right\}\quad\text{same.}
\end{aligned} \tag{2}$$

In light of different quantum numbers for the LH and RH quarks, their Lagrangian cannot have any mass terms $\psi^\dagger_L\psi_R$ or $\psi^\dagger_R\psi_L$. Instead, the physical quark masses arise from the Yukawa couplings of the quarks to the Higgs scalars H_i. In general, the Yukawa couplings of fermions to scalars (or pseudoscalars) have form

$$\text{true scalar:}\quad g_s\phi\times\overline{\Psi}\Psi \;=\; g_s\phi\times\bigl(\psi^\dagger_L\psi_R\,+\,\psi^\dagger_R\psi_L\bigr), \tag{3}$$

$$\text{pseudoscalar:} \quad ig_p\phi \times \overline{\Psi}\gamma^5\Psi \;=\; g_p\phi \times \left(i\psi_L^\dagger\psi_R - i\psi_R^\dagger\psi_L\right), \tag{4}$$

or for a complex scalar field Φ without parity symmetry

$$g\Phi \times \psi_L^\dagger\psi_R \;+\; g^*\Phi^* \times \psi_R^\dagger\psi_L \tag{5}$$

with a complex coupling constant $g = g_s + ig_p$. The theories with multiple fermionic and scalar fields may have different Yukawa couplings for different scalar and fermionic species, as long as they are invariant under all the required symmetries. For the electroweak symmetry at hand, the ψ_L are $SU(2)$ doublets while the ψ_R are singlets, so the bi-linears $\psi_L^\dagger\psi_R$ and $\psi_R^\dagger\psi_L$ are $SU(2)$ doublets, which may couple to the $SU(2)$ doublet of scalars such as the Higgs fields H^i or their conjugates H_i^*. Taking the $U(1)$ hypercharges of the up and down quarks into account, we see that $Y(\psi_L^\dagger\psi_R^u) = +\frac{1}{2}$ so this bilinear may couple to the H^*, while $Y(\psi_L^\dagger\psi_R^d) = -\frac{1}{2}$ so it may couple to the H, thus

$$\begin{aligned}\mathcal{L}_{\text{Yukawa}} \;=\; & -g_d H_i^* \times \psi_R^{d\dagger}\psi_L^i \;-\; g_d H^i \times \psi_{L,i}^\dagger\psi_R^d \\ & -\; g_u\epsilon_{ij}H^i \times \psi_R^{u\dagger}\psi_L^j \;-\; g_u\epsilon^{ij}H_i^* \times \psi_{L,j}^\dagger\psi_R^u\,.\end{aligned} \tag{6}$$

When a scalar field ϕ develops a non-zero VEV, the Yukawa couplings to this field give rise to the fermionic mass terms. For example, consider a toy model of a real scalar Φ and an initially-massless Dirac fermion Ψ with Lagrangian

$$\mathcal{L} \;=\; \tfrac{1}{2}(\partial_\mu\phi)^2 \;-\; V(\phi) \;+\; \overline{\Psi}i\partial\!\!\!/\Psi - \; g\phi\overline{\Psi}\Psi \tag{7}$$

where the potential $V(\phi)$ is minimized by $\phi \neq 0$, hence $\langle\phi\rangle \neq 0$. In terms of the shifted scalar field $\phi(x) = \langle\phi\rangle + \delta\phi(x)$, the Yukawa term in the Lagrangian (7) becomes

$$\mathcal{L}_{\text{Yukawa}} \;=\; -g\phi \times \overline{\Psi}\Psi \;=\; -g\left\langle\phi\right\rangle \times \overline{\Psi}\Psi \;-\; g\delta\phi \times \overline{\Psi}\Psi, \tag{8}$$

where the first term on the RHS is the *emergent mass term $m = g\left\langle\phi\right\rangle$ for the fermion Ψ*, while the second term is the Yukawa coupling to the physical scalar $\delta\phi$.

Likewise, *when the Higgs doublet of the GWS theory develops a non-zero Vacuum Expectation Value*

$$\langle H\rangle \;=\; \frac{v}{\sqrt{2}}\times\begin{pmatrix}0\\ 1\end{pmatrix},\qquad v\approx 247\ \mathrm{GeV},\quad H^i(x)\;=\;\langle H\rangle^i\;+\;\delta h^i(x),\tag{9}$$

the Yukawa couplings (6) of the up and down quarks to this VEV give rise to quark mass terms,

$$\mathcal{L}_{\text{Yukawa}}\;\longrightarrow\;\mathcal{L}_{\text{mass}}\;+\;\text{couplings to the physical Higgs field},\tag{10}$$

$$\begin{aligned}\mathcal{L}_{\text{mass}}\;&=\;\mathcal{L}_{\text{Yukawa}}\quad\text{for}\quad H\to\langle H\rangle\\ &=\;-g_d\,\frac{v}{\sqrt{2}}\times\left(\psi_R^{d\dagger}\psi_L^2\;+\;\psi_L^{2\dagger}\psi_R^d\right)\;-\;g_u\,\frac{v}{\sqrt{2}}\times\left(\psi_R^{u\dagger}\psi_L^1\;+\;\psi_L^{1\dagger}\psi_R^u\right)&(11)\\ &\equiv\;-m_d\times\overline{\Psi}^d\Psi^d\;-\;m_u\overline{\Psi}^u\Psi^u,&(12)\end{aligned}$$

where the Dirac fermions Ψ^u and Ψ^d comprise

$$\Psi^u\;=\;\begin{pmatrix}\psi_L^1\\ \psi_R^u\end{pmatrix},\qquad \Psi^d\;=\;\begin{pmatrix}\psi_L^2\\ \psi_R^d\end{pmatrix}\tag{13}$$

and their masses follow from the Higgs VEV and the Yukawa couplings as

$$m_u\;=\;g_u\times\frac{v}{\sqrt{2}}\,,\qquad m_d\;=\;g_d\times\frac{v}{\sqrt{2}}\,.\tag{14}$$

The other 4 quark flavors — charm, strange, top, and bottom — have similar quantum numbers to the up and down quarks. The left-handed quarks form $SU(2)$ doublets $(c,s)_L$ and $(t,b)_L$ with $Y=+\frac{1}{6}$ while the right-handed quarks are singlets with hypercharges $Y(c_R)=Y(T_R)=+\frac{2}{3}$ and $Y(s_R)=Y(b_R)=-\frac{1}{3}$, which lead to non-chiral electric charges

$$\begin{aligned}Q(c_{L\,\text{or}\,R})\;&=\;Q(t_{L\,\text{or}\,R})\;=\;Q(u_{L\,\text{or}\,R})\;=\;+\tfrac{2}{3},\\ Q(s_{L\,\text{or}\,R})\;&=\;Q(b_{L\,\text{or}\,R})\;=\;Q(d_{L\,\text{or}\,R})\;=\;-\tfrac{1}{3}.\end{aligned}\tag{15}$$

Again, the $SU(2)\times U(1)$ quantum numbers of these quarks forbid any mass terms $\psi_L^\dagger\psi_R$ or $\psi_R^\dagger\psi_L$ in the Lagrangian, but they allow the Yukawa couplings to the Higgs fields similar to (6).

The physical masses obtain from those Yukawa couplings when the Higgs scalar develops a non-zero VEV and breaks the $SU(2)\times U(1)$ symmetry down to the $U(1)_{\rm EM}$; similar to eq. (14),

$$m_s \;=\; g_s\times\frac{v}{\sqrt{2}}\,,\qquad m_c \;=\; g_c\times\frac{v}{\sqrt{2}}\,,\qquad m_b \;=\; g_b\times\frac{v}{\sqrt{2}}\,,\qquad m_t \;=\; g_t\times\frac{v}{\sqrt{2}}\,. \tag{16}$$

Note that the charge $=+\frac{2}{3}$ quarks u,c,t have exactly similar electroweak quantum numbers but very different values of the Yukawa couplings, $g_u\ll g_c\ll g_t$, and hence very different physical masses, $m_u\ll m_c\ll m_t$. Likewise, the charge $=-\frac{1}{3}$ quarks d,s,t have exactly similar electroweak quantum numbers but different Yukawa couplings, $g_d\ll g_s\ll g_b$, and hence different physical masses, $m_d\ll m_s\ll m_b$. Experimentally

$$m_u\approx 2.15\text{ MeV}\;\ll\; m_c\approx 1.28\text{ GeV}\;\ll\; m_t\approx 173\text{ GeV}, \tag{17}$$
$$m_d\approx 4.7\text{ MeV}\;\ll\; m_s\approx 94\text{ MeV}\;\ll\; m_b\approx 4.2\text{ GeV}, \tag{18}$$

but we do not have a good explanation of this hierarchical pattern. In the Standard Model, the Yukawa couplings are arbitrary parameters to be determined experimentally. Beyond the Standard Model, there have been all kinds of speculative explanations over the last 40+ years, but none of them can be supported by any experimental evidence whatsoever.

Besides the quarks, there are 3 species of charged leptons — the electron e^-, the muon μ^-, and the tau τ^- — and 3 species of neutrinos, ν_e, ν_μ, ν_τ. The left-handed fermions of these 6 species form three $SU(2)$ doublets $(\nu_e,e^-)_L$, $(\nu_\mu,\mu^-)_L$, and $(\nu_\tau,\tau^-)_L$ with $Y=-\frac{1}{2}$, so the bottom halves of these doublets have electric charges

$$Q(e^-_L)\;=\;Q(\mu^-_L)\;=\;Q(\tau^-_L)\;=\;Y\,-\,\tfrac{1}{2}\;=\;-1 \tag{19}$$

while the top-halves — the neutrinos — are electrically neutral,

$$Q(\nu_{eL})\;=\;Q(\nu_{\mu L})\;=\;Q(\nu_{\tau L})\;=\;Y\,+\,\tfrac{1}{2}\;=\;0. \tag{20}$$

The right-handed electron, muon, and tau are $SU(2)$ singlets with $Y=-1$, so their electric charge $Q=Y+0=-1$ is the same as for the left-handed e,μ,τ.

As to the right-handed neutrino fields, there are two theories: In one theory, the neutrino fields are left-handed Weyl spinors $\psi_L(\nu)$ rather than Dirac spinors, so the $\psi_R(\nu)$ simply do not exist! In the other theory, the $\psi_R(\nu)$ do exist, but they are $SU(2)$ singlets with $Y = 0$ and therefore do not have any weak interactions. Since they also do not have strong or EM interactions, this makes the RH neutrinos completely invisible to the experiment — and that's why we do not know if they exist or not. For the moment, let me focus on the simplest version without the $\psi_R(\nu)$; I'll come back to the other theory I'll in the separate set of notes on neutrino masses.

Similar to the quarks, the $SU(2) \times U(1)$ quantum numbers of the leptons do not allow any mass terms in the Lagrangian, but they do allow the Yukawa couplings of leptons to the Higgs fields,

$$\begin{aligned}\mathcal{L}_{\text{Yukawa}} \;=\; & -g_e H_i^* \times \psi_R^\dagger(e)\psi_L^i(\nu_e, e) \;-\; g_e H^i \times \psi_{L,i}^\dagger(\nu_e, e)\psi_R(e) \\ & -\; g_\mu H_i^* \times \psi_R^\dagger(\mu)\psi_L^i(\nu_\mu, \mu) \;-\; g_\mu H^i \times \psi_{L,i}^\dagger(\nu_\mu, \mu)\psi_R(\mu) \\ & -\; g_\tau H_i^* \times \psi_R^\dagger(\tau)\psi_L^i(\nu_\tau, \tau) \;-\; g_\tau H^i \times \psi_{L,i}^\dagger(\nu_\tau, \tau)\psi_R(\tau).\end{aligned} \tag{21}$$

When the Higgs field H_2 develop non-zero VEV $\frac{v}{\sqrt{2}}$, these Yukawa couplings give rise to the lepton masses; similar to the quarks,

$$m_e \;=\; g_e \times \frac{v}{\sqrt{2}}\,, \qquad m_\mu \;=\; g_\mu \times \frac{v}{\sqrt{2}}\,, \qquad m_\tau \;=\; g_\tau \times \frac{v}{\sqrt{2}}\,. \tag{22}$$

Experimentally, these masses are

$$m_e = 0.511 \text{ MeV} \;\ll\; m_\mu = 106 \text{ MeV} \;\ll\; m_\tau = 1777 \text{ MeV}. \tag{23}$$

Similar to the quarks, the masses of charged leptons form a hierarchy; we do no not know why.

Weak Currents

Altogether, the fermionic fields of the electroweak theory and their couplings to the bosonic gauge and Higgs fields can be summarized by the Lagrangian

$$\mathcal{L}_F \;=\; \sum_{\substack{\text{LH quarks}\\ \text{\& leptons}}} i\psi_L^\dagger \bar\sigma^\mu D_\mu \psi_L \;+\; \sum_{\substack{\text{RH quarks}\\ \text{\& leptons}}} i\psi_R^\dagger \sigma^\mu D_\mu \psi_R \;+\; \mathcal{L}_{\text{Yukawa}}\,. \tag{24}$$

In the first section of these notes I was focused on the Yukawa couplings that give rise to the fermion masses when the Higgs field gets its VEV, but now let's turn our attention to the interactions of quarks and leptons with the electroweak $SU(2)\times U(1)$ gauge fields. In the Lagrangian (24), the gauge interactions are hidden inside the covariant derivatives D_μ, so let's spell them out. Since these notes are about on the electroweak interactions rather than strong interactions, let me skip suppress the quarks' color indices and ignore their couplings to the gluon fields, thus:

- The left-handed quarks form $SU(2)$ doublets

$$\psi^i_L \;=\; \begin{pmatrix} u \\ d \end{pmatrix}_L \quad\text{or}\quad \begin{pmatrix} c \\ s \end{pmatrix}_L \quad\text{or}\quad \begin{pmatrix} t \\ b \end{pmatrix}_L \tag{25}$$

  of hypercharge $Y=+\frac{1}{6}$, so for the LH quark fields

$$D_\mu\psi^i_L \;=\; \partial_\mu\psi^i_L \;+\; \frac{ig_2}{2}\,W^a_\mu(\tau^a)^i{}_j\psi^j_L \;+\; \frac{ig_1}{6}\,B_\mu\psi^i_L\,.$$

- The left-handed leptons also form $SU(2)$ doublets

$$\psi^i_L \;=\; \begin{pmatrix} \nu_e \\ e^- \end{pmatrix}_L \quad\text{or}\quad \begin{pmatrix} \nu_\mu \\ \mu^- \end{pmatrix}_L \quad\text{or}\quad \begin{pmatrix} \nu_\tau \\ \tau^- \end{pmatrix}_L \tag{26}$$

  but of hypercharge $Y=-\frac{1}{2}$, so for the LH lepton fields

$$D_\mu\psi^i_L \;=\; \partial_\mu\psi^i_L \;+\; \frac{ig_2}{2}\,W^a_\mu(\tau^a)^i{}_j\psi^j_L \;-\; \frac{ig_1}{2}\,B_\mu\psi^i_L\,.$$

- The right handed quarks are $SU(2)$ singlets of hypercharges $Y=+\frac{2}{3}$ or $Y=-\frac{1}{3}$, thus

$$\begin{aligned} &\text{for } \psi_R \;=\; u_R \text{ or } c_R \text{ or } t_R\,, \quad D_\mu\psi_R \;=\; \partial_\mu\psi_R \;+\; \frac{2ig_1}{3}\,B_\mu\psi_R\,,\\ &\text{for } \psi_R \;=\; d_R \text{ or } s_R \text{ or } b_R\,, \quad D_\mu\psi_R \;=\; \partial_\mu\psi_R \;-\; \frac{ig_1}{3}\,B_\mu\psi_R\,. \end{aligned} \tag{27}$$

- The right-handed charged leptons are $SU(2)$ singlets of hypercharge $Y=-1$, thus

$$\text{for } \psi_R \;=\; e^-_R \text{ or } \mu^-_R \text{ or } \tau^-_R\,, \quad D_\mu\psi_R \;=\; \partial_\mu\psi_R \;-\; ig_1\,B_\mu\psi_R\,. \tag{28}$$

— Finally, if the right-handed neutrino fields exist at all, they are $SU(2)$ singlets and have zero hypercharge, thus

$$\text{for } \psi_R \;=\; \nu_R^e \text{ or } \nu_R^\mu \text{ or } \nu_R^\tau\,, \quad D_\mu\psi_R \;=\; \partial_\mu\psi_R \;+\; 0. \tag{29}$$

Now let's plug these covariant derivatives into the Lagrangian (24), extract the terms containing the $SU(2)\times U(1)$ gauge fields, and organize the fermionic fields interacting with those gauge fields into the currents according to

$$\mathcal{L} \;\supset\; -g_2 W_\mu^a J_{Ta}^\mu \;-\; g_1 B_\mu J_Y^\mu\,, \tag{30}$$

cf. eq. (21) from my notes on the bosonic sector on the electroweak theory. Since the right-handed quarks and leptons are $SU(2)$ singlets, the $SU(2)$ currents turn out to be purely left-handed,

$$J_{Ta}^\mu \;=\; \sum_{(u,d),(c,s),(t,b)}^{\text{LH quarks}} \psi_{L,i}^\dagger \left(\frac{\tau^a}{2}\right)^i_j \bar\sigma^\mu \psi_L^j \;+\; \sum_{(\nu_e,e),(\nu_\mu,\mu),(\nu_\tau,\tau)}^{\text{LH leptons}} \psi_{L,i}^\dagger \left(\frac{\tau^a}{2}\right)^i_j \bar\sigma^\mu \psi_L^j\,. \tag{31}$$

However, the $U(1)$ current has both left-handed and right handed contributions,

$$\begin{aligned} J_Y^\mu \;=\; &\sum_{u,c,t\ \text{quarks}} \left(\tfrac16\psi_L^\dagger\bar\sigma^\mu\psi_L \;+\; \tfrac23\psi_R^\dagger\sigma^\mu\psi_R\right) \;+\; \sum_{d,s,b\ \text{quarks}} \left(\tfrac16\psi_L^\dagger\bar\sigma^\mu\psi_L \;-\; \tfrac13\psi_R^\dagger\sigma^\mu\psi_R\right) \\ &+\; \sum_{e,\mu,\tau\ \text{leptons}} \left(-\tfrac12\psi_L^\dagger\bar\sigma^\mu\psi_L \;-\; \psi_R^\dagger\sigma^\mu\psi_R\right) \;+\; \sum_{\text{neutrinos}} \left(-\tfrac12\psi_L^\dagger\bar\sigma^\mu\psi_L \;+\; 0\right). \end{aligned} \tag{32}$$

In the my notes on the bosonic sector I had re-organized these 4 gauge currents into currents which couple to the specific electroweak gauge field, namely the electric current

$$J_{\text{EM}}^\mu \;=\; J_{T3}^\mu \;+\; J_Y^\mu \tag{33}$$

which couples to the EM field A_μ, the *charged weak currents*

$$J^{+\mu} \;=\; J_{T1}^\mu \;-\; iJ_{T2}^\mu \quad\text{and}\quad J^{-\mu} \;=\; J_{T1}^\mu \;+\; iJ_{T2}^\mu \tag{34}$$

which couple to the charged $W_\mu^\pm$ massive vector fields, and the *neutral weak current*

$$J_Z^\mu \;=\; J_{T3}^\mu \;-\; \sin^2\theta J_{\text{EM}}^\mu \tag{35}$$

which couples to the neutral massive vector field Z_μ^0. Now let's spell out all these currents in terms of the fermionic fields. For the charged currents, reorganizing the weak isospin currents (31) into the $J^{\pm\mu}$ amounts to combining the isospin Pauli matrices τ^a in the same way as the currents (34),

$$\tau^+ \equiv \tau^1 - i\tau^2 = \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}, \qquad \tau^- \equiv \tau^1 + i\tau^2 = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}. \tag{36}$$

Consequently, in eqs. (31) we have

$$\psi^\dagger_{L,i}\left(\frac{\tau^+}{2}\right)^i_j \bar\sigma^\mu \psi^j_L = \psi^\dagger_{L,2}\bar\sigma^\mu\psi^1_L\,, \qquad \psi^\dagger_{L,i}\left(\frac{\tau^-}{2}\right)^i_j \bar\sigma^\mu \psi^j_L = \psi^\dagger_{L,1}\bar\sigma^\mu\psi^2_L\,, \tag{37}$$

and therefore

$$\begin{aligned} J^{+\mu} &= \psi^\dagger_L(d)\bar\sigma^\mu\psi_L(u) + \psi^\dagger_L(s)\bar\sigma^\mu\psi_L(c) + \psi^\dagger_L(b)\bar\sigma^\mu\psi_L(t) \\ &\quad + \psi^\dagger_L(e)\bar\sigma^\mu\psi_L(\nu_e) + \psi^\dagger_L(\mu)\bar\sigma^\mu\psi_L(\nu_\mu) + \psi^\dagger_L(\tau)\bar\sigma^\mu\psi_L(\nu_\tau), \\ J^{-\mu} &= \psi^\dagger_L(u)\bar\sigma^\mu\psi_L(d) + \psi^\dagger_L(c)\bar\sigma^\mu\psi_L(s) + \psi^\dagger_L(t)\bar\sigma^\mu\psi_L(b) \\ &\quad + \psi^\dagger_L(\nu_e)\bar\sigma^\mu\psi_L(e) + \psi^\dagger_L(\nu_\mu)\bar\sigma^\mu\psi_L(\mu) + \psi^\dagger_L(\nu_\tau)\bar\sigma^\mu\psi_L(\tau). \end{aligned} \tag{38}$$

In terms of Dirac fermions for the quarks and leptons,

$$\psi^\dagger_L\bar\sigma^\mu\psi_L = \overline{\Psi}\gamma^\mu\frac{1-\gamma^5}{2}\Psi, \tag{39}$$

hence

$$\begin{aligned} J^{+\mu} &= \overline{\Psi}{}^d\gamma^\mu\frac{1-\gamma^5}{2}\Psi^u + \overline{\Psi}{}^s\gamma^\mu\frac{1-\gamma^5}{2}\Psi^c + \overline{\Psi}{}^b\gamma^\mu\frac{1-\gamma^5}{2}\Psi^t \\ &\quad + \overline{\Psi}{}^e\gamma^\mu\frac{1-\gamma^5}{2}\Psi^{\nu_e} + \overline{\Psi}{}^\mu\gamma^\mu\frac{1-\gamma^5}{2}\Psi^{\nu_\mu} + \overline{\Psi}{}^\tau\gamma^\mu\frac{1-\gamma^5}{2}\Psi^{\nu_\tau}, \\ J^{-\mu} &= \overline{\Psi}{}^u\gamma^\mu\frac{1-\gamma^5}{2}\Psi^d + \overline{\Psi}{}^c\gamma^\mu\frac{1-\gamma^5}{2}\Psi^s + \overline{\Psi}{}^t\gamma^\mu\frac{1-\gamma^5}{2}\Psi^b \\ &\quad + \overline{\Psi}{}^{\nu_e}\gamma^\mu\frac{1-\gamma^5}{2}\Psi^e + \overline{\Psi}{}^{\nu_\mu}\gamma^\mu\frac{1-\gamma^5}{2}\Psi^\mu + \overline{\Psi}{}^{\nu_\tau}\gamma^\mu\frac{1-\gamma^5}{2}\Psi^\tau. \end{aligned} \tag{40}$$

As promised, these charged weak currents are purely left-handed, so they completely violate the parity and the charge-conjugation symmetries. But please note that this left-handedness is in terms of chirality of the fermionic fields rather than helicities of the fermionic particles. In terms of helicities, *the quarks and the leptons participating in charged-current weak interactions are polarized left, but the antiquarks and the antileptons are polarized right;* the degree of polarization is $\beta = v/c$, which approaches 100% for the ultra-relativistic particles.

On the other hand, the electric current is left-right symmetric,

$$J^\mu_{\rm EM} \;=\; \tfrac{2}{3}\sum_{q=u,c,t}^{\rm quarks} \overline{\Psi}{}^q\gamma^\mu\Psi^q \;-\; \tfrac{1}{3}\sum_{q=d,s,b}^{\rm quarks} \overline{\Psi}{}^q\gamma^\mu\Psi^q \;-\; \sum_{\ell=e,\mu\tau}^{\rm leptons} \overline{\Psi}{}^\ell\gamma^\mu\Psi^\ell\,. \tag{41}$$

Finally, the neutral weak current has both left-handed and right-handed components but it is not left-right symmetric. In terms of Dirac spinor fields,

$$\begin{aligned} J^\mu_Z \;&=\; J^\mu_{T3}[\text{left-handed}] \;-\; \sin^2\theta \times J^\mu_{\rm EM}[\text{left-right symmetric}] \\ &= \sum_{q=u,c,t}^{\rm quarks} \overline{\Psi}{}^q\gamma^\mu\left(+\frac{1-\gamma^5}{4} - \frac{2}{3}\sin^2\theta\right)\Psi^q \;+\; \sum_{q=d,s,b}^{\rm quarks} \overline{\Psi}{}^q\gamma^\mu\left(-\frac{1-\gamma^5}{4} + \frac{1}{3}\sin^2\theta\right)\Psi^q \\ &\quad+ \sum_{\ell=e,\mu,\tau}^{\rm leptons} \overline{\Psi}{}^\ell\gamma^\mu\left(-\frac{1-\gamma^5}{4} + \sin^2\theta\right)\Psi^\ell \;+\; \sum_{\nu=\nu_e,\nu_\mu,\nu_\tau}^{\rm neutrinos} \overline{\Psi}{}^\nu\gamma^\mu\left(+\frac{1-\gamma^5}{4} - 0\right)\Psi^\nu\,. \end{aligned} \tag{42}$$

Flavor Mixing and the Cabibbo–Kobayashi–Maskawa Matrix

Actually, the charged weak currents are more complicated then I wrote down in eq. (40). Since we have 3 quark flavors of each charge $+\frac{2}{3}$ or $-\frac{1}{3}$, we need to be careful as to how they form 3 $SU(2)$ doublets. Normally, one defines the specific flavors of quarks as eigenstates of the quark mass matrix, but this definition does not respect the doublet structure: the $SU(2)$ partner of say the u quark is not the d quark but rather some linear combination of the d, s, b quarks, and likewise for the partners of the c and t quarks. Thus, the $SU(2)$ doublets are

$$\begin{pmatrix} u \\ d' \end{pmatrix},\quad \begin{pmatrix} c \\ s' \end{pmatrix},\quad \begin{pmatrix} t \\ b' \end{pmatrix},\quad \text{for}\quad \begin{pmatrix} d' \\ s' \\ b' \end{pmatrix} \;=\; V \times \begin{pmatrix} d \\ s \\ b \end{pmatrix} \tag{43}$$

where V is a unitary 3×3 matrix called the *Cabibbo–Kobayashi–Maskawa* matrix (CKM). In this section, I shall first explain where this matrix comes from, and then I'll tell you its physical consequences for the weak interactions.

In the un-broken $SU(2)\times U(1)$ theory the quarks are massless and we cannot tell which quark is u, which is c, *etc.*, *etc.*; we cannot even tell which left-handed Weyl field pairs up with which right-handed Weyl field into a Dirac spinor. We can use the $SU(2)$ symmetry to form doublets, but we are free to choose any basis we like for the 3 doublets — let's call them Q_α for $\alpha = 1,2,3$ — and we are free to change this basis by a unitary field re-definition,

$$\psi_L^i(Q_\alpha) \;\to\; \psi_L^i(Q'_\alpha) \;=\; \sum_\beta (\mathcal{U}^Q)_{\alpha,\beta} \times \psi_L^i(Q_\beta), \tag{44}$$

where $\mathcal{U}^Q$ is a unitary 3×3 matrix. Similarly, we may use any basis D_α for the 3 right-handed quarks of charge $-\frac{1}{3}$, any basis U_α for the 3 right-handed quarks of charge $+\frac{2}{3}$, and we are free to change these two bases by unitary transforms

$$\psi_R(U_\alpha) \;\to\; \psi_R(U'_\alpha) \;=\; \sum_\beta (\mathcal{U}^U)_{\alpha,\beta}\times\psi_R(U_\beta), \quad \psi_R(D_\alpha) \;\to\; \psi_R(D'_\alpha) \;=\; \sum_\beta (\mathcal{U}^D)_{\alpha,\beta}\times\psi_R(D_\beta), \tag{45}$$

where $\mathcal{U}^U$ and $\mathcal{U}^D$ are two independent unitary 3×3 matrices. However, we cannot mix the U_α with the D_α because of their different $U(1)$ hypercharges.

Likewise, we are free to use any basis L_α for the 3 doublets of left-handed leptons, any basis E_α for the 3 right-handed charged leptons, and we are free to changes all these bases by unitary transforms,

$$\psi_L^i(L_\alpha) \;\to\; \psi_L^i(L'_\alpha) \;=\; \sum_\beta (\mathcal{U}^L)_{\alpha,\beta}\times\psi_L^i(L_\beta), \quad \psi_R(E_\alpha) \;\to\; \psi_R(E'_\alpha) \;=\; \sum_\beta (\mathcal{U}^E)_{\alpha,\beta}\times\psi_R(E_\beta). \tag{46}$$

(I'll take care of the neutrinos in my notes on neutrino masses.)

The Yukawa couplings involve one Higgs field H^i or H^*_i and two fermion fields, — one left-handed, one right-handed — and for each choice of their $SU(2)\times U(1)$ quantum numbers, there are three ψ_L fields and three ψ_R fields. Consequently, there is a big lot of the Yukawa terms in the Lagrangian, namely

$$\begin{aligned}\mathcal{L}_{\text{Yukawa}} \;=\; &-\sum_{\alpha,\beta} Y^U_{\alpha\beta} \times \psi_R^\dagger(U_\alpha)\psi_L^i(Q_\beta) \times \epsilon_{ij}H^j \;-\; \sum_{\alpha,\beta} Y^D_{\alpha\beta} \times \psi_R^\dagger(D_\alpha)\psi_L^i(Q_\beta) \times H^*_i \\ &-\sum_{\alpha,\beta} Y^E_{\alpha\beta} \times \psi_R^\dagger(E_\alpha)\psi_L^i(L_\beta) \times H^*_i \;+\; \text{Hermitian conjugates,}\end{aligned} \tag{47}$$

where the $Y^U_{\alpha,\beta}$, the $Y^D_{\alpha,\beta}$, and the $Y^E_{\alpha,\beta}$ comprise three 3×3 complex matrices of the Yukawa coupling constants. And when the Higgs develops symmetry-breaking VEV, these matrices of Yukawa couplings give rise to the complex 3×3 *mass matrices*

$$M^U_{\alpha,\beta} \;=\frac{v}{\sqrt{2}}\times Y^U_{\alpha,\beta}\,,\qquad M^U_{\alpha,\beta} \;=\frac{v}{\sqrt{2}}\times Y^D_{\alpha,\beta}\,,\qquad M^E_{\alpha,\beta} \;=\frac{v}{\sqrt{2}}\times Y^E_{\alpha,\beta}\,, \tag{48}$$

$$\mathcal{L}_{\text{mass}} \;=\; -\sum_{\alpha,\beta} M^U_{\alpha\beta}\times\psi^\dagger_R(U_\alpha)\psi^1_L(Q_\beta)\;-\sum_{\alpha,\beta} M^D_{\alpha\beta}\times\psi^\dagger_R(D_\alpha)\psi^2_L(Q_\beta) \tag{49}$$

$$-\sum_{\alpha,\beta} M^E_{\alpha\beta}\times\psi^\dagger_R(E_\alpha)\psi^2_L(L_\beta)\;+\;\text{Hermitian conjugates.} \tag{50}$$

To get the physical masses of quarks and leptons, we need to diagonalize these mass matrices via suitable unitary transforms (44)–(46). In matrix notations, these transforms lead to

$$\bigl(Y^U\bigr)' \;=\; \mathcal{U}^U\times Y^U\times\bigl(\mathcal{U}^Q\bigr)^\dagger,\quad \bigl(Y^D\bigr)' \;=\; \mathcal{U}^D\times Y^D\times\bigl(\mathcal{U}^Q\bigr)^\dagger,\quad \bigl(Y^E\bigr)' \;=\; \mathcal{U}^E\times Y^E\times\bigl(\mathcal{U}^L\bigr)^\dagger, \tag{51}$$

and consequently

$$\bigl(M^U\bigr)' \;=\; \mathcal{U}^U\times M^U\times\bigl(\mathcal{U}^Q\bigr)^\dagger,\quad \bigl(M^D\bigr)' \;=\; \mathcal{U}^D\times M^D\times\bigl(\mathcal{U}^Q\bigr)^\dagger,\quad \bigl(M^E\bigr)' \;=\; \mathcal{U}^E\times M^E\times\bigl(\mathcal{U}^L\bigr)^\dagger. \tag{52}$$

Now, any complex matrix M can be written as a product $M = W_1 D W_2$ where W_1 and W_2 are unitary matrices while D is diagonal, real, and non-negative.⋆ Consequently, using appropriate unitary matrices $\mathcal{U}^E$ and $\mathcal{U}^Q$ we can make the charged leptons' mass matrix diagonal and real

$$M^E \;\to\; \bigl(M^E\bigr)' \;=\; \mathcal{U}^E\times M^E\times\bigl(\mathcal{U}^L\bigr)^\dagger \;=\; \begin{pmatrix} m_e & 0 & 0\\ 0 & m_\mu & 0\\ 0 & 0 & m_\tau\end{pmatrix}. \tag{53}$$

Note that it is in the transformed bases — where the $(M^E)'$ is diagonal — that the LH and

⋆ To prove, start with a polar decomposition $M = UH$ where U is unitary and $H = \sqrt{M^\dagger M}$ is hermitian and positive semi-definite. Then diagonalize the hermitian matrix H, *i.e.*, write it as $H = W^\dagger D W$ for some unitary matrix W. Consequently, $M = UW^\dagger D W = W_1 D W_2$ for $W_2 = W$ and $W_1 = UW^\dagger$.

RH Weyl fields combine into Dirac fields of the physical electron, muon, and the tau,

$$\begin{aligned}
\Psi_e &= \begin{pmatrix} \psi^2_L(L'_1) = \mathcal{U}^L_{1\beta}\psi^2_L(L_\beta) \\ \psi_R(E'_1) = \mathcal{U}^E_{1\beta}\psi_R(E_\beta) \end{pmatrix}, \\
\Psi_\mu &= \begin{pmatrix} \psi^2_L(L'_2) = \mathcal{U}^L_{2\beta}\psi^2_L(L_\beta) \\ \psi_R(E'_2) = \mathcal{U}^E_{2\beta}\psi_R(E_\beta) \end{pmatrix}, \\
\Psi_\tau &= \begin{pmatrix} \psi^2_L(L'_3) = \mathcal{U}^L_{3\beta}\psi^2_L(L_\beta) \\ \psi_R(E'_3) = \mathcal{U}^E_{3\beta}\psi_R(E_\beta) \end{pmatrix}.
\end{aligned} \tag{54}$$

Likewise, using the $\mathcal{U}^U$ and the $\mathcal{U}^Q$ unitary matrices we may diagonalize the mass matrix for the charge $+\frac{2}{3}$ quarks,

$$\begin{aligned}
M^U \ \to\ (M^U)' &= \mathcal{U}^U \times M^U \times (\mathcal{U}^Q)^\dagger = \begin{pmatrix} m_u & 0 & 0 \\ 0 & m_c & 0 \\ 0 & 0 & m_t \end{pmatrix}, \\
\Psi_u &= \begin{pmatrix} \psi^1_L(Q'_1) = \mathcal{U}^Q_{1\beta}\psi^1_L(Q_\beta) \\ \psi_R(U'_1) = \mathcal{U}^U_{1\beta}\psi_R(U_\beta) \end{pmatrix}, \\
\Psi_c &= \begin{pmatrix} \psi^1_L(Q'_2) = \mathcal{U}^Q_{2\beta}\psi^1_L(Q_\beta) \\ \psi_R(U'_2) = \mathcal{U}^U_{2\beta}\psi_R(U_\beta) \end{pmatrix}, \\
\Psi_t &= \begin{pmatrix} \psi^1_L(Q'_3) = \mathcal{U}^Q_{3\beta}\psi^1_L(Q_\beta) \\ \psi_R(U'_3) = \mathcal{U}^U_{3\beta}\psi_R(U_\beta) \end{pmatrix},
\end{aligned} \tag{55}$$

and similarly for the charge $-\frac{1}{3}$ quarks,

$$\begin{aligned}
M^D \ \to\ (M^D)' &= \mathcal{U}^D \times M^D \times (\widetilde{\mathcal{U}}^Q)^\dagger = \begin{pmatrix} m_d & 0 & 0 \\ 0 & m_s & 0 \\ 0 & 0 & m_b \end{pmatrix}, \\
\Psi_d &= \begin{pmatrix} \psi^2_L(Q'_1) = \widetilde{\mathcal{U}}^Q_{1\beta}\psi^2_L(Q_\beta) \\ \psi_R(U'_1) = \mathcal{U}^D_{1\beta}\psi_R(D_\beta) \end{pmatrix}, \\
\Psi_s &= \begin{pmatrix} \psi^2_L(Q'_2) = \widetilde{\mathcal{U}}^Q_{2\beta}\psi^1_L(Q_\beta) \\ \psi_R(U'_2) = \mathcal{U}^D_{2\beta}\psi_R(D_\beta) \end{pmatrix}, \\
\Psi_b &= \begin{pmatrix} \psi^2_L(Q'_3) = \widetilde{\mathcal{U}}^Q_{3\beta}\psi^1_L(Q_\beta) \\ \psi_R(U'_3) = \mathcal{U}^D_{3\beta}\psi_R(D_\beta) \end{pmatrix},
\end{aligned} \tag{56}$$

However, *it takes different unitary matrices $\mathcal{U}^Q \neq \widetilde{\mathcal{U}}^Q$ to diagonalize the up-type and down-type quark mass matrices, and that's what messes up the $SU(2)$ doublet structure!* Indeed, in terms of the upper components $\psi^1_L(Q_\alpha)$ of the original doublets, the left-handed u, c, t quarks of definite mass are linear combinations

$$\begin{pmatrix} u_L \\ c_L \\ t_L \end{pmatrix} = \mathcal{U}^Q \times \begin{pmatrix} \psi^1_L(Q_1) \\ \psi^1_L(Q_2) \\ \psi^1_L(Q_3) \end{pmatrix}, \tag{57}$$

so their $SU(2)$ partners are *similar* linear combinations of the lower components $\psi^2_L(Q_\alpha)$ of the original doublets,

$$\begin{pmatrix} d'_L \\ s'_L \\ b'_L \end{pmatrix} = \mathcal{U}^Q \times \begin{pmatrix} \psi^2_L(Q_1) \\ \psi^2_L(Q_2) \\ \psi^2_L(Q_3) \end{pmatrix}, \tag{58}$$

for the same $\mathcal{U}^Q$ matrix as the up-type quarks. On the other hand, the d, s, b quarks defined as mass eigenstates obtain from different linear combinations

$$\begin{pmatrix} d_L \\ s_L \\ b_L \end{pmatrix} = \widetilde{\mathcal{U}}^Q \times \begin{pmatrix} \psi^2_L(Q_1) \\ \psi^2_L(Q_2) \\ \psi^2_L(Q_3) \end{pmatrix}. \tag{59}$$

Comparing the sets of down-type quark fields, we immediately see that

$$\begin{pmatrix} d'_L \\ s'_L \\ b'_L \end{pmatrix} = \mathcal{U}^Q \times \widetilde{\mathcal{U}}^{Q\dagger} \times \begin{pmatrix} d_L \\ s_L \\ b_L \end{pmatrix}, \tag{60}$$

which gives us the Cabibbo–Kobayashi–Maskawa matrix

$$V_{\mathrm{CKM}} = \mathcal{U}^Q \times \widetilde{\mathcal{U}}^{Q\dagger}. \tag{61}$$

Now let's go back to the charged weak currents $J^{\pm\mu}$. Since they are gauge currents of the $SU(2)_W$, they connect a fermion in some $SU(2)$ doublet into the other fermion in exactly same doublet! Thus, the J^+ current would turn the u quark into its partner d', or the c quark into its partner s', *etc.*, and vice verse for the J^- current. In terms of the Dirac spinor fields, this means

$$\begin{aligned}
J^{-\mu}(\text{quarks}) &= \overline{\Psi}^u\gamma^\mu\frac{1-\gamma^5}{2}\Psi^{d'} + \overline{\Psi}^c\gamma^\mu\frac{1-\gamma^5}{2}\Psi^{s'} + \overline{\Psi}^t\gamma^\mu\frac{1-\gamma^5}{2}\Psi^{b'}\\
&= \sum_{\alpha=u,c,t}\sum_{\beta=d,s,b} V_{\alpha,\beta}\times\overline{\Psi}^\alpha\gamma^\mu\frac{1-\gamma^5}{2}\Psi^\beta,\\
J^{+\mu}(\text{quarks}) &= \overline{\Psi}^{d'}\gamma^\mu\frac{1-\gamma^5}{2}\Psi^{u} + \overline{\Psi}^{s'}\gamma^\mu\frac{1-\gamma^5}{2}\Psi^{c} + \overline{\Psi}^{b'}\gamma^\mu\frac{1-\gamma^5}{2}\Psi^{t}\\
&= \sum_{\alpha=u,c,t}\sum_{\beta=d,s,b} V^*_{\alpha,\beta}\times\overline{\Psi}^\beta\gamma^\mu\frac{1-\gamma^5}{2}\Psi^\alpha,
\end{aligned}\tag{62}$$

where $V_{\alpha,\beta}$ is the CKM matrix.

The CKM matrix is very important for the physics of weak interactions. For example, without this matrix the strange particles like the K-mesons or Λ-baryons would be stable because the s quark would not be able to decay. Indeed, the $SU(2)$ partner of the s quark is the c quark, so without the CKM matrix the only flavor-changing processes involving the s quark would be $s \leftrightarrow c$. However, the c quark is much heavier than s, so the decay can only go from c to s but not from s to c. But thanks to the CKM matrix — specifically, to the non-zero matrix element $V_{u,s}$ — the s quark may also decay to the u quark (which is lighter than s), albeit with a reduced amplitude $\propto V_{u,s} \approx 0.22$.

There are many other interesting flavor-changing weak processes involving the charged currents and the CKM matrix. I wish I could spend a few weeks talking about them, but alas I do not have the time for this in my QFT class. I hope professor Çan Kiliç would explain the subject in some detail in his Phenomenology class, whenever he gets to teach it next time. But in these notes, I have to move on to the next subject.

Eq. (62) gives the charged weak currents of the quarks, but what about the leptons? Again, we need to pick the bases for the 3 charged leptons and for the neutrinos, and if the two bases disagree with the $SU(2)$ doublet structure, we would get a CKM-like matrix for the leptons. For the charged leptons, the mass is important, so people always use the basis of mass eigenstates (e, μ, τ) as in eqs. (53) and (54). But the neutrino masses are so small, they only matter in long-baseline interferometry experiments, so for all other purposes people use the interaction basis $(\nu_e, \nu_\mu, \nu_\tau)$ of species *defined as the $SU(2)$ partners of the corresponding charged leptons* (e, μ, τ). In this basis, there are no CKM-like matrices and

$$\begin{aligned} J^{+\mu}(\text{leptons}) &= \overline{\Psi}^e \gamma^\mu \frac{1-\gamma^5}{2} \Psi^{\nu_e} + \overline{\Psi}^\mu \gamma^\mu \frac{1-\gamma^5}{2} \Psi^{\nu_\mu} + \overline{\Psi}^\tau \gamma^\mu \frac{1-\gamma^5}{2} \Psi^{\nu_\tau}, \\ J^{-\mu}(\text{leptons}) &= \overline{\Psi}^{\nu_e} \gamma^\mu \frac{1-\gamma^5}{2} \Psi^{e} + \overline{\Psi}^{\nu_\mu} \gamma^\mu \frac{1-\gamma^5}{2} \Psi^{\mu} + \overline{\Psi}^{\nu_\tau} \gamma^\mu \frac{1-\gamma^5}{2} \Psi^{\tau}. \end{aligned} \tag{63}$$

On the other hand, in this basis the neutrino mass matrix is non-diagonal, which makes neutrinos slowly oscillate from one species to another. I shall discuss this issue in separate set of notes.

Now consider the neutral weak current J_Z^μ. The unitary transforms that diagonalize the fermion's mass matrices can only mix up fields with similar chiralities (ψ_L and ψ_R) and electric charges. In the Standard Model, this limits the mixing to fermions that have both similar weak isospins T^3 and similar hypercharges Y, which makes for similar contributions to the neutral weak current

$$J_Z^\mu \supset \sum_\alpha^{\text{LH species}} (T^3 - \sin^2\theta Q_{\text{el}}) \psi_L^\dagger(\alpha) \bar{\sigma}^\mu \psi_L(\alpha) \quad \text{or} \quad \sum_\alpha^{\text{RH species}} (T^3 - \sin^2\theta Q_{\text{el}}) \psi_R^\dagger(\alpha) \sigma^\mu \psi_R(\alpha). \tag{64}$$

The sums here are invariant under all unitary field redefinitions that mix only fields with similar $T^3 - \sin^2 Q_{\text{el}}$, so regardless of such redefinitions the neutral weak current remains diagonal. Specifically,

$$\begin{aligned} J_Z^\mu &= J_{T3}^\mu[\text{left-handed}] - \sin^2\theta \times J_{\text{EM}}^\mu[\text{left-right symmetric}] \\ &= \sum_{q=u,c,t}^{\text{quarks}} \overline{\Psi}^q \gamma^\mu \left(+ \frac{1-\gamma^5}{4} - \frac{2}{3}\sin^2\theta \right) \Psi^q + \sum_{q=d,s,b}^{\text{quarks}} \overline{\Psi}^q \gamma^\mu \left(- \frac{1-\gamma^5}{4} + \frac{1}{3}\sin^2\theta \right) \Psi^q \\ &\quad + \sum_{\ell=e,\mu,\tau}^{\text{leptons}} \overline{\Psi}^\ell \gamma^\mu \left(- \frac{1-\gamma^5}{4} + \sin^2\theta \right) \Psi^\ell + \sum_{\nu=\nu_e,\nu_\mu,\nu_\tau}^{\text{neutrinos}} \overline{\Psi}^\nu \gamma^\mu \left(+ \frac{1-\gamma^5}{4} - 0 \right) \Psi^\nu , \end{aligned} \tag{42}$$

and *there are no flavor-changing neutral weak currents in the Standard Model.*

Note that this is a peculiar property of the Standard Model where all fermions of the same electric charge and chirality also have the same T^3. Historically, before the Standard

Model was fully developed and confirmed experimentally, people used to consider models with different quantum numbers for different quarks. In particular, back in the 1960's when only 3 quark flavors u, d, s were known, people assumed the left-handed s quark was un-paired $SU(2)$ singlet (with $Y = -\frac{1}{3}$ to give it the right electric charge). The mass matrix somehow mixed the two charge $-\frac{1}{3}$ quarks d and s, so the $SU(2)$ doublet was $(u, d')_L$ while the singlet was s'_L, where

$$d' = d\times\cos\theta_c + s\times\sin\theta_c\,, \qquad s' = s\times\cos\theta_c - d\times\sin\theta_c\,, \qquad \theta_c = \theta_{\text{Cabibbo}} \approx 13^\circ. \quad (65)$$

In such a model, the s'_L and the d'_L have different $T^3 - \sin^2\theta Q_{\text{el}}$, so their mixing makes for off-diagonal terms in the J^μ_Z:

$$\begin{aligned} J^\mu_Z[\text{quarks}] &= \overline{\Psi}^u\gamma^\mu\left(\frac{1-\gamma^5}{4} - \frac{2}{3}\sin^2\theta\right)\Psi^u + \overline{\Psi}^d\gamma^\mu\left(\cos^2\theta_c\frac{1-\gamma^5}{4} + \frac{1}{3}\sin^2\theta\right)\Psi^d \\ &\quad + \overline{\Psi}^s\gamma^\mu\left(\sin^2\theta_c\frac{1-\gamma^5}{4} + \frac{1}{3}\sin^2\theta\right)\Psi^s \\ &\quad + \cos\theta_c\sin\theta_c\times\left(\overline{\Psi}^d\gamma^\mu\frac{1-\gamma^5}{4}\Psi^s + \overline{\Psi}^s\gamma^\mu\frac{1-\gamma^5}{4}\Psi^d\right). \end{aligned} \quad (66)$$

Physically, the off-diagonal terms on the bottom line here mean the $s \leftrightarrow d$ flavor changing neutral current, which would lead to processes like the $K^0 \to \mu^+\mu^-$ decay. But experimentally, there are no such decays, nor any other signatures of the flavor-changing neutral currents. This made Glashow, Illiopoulos, and Maiani conjecture in 1970 that the s quark (or rather the s') should be a member of a doublet just like the d' quark — which would give them the same T^3 and hence keep the neutral weak current flavor-diagonal — and consequently there must be a fourth quark flavor c to form the (c, s') doublet. And in 1974 this fourth flavor (called the 'charm') was experimentally discovered at SLAC and BNL.

Later, when the fifth flavor b was discovered in 1977, most physicists expected it to also be a part of the doublet, so everybody was looking for the sixth flavor t. This expectation turned out to be correct, and the t quark was duly discovered in 1995. The delay was due to the very large mass of the top quark, $m_t \approx 173$ GeV, much heavier that the other 5 flavors.

CP violation

Like any chiral gauge theory, the weak interactions do not have the parity symmetry **P** or the charge conjugation symmetry **C**. In particular, the charged currents involve only the left-chirality Weyl spinors, which in particle terms mean left-helicity quarks and leptons but right-helicity anti-quarks or anti-leptons.

However, the chirality is perfectly consistent with the combined **CP** symmetry, which does not mix the ψ_L and the ψ_R fields; instead it acts as

$$\mathbf{CP}: \quad \psi_L(\mathbf{x},t) \ \to \pm\sigma_2\psi_L^*(-\mathbf{x},t), \quad \psi_R(\mathbf{x},t) \ \to \pm\sigma_2\psi_R^*(-\mathbf{x},t). \tag{67}$$

By the CPT theorem, the **CP** symmetry is equivalent to the time-reversal (or rather motion-reversal) symmetry **T**, so it would be nice to have it as an exact symmetry of Nature. But in 1964, Cronin and Fitch have discovered that weak decays of the neutral K-mesons are only approximately CP-symmetric, but sometimes a CP-odd state of the kaon decays into a CP-even pair of pions. Later experiments found CP violations in other weak processes involving mesons containing b quarks or c quarks.

All the experimentally measured CP-violating effects can be explained by the imaginary parts $\mathrm{Im}(V_{\alpha,\beta})$ of the CKM matrix elements. In General, the relation between the CP violation and the CKM matrix is rather complicated and involves interference between different orders of perturbations theory; at the tree level, there is no CP violation. I am not going to work out such complicated issues in these notes; instead, let me simply show that complex CKM matrix violates the CP symmetry of the electroweak Lagrangian.

Since the neutral weak current does not care about the CKM matrix, let me focus on the charged currents. Under **CP**, the charged vector fields $W_\mu^\pm(x)$ transform as

$$\mathbf{CP}: \quad W_0^\pm(\mathbf{x},t) \ \to \ -W_0^\mp(-\mathbf{x},+t), \quad W_i^\pm(\mathbf{x},t) \ \to \ +W_i^\mp(-\mathbf{x},+t), \tag{68}$$

where the exchange $W^+ \leftrightarrow W^-$ is due to charge conjugation while different signs for 3-scalar and 3-vector components are due to reflection $\mathbf{x} \to -\mathbf{x}$ of the space coordinates. Consequently, in a **CP** symmetric theory we would need a similar relation for the charged currents,

$$\mathbf{CP}:\quad J^{0\pm}(\mathbf{x},t)\ \to\ -J^{0\pm}(-\mathbf{x},+t),\quad J^{i\pm}(\mathbf{x},t)\ \to\ +J^{i\mp}(-\mathbf{x},+t), \tag{69}$$

In terms of fermions, the charged weak currents are sums of left-handed currents terms of general form

$$j^\mu_L\ =\ \psi_L^{1\dagger}\bar\sigma^\mu\psi_L^2\ =\ \overline{\Psi}^1\gamma^\mu\frac{1-\gamma^5}{2}\Psi^2\,, \tag{70}$$

so let's work out how such terms transform under **CP**. Assuming the Weyl fermions ψ^1_L and ψ^2_L have the similar intrinsic CP signs as members of the same $SU(2)$ doublet, we have

$$\begin{aligned}\mathbf{CP}:\ \psi_L^{1\dagger}\bar\sigma^\mu\psi_L^2\ &=\ +(\psi^1_L)^\top\sigma_2\times\bar\sigma^\mu\times\sigma_2(\psi^2_L)^*\\ &=\ +(\psi^1_L)^\top\times\left(\sigma_2\bar\sigma^\mu\sigma_2\ =\ (\sigma^\mu)^\top\right)\times(\psi^2_L)^*\\ &=\ -\psi_L^{2\dagger}\sigma^\mu\psi^1_L\\ &=\ \psi_L^{2\dagger}\bar\sigma^\mu\psi^1_L\times\begin{cases}+1 & \text{for } \mu=1,2,3,\\ -1 & \text{for } \mu=0.\end{cases}\end{aligned} \tag{71}$$

The μ dependence of the overall sign here — which comes from comparing $-\sigma^\mu$ to $+\bar\sigma^\mu$ — is in perfect agreement with eq. (69). In Dirac notations, eq (71) amounts to

$$\mathbf{CP}:\ \overline{\Psi}^1\gamma^\mu\frac{1-\gamma^5}{2}\Psi^2\ \to\ \overline{\Psi}^2\gamma^\mu\frac{1-\gamma^5}{2}\Psi^1\times\begin{cases}+1 & \text{for } \mu=1,2,3,\\ -1 & \text{for } \mu=0.\end{cases} \tag{72}$$

Besides the μ–dependent sign, the **CP** exchanges the two fermionic species $\Psi^1\leftrightarrow\Psi^2$ involved in the current j^μ_L. For the leptonic charged weak currents (63), this exchange leads to $J^{+\mu}\leftrightarrow J^{-\mu}$, exactly as in eq. (69); indeed,

$$J^{+\mu}\ \supset\ \overline{\Psi}^e\gamma^\mu\frac{1-\gamma^5}{2}\Psi^{\nu_e}\quad\text{while}\quad J^{-\mu}\ \supset\ \overline{\Psi}^{\nu_e}\gamma^\mu\frac{1-\gamma^5}{2}\Psi^e,\quad \textit{etc., etc.} \tag{73}$$

Consequently, the interactions

$$\mathcal{L}\ \supset =\ -\frac{g_2}{\sqrt{2}}\times\left(W^+_\mu J^{\mu-}_{\text{leptonic}}\,+\,W^-_\mu J^{\mu+}_{\text{leptonic}}\right) \tag{74}$$

of the leptons with the vector fields $W^\pm_\mu$ are invariant under **CP**.

But for the charged currents of the quarks, we have

$$\begin{aligned} J^{-\mu}(\text{quarks}) &= \sum_{\alpha=u,c,t}\ \sum_{\beta=d,s,b} V_{\alpha,\beta}\times\overline{\Psi}^{\alpha}\gamma^{\mu}\frac{1-\gamma^5}{2}\Psi^{\beta}, \\ J^{+\mu}(\text{quarks}) &= \sum_{\alpha=u,c,t}\ \sum_{\beta=d,s,b} V^{*}_{\alpha,\beta}\times\overline{\Psi}^{\beta}\gamma^{\mu}\frac{1-\gamma^5}{2}\Psi^{\alpha}, \end{aligned} \tag{62}$$

which transform into

$$\begin{aligned} &\mathbf{CP}:\ J^{-\mu}(\text{quarks}) \to \pm(\mu)\times\sum_{\beta=d,s,b} V_{\alpha,\beta}\times\overline{\Psi}^{\beta}\gamma^{\mu}\frac{1-\gamma^5}{2}\Psi^{\alpha}, \\ &\qquad \text{which is almost like}\quad \pm(\mu)\times J^{+\mu}(\text{quarks}),\ \text{except for } V_{\alpha,\beta} \text{ instead of } V^{*}_{\alpha,\beta}; \\ &\mathbf{CP}:\ J^{+\mu}(\text{quarks}) \to \pm(\mu)\times\sum_{\beta=d,s,b} V^{*}_{\alpha,\beta}\times\overline{\Psi}^{\alpha}\gamma^{\mu}\frac{1-\gamma^5}{2}\Psi^{\beta}, \\ &\qquad \text{which is almost like}\quad \pm(\mu)\times J^{-\mu}(\text{quarks}),\ \text{except for } V^{*}_{\alpha,\beta} \text{ instead of } V_{\alpha,\beta}; \end{aligned} \tag{75}$$

Consequently, the net effect of **CP** on the interactions

$$\mathcal{L} \supset = -\frac{g_2}{\sqrt{2}}\times\left(W^{+}_{\mu}J^{\mu-}_{\text{quark}} + W^{-}_{\mu}J^{\mu+}_{\text{quark}}\right) \tag{76}$$

of the $W^{\pm}_{\mu}$ with the quarks is equivalent to complex conjugating the CKM matrix,

$$\mathbf{CP}:\quad V_{\alpha,\beta} \leftrightarrow V^{*}_{\alpha,\beta}\,. \tag{77}$$

Thus, the weak interactions of quarks (and hence hadrons) are **CP** symmetric if and only if the CKM matrix is real.

Caveat: The specific action of the **CP** symmetry can be modified by changing the phase conventions of the particle and antiparticle states and the corresponding fields. For example, if we change the phase of a Dirac spinor field

$$\Psi(x) \to \Psi'(x) = e^{i\theta}\Psi(x), \tag{78}$$

then the **CP** action on that field

$$\mathbf{CP}:\Psi(\mathbf{x},t)\ \rightarrow\ \gamma^0\gamma^2\Psi(-\mathbf{x},t) \tag{79}$$

becomes

$$\mathbf{CP}:\Psi'(\mathbf{x},t)\ \rightarrow\ \gamma^0\gamma^2\Psi'(-\mathbf{x},t), \tag{80}$$

which in terms of the original $\Psi(x)$ field becomes

$$\mathbf{CP}:\Psi(\mathbf{x},t)\ \rightarrow\ e^{-2i\theta}\gamma^0\gamma^2\Psi^*(-\mathbf{x},t), \tag{81}$$

with an extra phase factor $e^{-2i\theta}$.

In the context of quarks in the GWS theory, redefinitions of the quark fields must keep the quark mass matrices M^U and M^D real and diagonal. Thus, we must preserve the pairings of the LH and RH Weyl spinors into Dirac spinors, but we may multiply each such Dirac spinor by a separate phase factor:

$$\Psi^u\to e^{i\theta_u}\Psi^u,\quad \Psi^c\to e^{i\theta_c}\Psi^c,\quad \Psi^t\to e^{i\theta_t}\Psi^t;\quad \Psi^d\to e^{i\theta_d}\Psi^d,\quad \Psi^s\to e^{i\theta_s}\Psi^s,\quad \Psi^b\to e^{i\theta_b}\Psi^b. \tag{82}$$

Consequently, the matrix elements of the Cabibbo–Kobayashi–Maskawa matrix also change their phases according to

$$V_{\alpha,\beta}\ \rightarrow\ e^{i\theta_\alpha-i\theta_\beta}\times V_{\alpha,\beta}\,. \tag{83}$$

At the same time, the CP symmetry is also redefined to accommodate the new phases of the quark fields. In fact, this redefinition completely parallels the redefined CKM matrix so that in the new basis it is equivalent to complex conjugation of the new CKM matrix,

$$\mathbf{CP}:\ V_{\alpha,\beta}\ \leftrightarrow\ V^*_{\alpha,\beta}\,. \tag{84}$$

Therefore, the weak interactions are invariant under *some kind of a* **CP** *symmetry* if and only if the CKM matrix can be made real by a rephasing (82) of the quark flavors.

For two quark doublets (u,d) and (c,s) — but no (t,b) — this is automatically true. Indeed, for two doublets the CKM matrix is a 2×2 unitary matrix, which may be parametrized by 1 real angle (the Cabibbo angle) and 3 complex phases, for example

$$V = \begin{pmatrix} e^{i(a+b+c)}\cos\theta_c & e^{i(a+b)}\sin\theta_c \\ -e^{i(a+c)}\sin\theta_c & e^{i(a)}\cos\theta_c \end{pmatrix}. \tag{85}$$

At the same time, there 4 quark flavors whose phases we can change, but since only the differences between quark phases affect the CKM matrix, so we may adjust $4-1=3$ of its complex phases. In particular, we may set $a=b=c=0$ in eq. (85) and get a real matrix

$$V = \begin{pmatrix} \cos\theta_c & \sin\theta_c \\ -\sin\theta_c & \cos\theta_c \end{pmatrix}. \tag{86}$$

Thus, for just 2 quark doublets (u,d) and (c,s), the weak interactions preserve CP!

But for three quark doublets (u,d), (c,s), and (t,b), the parameter counting is quite different: The CKM matrix V is a 3×3 unitary matrix, which may be parametrized by 3 real angles — similar to a real orthogonal $SO(3)$ matrix — and 6 complex phases. At the same time, we may change the phases of 6 quark flavors, but since only the 5 independent differences between such phases affect the CKM matrix, we may eliminate 5 complex phases of the V. This leaves us with $6-5=1$ complex phase we cannot eliminate, and it is this one remaining phase which breaks the CP symmetry of the weak interactions!

Back in 1973, only 4 quark flavors u,d,s,c were known — in fact, even the charm quark was predicted but not yet discovered experimentally — and the origin of the weak CP violation was a complete mystery (although there were many far-out speculations). At that time, Kobayashi and Maskawa speculated that maybe there is a third quark doublet (t,b) similar to the first two; in this case, the flavor mixing matrix would be 3×3 instead of 2×2, so one of its complex phases could not be eliminated by field redefinition, and that would be a source of CP violation. Their speculation turned out to be correct, and in 2008 Kobayashi and Maskawa got their Nobel prizes.

In the original Kobayashi–Maskawa convention, the quark phases were chosen such that the $V_{\alpha,\beta}$ matrix elements involving the lightest u and d quarks were all real, thus

$$V = \begin{pmatrix} \text{real} & \text{real} & \text{real} \\ \text{real} & \text{complex} & \text{complex} \\ \text{real} & \text{complex} & \text{complex} \end{pmatrix}. \tag{87}$$

Nowdays, the *standard convention* is to associate the CP-violating phase with the smalles mixing angle between the first and the third generations, thus

$$V = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_{23} & \sin\theta_{23} \\ 0 & -\sin\theta_{23} & \cos\theta_{23} \end{pmatrix} \times \begin{pmatrix} \cos\theta_{13} & 0 & \sin\theta_{13}e^{-\delta_{13}} \\ 0 & 1 & 0 \\ -\sin\theta_{13}e^{+\delta_{13}} & 0 & \cos\theta_{13} \end{pmatrix} \times \begin{pmatrix} \cos\theta_{12} & \sin\theta_{12} & 0 \\ -\sin\theta_{12} & \cos\theta_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{88}$$

Experimentally, $\theta_{12} = 12.975^\circ \pm 0.025^\circ$ (the Cabibbo angle), $\theta_{23} = 2.42^\circ \pm 0.04^\circ$, $\theta_{13} = 0.207^\circ \pm 0.006^\circ$, and $\delta_{13} = 72.5^\circ \pm 2.7^\circ$, see the particle data book, chapter 12 for details. Note that the CP violating phase δ_{13} itself is not small, but its effect is small due to the smallness of the θ_{13} mixing angle.

In a basis-independent way, a useful parameter of CP violation is the Jarlskog invariant J defined as

$$\text{for } \alpha,\beta = u,c,t \text{ but } \alpha\neq\beta, \text{ and } i,j = d,s,b \text{ but } i\neq j: \quad \operatorname{Im}\bigl(V_{\alpha,i}V_{\beta,j}V^*_{\alpha,j}V^*_{\beta,i}\bigr) = \pm J. \tag{89}$$

In the standard convention

$$J = \cos\theta_{12}\sin\theta_{12} \times \cos\theta_{23}\sin\theta_{23} \times \cos^2\theta_{13}\sin\theta_{13} \times \sin(\delta_{13}), \tag{90}$$

and experimentally $J = (3.18 \pm 0.15) \times 10^{-5}$.

Historically, the CP violation was discovered in 1964 by James Cronin and Val Fitch in decays of the neutral K mesons. And then for almost 50 years, it was the only place CP violation could be seen.⋆ But eventually, in 2013 the LHCb experiment saw CP violation in the decays of B_s mesons (made from b quark and $\bar{s}$ antiquark), and this year (2019) they also saw it in the decays of D^0 mesons (made from c quark and $\bar{u}$ antiquark). Thus far, all the observed CP violating effects seem to be consistent with the Kobayashi–Maskawa mechanism and nothing but Kobayashi–Maskawa.

⋆ Please see 2009 lecture notes by prof. Mark Thomson at Cambridge University (pages 424–428) for a simple explanation of CP violation in the neutral kaon system.