## Gauge Dependence

In QED, the on-shell physical amplitudes do not depend on the gauge-fixing condition, but that's unfortunately not true for the off-shell amplitudes. Even the UV divergences - and hence the counterterms depend on the gauge-fixing conditions. In particular, in the Lorenz-invariant gauges where the photon propagator is

$$
\begin{equation*}
\text { ఋ心n }=\frac{-i}{k^{2}+i 0}\left[g^{\mu \nu}+(\xi-1) \frac{k^{\mu} k^{\nu}}{k^{2}+i 0}\right] \tag{1}
\end{equation*}
$$

the counterterms depend on the $\xi$ parameter. In these notes, we shall focus on the $\xi$ dependence of the $\delta_{1}$ and the $\delta_{2}$ counterterms.

Let's start with the one-loop $\delta_{1}$ counterterm which cancels the UV divergence of the vertex correction


Evaluating this diagram for the general $\xi$ gauge, we get

$$
\begin{align*}
i e \Gamma_{1 \text { loop }}^{\mu}\left(p^{\prime}, p\right)= & \int_{\text {reg }} \frac{d^{4} k}{(2 \pi)^{4}} i e \gamma_{\nu} \times \frac{i}{\not p^{\prime}+\not k-m+i 0} \times i e \gamma^{\mu} \times \frac{i}{\not p+\not k-m+i 0} \times i e \gamma_{\lambda} \times \\
& \times \frac{-i}{k^{2}+i 0}\left[g^{\lambda \nu}+(\xi-1) \frac{k^{\lambda} k^{\nu}}{k^{2}+i 0}\right] \\
= & e^{3} \int_{\text {reg }} \frac{d^{4} k}{(2 \pi)^{4}} \frac{1}{k^{2}+i 0} \times \gamma^{\nu} \frac{1}{\not p^{\prime}+\not k-m+i 0} \gamma^{\mu} \frac{1}{\not p+\not k-m+i 0} \gamma_{\nu}  \tag{3}\\
& +(\xi-1) e^{3} \int_{\text {reg }} \frac{d^{4} k}{(2 \pi)^{4}} \frac{1}{\left(k^{2}+i 0\right)^{2}} \times \not k \frac{1}{\not p^{\prime}+\not k-m+i 0} \gamma^{\mu} \frac{1}{\not p+\not k-m+i 0} \nless \\
= & i e \Gamma_{F}^{\mu}\left(p^{\prime}, p\right)+(\xi-1) \times i e \Delta \Gamma^{\mu}\left(p^{\prime}, p\right)
\end{align*}
$$

where $\Gamma_{F}^{\mu}$ stands for the $\Gamma_{1 \text { loop }}^{\mu}$ which obtains in the Feynman gauge $\xi=0-$ see my previous set
of notes for detail, - while

$$
\begin{equation*}
\Delta \Gamma^{\mu}\left(p^{\prime}, p\right)=-i e^{2} \int_{\text {reg }} \frac{d^{4} k}{(2 \pi)^{4}} \frac{1}{\left(k^{2}+i 0\right)^{2}} \times \not k \frac{1}{\not p^{\prime}+\not k-m+i 0} \gamma^{\mu} \frac{1}{\not p+\not k-m+i 0} \not k \tag{4}
\end{equation*}
$$

is the gauge-dependent correction. Fortunately, this correction drastically simplifies for the onshell electrons when $\Delta \Gamma^{\mu}$ appears in the context of $\bar{u}\left(p^{\prime}\right) \Delta \Gamma^{\mu} u(p)$. Indeed, in this context

$$
\frac{1}{\not p+\not k-m+i 0} \not k=1-\frac{1}{\not p+\not k-m+i 0}(\not p-m) \cong 1
$$

because $(p p-m) u(p)=0$, and likewise

$$
\not k \frac{1}{\not p^{\prime}+\not k-m+i 0}=1-\left(\not p^{\prime}-m\right) \frac{1}{p^{\prime}+\not \nless-m+i 0} \cong 1 .
$$

Consequently, eq. (4) simplifies to

$$
\begin{equation*}
\Delta \Gamma^{\mu}=e^{2} \gamma^{\mu} \times \int_{\text {reg }} \frac{d^{4} k}{(2 \pi)^{4}} \frac{-i}{\left(k^{2}+i 0\right)^{2}} \tag{5}
\end{equation*}
$$

which does not depend on any momenta, $p, p^{\prime}$, or $q$, but only on the UV and the IR regulators.
Finally, since the complete dressed vertex involves not only the loop diagram (2) but also the $\delta_{1}$ counterterm, we see that the gauge-dependent correction (5) can be completely canceled by the gauge-dependent correction to the $\delta_{1}$, namely

$$
\begin{equation*}
\delta_{1}(\xi)=\delta_{1}^{\text {Feynman gauge }}-(\xi-1) \times e^{2} \int_{\text {reg }} \frac{d^{4} k}{(2 \pi)^{4}} \frac{-i}{\left(k^{2}+i 0\right)^{2}} \tag{6}
\end{equation*}
$$

Now consider the $\delta_{2}$ counterterm. which together with the $\delta_{m}$ counterterm cancels the UV divergence of the electron's self energy

$$
\begin{equation*}
\Sigma_{\text {net }}(\not p)=\Sigma_{\text {loops }}(\not p)+\delta_{m}-\Delta_{2} \not p . \tag{7}
\end{equation*}
$$

At the one loop level,

which evaluates to

$$
\begin{align*}
-i \Sigma^{1 \text { loop }}(\not p)= & \int_{\text {reg }} \frac{d^{4} k}{(2 \pi)^{4}} i e \gamma_{\lambda} \frac{i}{\not k+\not p-m_{e}+i 0} \times i e \gamma_{\nu} \times \frac{-i}{k^{2}+i 0}\left[g^{\lambda \nu}+(\xi-1) \frac{k^{\lambda} k^{\nu}}{k^{2}+i 0}\right] \\
= & -e^{2} \int_{\text {reg }} \frac{d^{4} k}{(2 \pi)^{4}} \frac{1}{k^{2}+i 0} \times \gamma^{\nu} \frac{1}{\not k+\not p-m_{e}+i 0} \gamma_{\nu}  \tag{9}\\
& +(\xi-1) \times\left(-e^{2}\right) \int_{\text {reg }} \frac{d^{4} k}{(2 \pi)^{4}} \frac{1}{\left(k^{2}+i 0\right)^{2}} \times \not k \frac{1}{\not k+\not p-m_{e}+i 0} \nLeftarrow \\
= & -i \Sigma_{F}(\not p)-i(\xi-1) \times \Delta \Sigma(\not p),
\end{align*}
$$

where $\Sigma_{F}(\not \not p)$ is the $\Sigma_{1 \text { loop }}$ which obtains in the Feynman gauge - and which you should calculate in homework set\#17, - while

$$
\begin{equation*}
\Delta \Sigma(\not p)=-i e^{2} \int_{\text {reg }} \frac{d^{4} k}{(2 \pi)^{4}} \frac{1}{\left(k^{2}+i 0\right)^{2}} \times \not k \frac{1}{\not k+\not p-m_{e}+i 0} \not k \tag{10}
\end{equation*}
$$

is the gauge-dependent correction. To compensate for this correction, we should also correct the $\delta_{2}$ and $\delta_{m}$ counterterms to assure that $\Sigma_{\text {net }}$ and $d \Sigma_{\text {net }} / d \not p$ both vanish at $\not p=m$, hence

$$
\begin{equation*}
\delta_{2}=\delta_{2}^{\text {Feynman gauge }}+(\xi-1) \Delta \delta_{2}, \quad \delta_{m}=\delta_{m}^{\text {Feynman gauge }}+(\xi-1) \Delta \delta_{m} \tag{11}
\end{equation*}
$$

for

$$
\begin{equation*}
\Delta \delta_{2}=\left.\frac{d \Delta \Sigma}{d \not p}\right|_{p \equiv m} \quad \text { and } \quad \Delta \delta m-m \Delta \delta_{2}=-\Delta \Sigma(\not p=m) . \tag{12}
\end{equation*}
$$

Taking the derivative of $\Delta \Sigma$ from eq. (10), we get

$$
\begin{equation*}
\frac{d \Delta \Sigma}{d \not p}=e^{2} \int_{\text {reg }} \frac{d^{4} k}{(2 \pi)^{4}} \frac{-i}{\left(k^{2}+i 0\right)^{2}} \times \not \not / \frac{-1}{\left(\not k+\not p-m_{e}+i 0\right)^{2}} \not k, \tag{13}
\end{equation*}
$$

where

$$
\begin{align*}
\not k \frac{1}{\left(\not k+\not p-m_{e}+i 0\right)^{2}} \not k & =\left(1-(\not p-m) \frac{1}{\not p+\nvdash-m+i 0}\right) \times\left(1-\frac{1}{\not p+\not k-m+i 0}(\not p-m)\right) \\
& \rightarrow 1 \text { for } \not p=m . \tag{14}
\end{align*}
$$

Consequently,

$$
\begin{equation*}
\Delta \delta_{2}=\left.\frac{d \Delta \Sigma}{d \not p}\right|_{p=m}=-e^{2} \int_{\text {reg }} \frac{d^{4} k}{(2 \pi)^{4}} \frac{-i}{\left(k^{2}+i 0\right)^{2}} \tag{15}
\end{equation*}
$$

As to the $\Delta \delta_{m}$ corrections to the mass counterterm, I leave its calculation to your next homework.
Finally, comparing eqs. (6) and (15), we see that the gauge0dependent corrections to the $\delta_{1}$ and $\delta_{2}$ counterterms are exactly the same,

$$
\begin{equation*}
\Delta \delta_{1}=\Delta \delta_{2}=-e^{2} \int_{\text {reg }} \frac{d^{4} k}{(2 \pi)^{4}} \frac{-i}{\left(k^{2}+i 0\right)^{2}} \tag{16}
\end{equation*}
$$

Therefore, once we verify the Ward identity $\delta_{1}=\delta_{2}$ in the Feynman gauge, it follows that

$$
\begin{equation*}
\delta_{1}(\xi)=\delta_{2}(\xi) \quad \text { in any gauge. } \tag{17}
\end{equation*}
$$

