Quantization of Yang–Mills Theories

Consider pure Yang–Mills theory with some simple gauge group G. Classically, the only fields of the theory are the gauge fields $A^a_\mu(x)$ in the adjoint multiplet of G. The Euclidean Lagrangian is

$$\mathcal{L}_E \;=\; +\frac{1}{4}\sum_a \bigl(F^a_{\mu\nu}\bigr)^2 \tag{1}$$

where

$$F^a_{\mu\nu} \;=\; \partial_\mu A^a_\nu \;-\; \partial_\nu A^a_\mu \;-\; g f^{abc} A^b_\mu A^c_\nu \tag{2}$$

are the non-abelian tension fields, g is the gauge coupling constant, and f^{abc} are the structure constants of the Lie algebra of G. That is, the generators T^a of G obey $[T^a, T^b] = i f^{abc} T^c$.

In perturbation theory we decompose the Lagrangian into quadratic, cubic, and quartic terms,

$$\begin{aligned}
\mathcal{L} \;&=\; \mathcal{L}_2 \;+\; g\mathcal{L}_3 \;+\; g^2\mathcal{L}_4\,,\\
\mathcal{L}_2 \;&=\; \tfrac{1}{4}\bigl(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu\bigr)^2 \;=\; \tfrac{1}{2}\bigl(\partial_\mu A^a_\nu\bigr)^2 \;-\; \tfrac{1}{2}\bigl(\partial_\mu A^a_\mu\bigr)^2,\\
\mathcal{L}_3 \;&=\; -\tfrac{1}{2}\bigl(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu\bigr) \times f^{abc} A^b_\mu A^c_\nu\,,\\
\mathcal{L}_4 \;&=\; \tfrac{1}{4}\bigl(f^{abc} A^b_\mu A^c_\nu\bigr)^2.
\end{aligned} \tag{3}$$

In the Feynman rules, the propagators should come from the quadratic part $\mathcal{L}_2$ while the vertices should come from the cubic and the quartic parts. But the quadratic part here looks like $|G|$ species of photons and it suffers from exactly the same quantization problem as the QED: the Euclidean path integral over the free $A^a_\mu(x)$ fields diverges for for generic sources $J^a_\mu(x)$ and does not give us a valid propagator.

Just as in QED, the solution to this problem is to fix a gauge. That is, for every configuration $A^a_\mu(x)$ of the gauge fields, we replace it with a gauge-equivalent configuration ${}^{\Lambda}\!A^a_\mu(x)$ which obeys some simple constraint at every point x, for example the Landau gauge constraint

$$\partial_\mu {}^{\Lambda}\!A^a_\mu(x) \;\equiv\; 0 \quad \forall x,\, \forall a. \tag{4}$$

Consequently, the naive path integral of the YM theory becomes

$$\begin{aligned}Z^{\text{naive}}[J] &= \iiint \mathcal{D}[A^a_\mu(x)]\, \exp\bigl(-S_E[A,J]\bigr) \\ &\to \iiint \mathcal{D}[A^a_\mu(x)]\, \exp\bigl(-S_E[A,J]\bigr) \times \iiint \mathcal{D}[\Lambda^a(x)]\, \Delta[\partial_\mu {}^{\Lambda}\!A^a_\mu(x)] \times \text{Det}[FP]\end{aligned} \tag{5}$$

where ${}^{\Lambda}\!A^a_\mu(x)$ obtains from the $A^a_\mu(x)$ via the gauge transform parametrized by the $\Lambda^a(x)$ (I'll write an explicit formula in a moment), and $\text{Det}[FP]$ is the Fadde'ev–Popov determinant,

$$\text{Det}[FP] \;=\; \text{Det}\left[\frac{\delta(\partial_\mu {}^{\Lambda}\!A^a_\mu)}{\delta\Lambda^b}\right]. \tag{6}$$

Formally, on the second line of eq. (5), the integral is over un-constrained YM potentials $A^a_\mu(x)$ and also over *independent* gouge transform parameters $\Lambda^a(x)$; it's the $\Delta[\partial_\mu {}^{\Lambda}\!A^a_\mu]$ factor which enforces the Landau gauge condition. Consequently, we may integrate over the $A^a_\mu(x)$ before integrating over the $\Lambda^a(x)$, thus

$$\begin{aligned}Z^{\text{naive}}[J] &= \iiint \mathcal{D}[\Lambda^a(x)] \iiint \mathcal{D}[A^a_\mu(x)]\, \exp\bigl(-S_E[A,J]\bigr) \times \Delta[\partial_\mu {}^{\Lambda}\!A^a_\mu(x)] \times \text{Det}[FP] \\ &= \iiint \mathcal{D}[\Lambda^a(x)]\, \widehat{Z}[J,\Lambda]\end{aligned} \tag{7}$$

where

$$\widehat{Z}[J,\Lambda] \;=\; \iiint \mathcal{D}[A^a_\mu(x)]\, \exp\bigl(-S_E[A,J]\bigr) \times \Delta[\partial_\mu {}^{\Lambda}\!A^a_\mu(x)] \times \text{Det}[FP]. \tag{8}$$

The YM action is gauge invariant, so in the integral (8) we may replace

$$S_E(A,J) \;\to\; S_E[{}^{\Lambda}\!A,J]. \tag{9}$$

Also, the integral (8) is over the un-constrained $A^a_\mu(x)$ and it's taken for a fixed $\Lambda^a(x)$, so presuming the functional integral's measure is gauge invariant, we may change the integration variable from $A^a_\mu(x)$ to ${}^{\Lambda}\!A^a_\mu(x)$,

$$\mathcal{D}[A^a_\mu(x)] \;\to\; \mathcal{D}[{}^{\Lambda}\!A^a_\mu(x)], \tag{10}$$

thus

$$\widehat{Z}[J,\Lambda] = \iiint \mathcal{D}[{}^{\Lambda}\!A^a_\mu(x)]\, \exp\bigl(-S_E[{}^{\Lambda}\!A,J]\bigr) \times \Delta[\partial_\mu {}^{\Lambda}\!A^a_\mu(x)] \times \text{Det}[FP]. \tag{11}$$

Note that the only Λ dependence of this functional integral is via the integration variable

${}^{\Lambda}\!A^a_\mu$. However, since we integrate over all possible ${}^{\Lambda}\!A^a_\mu(x)$, we may just as well rename this variable ${}^{\Lambda}\!A^a_\mu(x) \to A^a_\mu(x)$, thus

$$\widehat{Z}[J,\cancel{\Lambda}] = \iiint \mathcal{D}[A^a_\mu(x)] \, \exp\bigl(-S_E[A,J]\bigr) \times \Delta[\partial_\mu A^a_\mu(x)] \times \mathrm{Det}[FP]. \tag{12}$$

This formula makes $\widehat{Z}$ manifestly Λ-independent! Consequently, eq. (7) becomes

$$Z^{\text{naive}}[J] \;=\; \widehat{Z}[J \,\text{only}] \times \iiint \mathcal{D}[\Lambda^a(x)]. \tag{13}$$

where the second factor is a constant. Moreover, this constant stems from integration over the physically redundant degrees of freedom of the YM potentials $A^a_\mu(x)$, so we should simply discard it. In other words, we *redefine* the partition function of the YM theory as $\hat{Z}$ rather than Z^{naive},

$$Z[J] \;=\; \widehat{Z}[J,\text{only}] \;=\; \iiint \mathcal{D}[A^a_\mu(x)] \, \exp\bigl(-S_E[A,J]\bigr) \times \Delta[\partial_\mu A^a_\mu(x)] \times \mathrm{Det}[FP]. \tag{14}$$

The above formulae seem to work exactly as in QED, but the devil is in the details: The non-abelian gauge transforms are more complicated, which makes the Fadde'ev–Popov determinant depend on the vector fields A^a_μ. Indeed, the non-abelian gauge transforms do not merely shift the A_μ by $\partial_\mu\Lambda(x)$ but also rotate the components A^a_μ into each other. For the infinitesimal gauge transform parameters $\Lambda^a(x)$,

$$\delta A^a_\mu(x) \;=\; -\partial_\mu\Lambda^a(x) \;-\; g f^{abc}\Lambda^b(x) A^c_\mu(x) \;=\; -D_\mu\Lambda^a(x), \tag{15}$$

while the finite gauge transforms are best written in matrix notations for the symmetry group G: The transform is parametrized by the x–dependent symmetry matrix $U(x) = \exp(ig\Lambda^a(x)T^a)$, while the matrix-valued vector field $\mathcal{A}_\mu(x) = gA^a_\mu(x)T^a$ transforms as

$$\mathcal{A}_\mu(x) \;\longrightarrow\; U(x)\times\mathcal{A}_\mu(x)\times U^{-1}(x) \;+\; i\partial_\mu U(x)\times U^{-1}(x). \tag{16}$$

Fortunately, the Fadde'ev–Popov determinant does not depend on the finite gauge transform that gets us from some original $A^a_\mu(x)$ to the ${}^{\Lambda}\!A^a_\mu(x)$ that obey the Landau gauge constraint.

All we need are the infinitesimal variations of that gauge transform, and we can build them in two stages:

$$\begin{array}{rcl} \text{first,} & A^a_\mu(x) \xrightarrow[\text{finite}]{} {}^{\Lambda}\!A^a_\mu(x)\,, & \text{which obeys } \partial_\mu {}^{\Lambda}\!A^a_\mu(x) \equiv 0, \\ \text{second} & {}^{\Lambda}\!A^a_\mu(x) \xrightarrow[\text{infi}]{} {}^{\Lambda+\delta\lambda}\!A^a_\mu(x) & = \ {}^{\Lambda}\!A^a_\mu(x) \ - \ D_\mu\delta\Lambda^a(x). \end{array} \tag{17}$$

The Fadde'ev–Popov determinant depends only on the second stage here; indeed, once we gauge-fix the ${}^{\Lambda}\!A^a_\mu(x)$, further infinitesimal gouge transformation $\delta\Lambda^a(x)$ changes the field by

$$\delta\,{}^{\Lambda}\!A^a_\mu(x) \ = \ -D_\mu\delta\Lambda^a(x), \tag{18}$$

hence

$$\frac{\delta\bigl(\partial_\mu {}^{\Lambda+\delta\lambda}\!A^a_\mu(x)\bigr)}{\delta\Lambda^b(y)} \ = \ -\partial_\mu D_\mu \delta^{ab}\delta^{(4)}(x-y). \tag{19}$$

Thus, the Fadde'ev–Popov determinant is

$$\mathrm{Det}[FP] \ = \ \mathrm{Det}\left[\frac{\delta\bigl(\partial_\mu {}^{\Lambda+\delta\lambda}\!A^a_\mu(x)\bigr)}{\delta\Lambda^b(y)}\right] \ = \ \mathrm{Det}\bigl[(-\partial_\mu D_\mu)^a{}_b\bigr], \tag{20}$$

where the differential operator — a product of an ordinary derivative ∂_μ and a covariant derivative D_μ — act on an *adjoint* multiplet of scalar fields $\varphi^a(x)$ in the 4D Euclidean space,

$$[-\partial_\mu D_\mu\varphi]^a \ = \ -\partial^2\varphi^a \ + \ \partial_\mu\bigl(f^{abc}A^b_\mu\varphi^c\bigr). \tag{21}$$

The non-abelian second term here makes for a big difference between the YM theories and the electromagnetism: *in the YM case, the Fadde'ev–Popov determinant depends on the vector field $A^a_\mu(x)$*.

We may re-implement the Fadde'ev–Popov determinant (20) using a fermionic path integral. Indeed, the determinant of any matrix $\mathcal{O}_{ij}$ of differential operators obtains from the fermionic path integral

$$\iiint\mathcal{D}[\overline{\psi}_i(x)]\iiint\mathcal{D}[\psi_j(x)]\,\exp\left(-\int d^4x_e\,\overline{\psi}_i\mathcal{O}_{ij}\psi_j\right) \ = \ \mathrm{Det}[\mathcal{O}_{ij}]. \tag{22}$$

The number of ψ_i and the $\overline{\psi}_j$ fields here depends on the matrix size of the $\mathcal{O}_{ij}$, and their indices should be of the same type. In particular, for the Fadde'ev–Popov determinant (20) the fermionic fields should carry the adjoint indices of the gauge symmetry, thus

$$\mathrm{Det}[FP] \;=\; \iiint\mathcal{D}[\bar{c}^a(x)] \iiint\mathcal{D}[c^a(x)]\, \exp\left(+\int d^4x_e\, \bar{c}^a\partial_\mu D_\mu c^a \;=\; -\int d^4x_e\, \partial_\mu\bar{c}^a D_\mu c^a\right). \quad (23)$$

On the other hand, since the operator $-\partial_\mu D_\mu$ does not have any Dirac indices, the fermionic fields $c^a(x)$ and $\bar{c}^a(x)$ — called the *Fadde'ev–Popov ghost fields* — are spinless scalar fields despite their fermionic statistics! This violates the spin-statistics theorem, so quanta of the ghost fields are not physical particles, and their Hilbert space has negative norm.

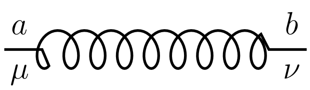
In terms of the ghosts fields, the partition function (12) for the Yang–Mills theory becomes

$$\begin{aligned} Z \;=\; &\iiint\mathcal{D}[A^a_\mu(x)]\,\Delta[\partial_\mu A^a_\mu(x)] \iiint\mathcal{D}[\bar{c}^a(x)] \iiint\mathcal{D}[c^a(x)] \\ &\exp\left(-\int d^4x_e\left(\tfrac{1}{4}F^a_{\mu\nu}F^a_{\mu\nu} \;-\; J^a_\mu A^a_\mu \;+\; \partial_\mu\bar{c}^a D_\mu c^a\right)\right). \end{aligned} \quad (24)$$

In other words, the quantum theory has both vector and ghost fields, its effective Euclidean Lagrangian is

$$\mathcal{L}_{\mathrm{eff}} \;=\; \tfrac{1}{4}F^a_{\mu\nu}F^a_{\mu\nu} \;+\; \partial_\mu\bar{c}^a D_\mu c^a, \quad (25)$$

and the vector fields are constrained by the Landau gauge condition $\partial_\mu A^a_\mu(x) \equiv 0$. Thanks to this condition, the theory has well-defined vector propagators:



$$\begin{aligned} &= \frac{\delta^{ab}}{k^2}\left(\delta_{\mu\nu} \;-\; \frac{k_\mu k_\nu}{k^2}\right) \qquad \text{(Euclidean)} \\ &= \frac{-i\delta^{ab}}{k^2+i0}\left(g_{\mu\nu} \;-\; \frac{k_\mu k_\nu}{k^2+i0}\right) \qquad \text{(Minkowski).} \end{aligned} \quad (26)$$

Sometimes it is more convenient to use the Feynman gauge or a more general ξ gauge. To change the gauge, we proceed similar to QED. First, we modify the right hand side of the Landau gauge constraint and demand $\partial_\mu A^a_\mu(x) \equiv \omega^a(x)$ for a fixed $\omega^a(x)$. This change does not affect the Fadde'ev–Popov determinant, so the partition function becomes

$$Z[J,\omega] \;=\; \iiint\!\mathcal{D}[A^a_\mu(x)]\,\Delta[\partial_\mu A^a_\mu(x)-\omega^a(x)]\iiint\!\mathcal{D}[\bar c^a(x)]\iiint\!\mathcal{D}[c^a(x)]\;\exp\left(-\int d^4x_e\left(\tfrac14 F^a_{\mu\nu}F^a_{\mu\nu}\;-\;J^a_\mu A^a_\mu\;+\;\partial_\mu\bar c^a D_\mu c^a\right)\right). \tag{27}$$

By gauge invariance of the original theory, this partition function does not depend on the $\omega^a(x)$, so we may just as well average it over the ω configurations with some Gaussian weight. In other words, we add to the theory a non-propagating auxiliary field — or rather an adjoint multiplet of auxiliary fields $\omega^a(x)$ — with a quadratic Lagrangian

$$\mathcal{L}_\omega \;=\; \frac{1}{2\xi}\,\omega^a\omega^a\,, \tag{28}$$

Consequently, the partition function becomes

$$\begin{aligned} Z[J] \;&=\; \iiint\!\mathcal{D}[\omega^a(x)]\;\exp\left(\frac{-1}{2\xi}\int d^4x_E\,\omega^a\omega^a\right)\times Z[J,\omega]\\ &=\; \iiint\!\mathcal{D}[\omega^a(x)]\iiint\!\mathcal{D}[A^a_\mu(x)]\,\Delta[\partial_\mu A^a_\mu(x)-\omega^a(x)]\iiint\!\mathcal{D}[\bar c^a(x)]\iiint\!\mathcal{D}[c^a(x)]\\ &\qquad \exp\left(-\int d^4x_E\left(\tfrac14 F^a_{\mu\nu}F^a_{\mu\nu}\;-\;J^a_\mu A^a_\mu\;+\;\partial_\mu\bar c^a D_\mu c^a\;+\;\frac{1}{2\xi}\,\omega^a\omega^a\right)\right). \end{aligned} \tag{29}$$

But in the last integral, we may use the $\Delta[A^a_\mu(x)-\omega^a(x)]$ functional to eliminate the auxiliary fields ω^a instead of constraining the vector fields, thus

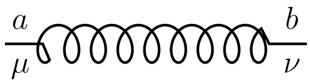
$$Z[J] \;=\; \iiint\!\mathcal{D}[A^a_\mu(x)]\iiint\!\mathcal{D}[\bar c^a(x)]\iiint\!\mathcal{D}[c^a(x)]\;\exp\left(-\int d^4x_E\big(\mathcal{L}_{\rm net}-J^a_\mu A^a_\mu\big)\right) \tag{30}$$

where the net Euclidean Lagrangian is now

$$\mathcal{L}_{\rm net} \;=\; \tfrac14 F^a_{\mu\nu}F^a_{\mu\nu}\;+\;\frac{1}{2\xi}\left(\partial_\mu A^a_\mu\right)^2\;+\;\partial_\mu\bar c^a D_\mu c^a \tag{31}$$

and the vector fields are no longer constrained. Instead, we have the gauge-fixing term $(\partial_\mu A^a_\mu)^2/2\xi$ in the Lagrangian. Adding this term to the quadratic part of the original YM

Lagrangian, we arrive at the ξ–gauge propagators for the vector fields,



$$\begin{aligned} &= \frac{\delta^{ab}}{k^2}\left(\delta_{\mu\nu} + (\xi-1)\times\frac{k_\mu k_\nu}{k^2}\right) && \text{(Euclidean)} \\ &= \frac{-i\delta^{ab}}{k^2+i0}\left(g_{\mu\nu} + (\xi-1)\times\frac{k_\mu k_\nu}{k^2+i0}\right) && \text{(Minkowski).} \end{aligned} \tag{32}$$

The Feynman gauge is the special case of this gauge for $\xi = 1$.