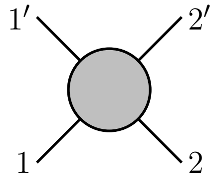
Mandelstam Variables

Consider any kind of a 2 particles → 2 particles process



(1)

The 4-momenta p_1^μ, p_2^μ, $p_1'^\mu$, and $p_2'^\mu$ of the 2 incoming and 2 outgoing particles satisfy 8 constraints: the on-shell conditions for each particle

$$p_1^2 \;=\; m_1^2, \quad p_2^2 = m_2^2, \quad p_1'^2 \;=\; m_1'^2, \quad p_2'^2 \;=\; m_2'^2, \tag{2}$$

and the net 4-momentum conservation

$$p_1^\mu \;+\; p_2^\mu \;=\; p_1'^\mu \;+\; p_2'^\mu. \tag{3}$$

Altogether, this gives us $4 \times 4 - 8 = 8$ independent momentum variables, and the number of independent Lorentz-invariant combinations of these variables is only $8 - 6 = 2$.

However, for practical purposes it's is often convenient to use 3 Lorentz-invariant variables with a fixed sum,

$$\begin{aligned} s \;&=\; (p_1 + p_2)^2 \;=\; (p_1' + p_2')^2, \\ t \;&=\; (p_1 - p_1')^2 \;=\; (p_2' - p_2)^2, \\ u \;&=\; (p_1 - p_2')^2 \;=\; (p_1' - p_2)^2. \\ s \;+\; t \;+\; u \;&=\; m_1^2 \;+\; m_2^2 \;+\; m_1'^2 \;+\; m_2'^2\,. \end{aligned} \tag{4}$$

Indeed,

$$\begin{aligned} s \;+\; t \;+\; u \;&=\; (p_1 + p_2)^2 \;+\; (p_1 - p_1')^2 \;+\; (p_1 - p_2')^2 \\ &=\; 3p_1^2 \;+\; p_2^2 \;+\; p_1'^2 \;+\; p_2'^2 \;+\; 2(p_1 p_2) \;-\; 2(p_1 p_1') \;-\; 2(p_1 p_2') \\ &=\; p_1^2 \;+\; p_2^2 \;+\; p_1'^2 \;+\; p_2'^2 +\; 2p_1 \times (p_1 + p_2 - p_1' - p_2' = 0) \\ &=\; p_1^2 \;+\; p_2^2 \;+\; p_1'^2 \;+\; p_2'^2 \\ &=\; m_1^2 \;+\; m_2^2 \;+\; m_1'^2 \;+\; m_2'^2\,. \end{aligned} \tag{5}$$

The s, t, and u are called Mandelstam variables after Stanley Mandelstam who introduced them back in 1958.

In the center-of-mass frame where $\mathbf{p}_1 + \mathbf{p}_2 = \mathbf{p}'_1 + \mathbf{p}'_2 = 0$, $\sqrt{s}$ is the total energy of the colliding particles, $\sqrt{s} = E_1 + E_2 = E'_1 + E'_2$. Also, for an elastic collision in the CM frame, t parametrizes the scattering angle according to $t = -(\mathbf{p}'_1 - \mathbf{p}_1)^2 = -\mathbf{p}^2 \times (1 - \cos\theta)$. Hence, the Lorentz-invariant definitions (4) translate the CM-frame energy and the CM-frame scattering angle to any other frame of reference.

All Lorentz-invariant combinations of the four momenta p_1^μ, p_2^μ, $p_1'^\mu$, and $p_2'^\mu$ can be expressed in terms of the Mandelstam variables. For example, the Lorentz products $k^\mu k'_\mu$ of any two momenta are

$$\begin{aligned}
2(p_1p_2) &= (p_1+p_2)^2 - p_1^2 - p_2^2 = s - m_1^2 - m_2^2\,,\\
2(p'_1p'_2) &= (p'_1+p'_2)^2 - p_1'^2 - p_2'^2 = s - m_1'^2 - m_2'^2\,,\\
2(p_1p'_1) &= p_1^2 + p_1'^2 - (p_1-p'_1)^2 = m_1^2 + m_1'^2 - t,\\
2(p_2p'_2) &= p_2^2 + p_2'^2 - (p_2-p'_2)^2 = m_2^2 + m_2'^2 - t,\\
2(p_1p'_2) &= p_1^2 + p_2'^2 - (p_1-p'_2)^2 = m_1^2 + m_2'^2 - u,\\
2(p_2p'_1) &= p_2^2 + p_1'^2 - (p_2-p'_1)^2 = m_2^2 + m_1'^2 - u.
\end{aligned} \tag{6}$$

In particular, for an elastic scattering of 2 same-mass particles

$$\begin{aligned}
s + t + u &= 4m^2,\\
2(p_1p_2) = 2(p'_1p'_2) &= s - 2m^2,\\
2(p_1p'_1) = 2(p_2p'_2) &= 2m^2 - t,\\
2(p_1p'_2) = 2(p_2p'_1) &= 2m^2 - u.
\end{aligned} \tag{7}$$

For future reference, let me give you similar formulae for the $e^-e^+ \to \mu^-\mu^+$ pair-production,

$$\begin{aligned}
s + t + u &= 2M_\mu^2 + 2m_e^2 \approx 2M_\mu^2,\\
2(p_1p_2) &= s - 2m_e^2 \approx s,\\
2(p'_1p'_2) &= s - 2M_\mu^2,\\
2(p_1p'_1) = 2(p_2p'_2) &= M_\mu^2 + m_e^2 - t \approx M_\mu^2 - t,\\
2(p_1p'_2) = 2(p_2p'_1) &= M_\mu^2 + m_e^2 - u \approx M_\mu^2 - u.
\end{aligned} \tag{8}$$

and for the $e^-e^+ \to \gamma\gamma$ annihilation process $p_- + p_+ \to k_1 + k_2$,

$$
\begin{aligned}
s \ + \ t \ + \ u \ &= \ 2m_e^2\,, \\
2(p_-p_+) \ &= \ s - \ 2m_e^2\,, \\
2(k_1k_2) \ &= \ s, \\
2(p_-k_1) \ = \ 2(p_+k_2) \ &= \ m_e^2 \ - \ t, \\
2(p_-k_2) \ = \ 2(p_+k_1) \ &= \ m_e^2 \ - \ u.
\end{aligned}
\tag{9}
$$