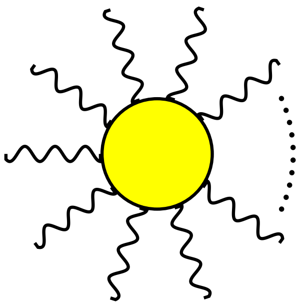
Ward–Takahashi Identities

There is a large family of Ward–Takahashi identities. Let's start with two series of basic identities for off-shell amplitudes involving 0 or 2 electrons and any number of photons.

- No electrons, N photons amplitudes

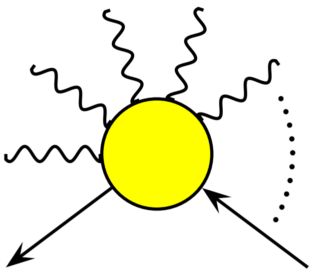


$$= \; iV_N^{\mu_1\cdots\mu_N}(k_1,\ldots,k_N) \;\xrightarrow[\text{shorthand}]{}\; iV_N^{1,\ldots,N}.$$

The V_N are amputated amplitudes, meaning no external leg bubbles in the diagrams, and the external legs themselves are not included in the amplitudes. The Ward–Takahashi identities for the V_N are simply

$$\forall i, \quad (k_i)_{\mu_i} \times V_N^{\mu_1\cdots\mu_N}(k_1,\ldots,k_N) \;=\; 0. \qquad (1)$$

- Two electrons, N photons amplitudes



$$= \; S_N^{\mu_1\cdots\mu_N}(p',p;k_1,\ldots k_N) \;\xrightarrow[\text{shorthand}]{}\; S_N^{1,\ldots,N}(p',p).$$

This time, only the photonic external legs are amputated, but the external legs for the incoming and outgoing electrons are included in the amplitudes. By convention, the photon momenta k_i are incoming, while the electronic momenta follow the charge arrows: p is incoming while p' is outgoing, hence $p' - p = k_1 + \cdots k_N$. For the S_N amplitudes, the Ward–Takahashi identities are more complicated:

$$\forall i, \quad (k_i)_{\mu_i} \times S_N^{1,\ldots,N}(p',p) \;=\; eS_{N-1}^{1,\ldots,\cancel{i},\ldots,N}(p',p+k_i) \;-\; eS_{N-1}^{1,\ldots,\cancel{i},\ldots,N}(p'-k_i,p). \qquad (2)$$

OUTLINE:

1. Proof of (2) at the tree level.
2. Proof of (1) at the one-loop level.
3. Proof of both identities for multi-loop amplitudes.
4. Ward–Takahashi identities, renormalization, and counterterms.
5. Other Ward–Takahashi identities.
6. Ward–Takahashi identities and the electric current conservation.

(1) **Lemma 1:** *the identity* (2) *holds at the tree level.*

Proof by induction in N: first prove (2) for $N = 1$ and $N = 2$, then show that if the identity holds for some N, it also holds for $N + 1$.

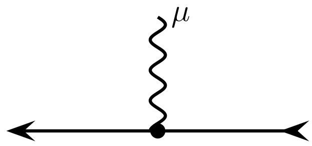
Let's start with $N = 1$. At the tree level

$$S_0(p' = p) \;=$$

$$= \frac{i}{\not{p} - m} \tag{3}$$

while

$$S_1^{\mu}(p', p; k) \;=$$



$$= \frac{i}{\not{p}' - m}\left(ie\gamma^{\mu}\right)\frac{i}{\not{p} - m}\,. \tag{4}$$

Multiplying this expression by the k_μ produces

$$k_\mu \times S_1^{\mu} \;=\; -ie\,\frac{1}{\not{p}' - m}\left(k_\mu \times \gamma^{\mu} = \not{k}\right)\frac{1}{\not{p} - m}\,, \tag{5}$$

but thanks to momentum conservation

$$k^{\mu} \;=\; p'^{\mu} \;-\; p^{\mu} \quad\Longrightarrow\quad \not{k} \;= \not{p}' \;-\; \not{p} \;=\; (\not{p}' \;-m) \;-\; (\not{p} - m). \tag{6}$$

Consequently

$$\frac{1}{\not{p}' - m} \times \not{k} \times \frac{1}{\not{p} - m} \;=\; \frac{1}{\not{p} - m} \;-\; \frac{1}{\not{p}' - m} \tag{7}$$

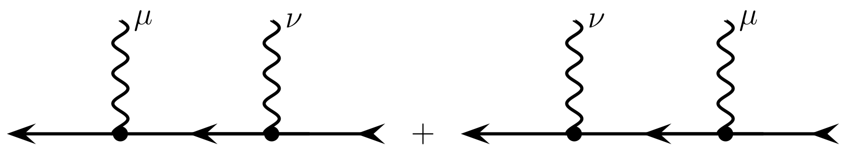
and therefore

$$
\begin{aligned}
k_\mu \times S_1^\mu(p',p;k) &= \frac{-ie}{\not{p}-m} \;+\; \frac{ie}{\not{p}'-m} \\
&= -eS_0(p,p) \;+\; eS_0(p',p') \\
&= -eS_0(p'-k,p) \;+\; eS_0(p',p+k) \quad \text{since } p'-p=k.
\end{aligned} \tag{8}
$$

This proves the tree-level WT identity (2) for $N=1$.

For $N=2$, there are two tree diagrams for the S_2 amplitude, and we must add them up to make the WT identity work — each diagram by itself does not satisfy any useful WT-like identities. Indeed, at the tree level

$S_2^{\mu\nu}(p',p;\,k_1,k_2) \;=$



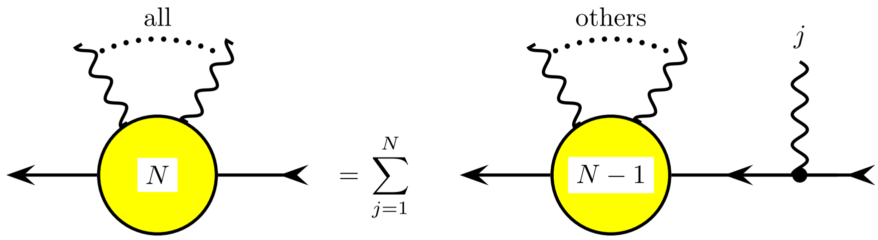
$$
\begin{aligned}
&= \frac{i}{\not{p}'-m}\left(ie\gamma^\mu\right)\frac{i}{\not{p}'-\not{k}_1-m}\left(ie\gamma^\nu\right)\frac{i}{\not{p}-m} \\
&\quad + \frac{i}{\not{p}'-m}\left(ie\gamma^\nu\right)\frac{i}{\not{p}+\not{k}_1-m}\left(ie\gamma^\mu\right)\frac{i}{\not{p}-m}\,.
\end{aligned} \tag{9}
$$

Multiplying this expression by the $(k_1)_\mu$ and using eqs. (7), we obtain

$$
\begin{aligned}
(k_1)_\mu \times S_2^{\mu\nu}(p',p;\,k_1,k_2) &= \frac{i}{\not{p}'-m}\left(ie\,\not{k}_1\right)\frac{i}{\not{p}'-\not{k}_1-m}\left(ie\gamma^\nu\right)\frac{i}{\not{p}-m} \\
&\quad + \frac{i}{\not{p}'-m}\left(ie\gamma^\nu\right)\frac{i}{\not{p}+\not{k}_1-m}\left(ie\,\not{k}_1\right)\frac{i}{\not{p}-m} \\
&= \left(\cancel{\frac{ie}{\not{p}'-m}} - \frac{ie}{\not{p}'-\not{k}_1-m}\right)\times\left(ie\gamma^\nu\right)\frac{i}{\not{p}-m} \\
&\quad + \frac{i}{\not{p}'-m}\left(ie\gamma^\nu\right)\times\left(\frac{ie}{\not{p}+\not{k}_1-m} - \cancel{\frac{ie}{\not{p}-m}}\right) \\
&= e\,\frac{i}{\not{p}'-m}\left(ie\gamma^\nu\right)\frac{i}{\not{p}+\not{k}_1-m} \;-\; e\,\frac{i}{\not{p}'-\not{k}_1-m}\left(ie\gamma^\nu\right)\frac{i}{\not{p}-m} \\
&= e\times S_1^\nu(p',p+k_1;k_2) \;-\; e\times S_1^\nu(p'-k_1,p;k_2),
\end{aligned} \tag{10}
$$

which proves the Lemma for $N=2$.

For $N > 2$ there are $N!$ tree diagrams according to $N!$ orderings of the N photons' vertices along the electron line. To make the WT identities work for all N photons we must sum all the $N!$ diagrams, although fewer diagrams will make the identity work for any one particular photon.[⋆] But instead of writing down all the $N!$ diagrams, let me simply organize them into N blocks of $(N-1)!$ diagrams according to which photon's vertex is closest to the incoming end of the electron line. Diagrammatically,



(11)

which gives us a recursive formula for the tree-level S_N amplitudes,

$$S_N^{1,\ldots,N}(p',p;) \;=\; \sum_{j=1}^{N} S_{N-1}^{\ldots\cancel{j}\ldots}(p',p+k_j) \times \left(ie\gamma^{\mu_j}\right)\frac{i}{\not{p}-m}\,. \tag{12}$$

This recursive formula will help prove the induction step: suppose all the S_{N-1} amplitudes on the RHS of eq. (12) obey the WT identity (2), then the S_N amplitude on the LHS also obey the WT identity. Indeed, multiplying both sides of eq. (12) by the $(k_i)_{\mu_i}$ we obtain

$$\begin{aligned}(k_i)_{\mu_i} \times S_N^{1,\ldots,N}(p',p) \;=\; &\sum_{i\neq j}(k_i)_{\mu_i} \times S_{N-1}^{\ldots\cancel{j}\ldots}(p',p+k_j) \times \left(ie\gamma^{\mu_j}\right)\frac{i}{\not{p}-m}\\ &+ S_{N-1}^{\ldots\cancel{i}\ldots}(p',p+k_i) \times \left(ie\,\not{k}_i\right)\frac{i}{\not{p}-m}\end{aligned} \tag{13}$$

where on the RHS I have separated the $j=i$ term in the $\sum_j$ from the other terms. For each

⋆ Specifically, pick any one ordering of the $N-1$ photons for the S_{N-1} amplitudes on the RHS of the identity (2), say $1,2,\ldots,(N-1)$. Then to make the identity work, the S_N on the LHS of the identity should sum over N orderings — for all possible insertions of the extra photon (whose k_μ multiplies the S_N) into the fixed order of the other photons, namely $(N,1,2,\ldots(N-1))$, $(1,N,2,3,\ldots,(N-1))$, all the way to $(1,2,\ldots,(N-2),N,(N-1))$, and finally $(1,2,\ldots,(N-1),N)$.

$j \neq i$ term we may use the induction hypotheses for the S_{N-1} amplitudes, thus

$$(k_i)_{\mu_i} \times S_{N-1}^{\cdots\cancel{j}\cdots}(p', p+k_j) \;=\; eS_{N-2}^{\cdots\cancel{i}\cdots\cancel{j}\cdots}(p', p+k_j+k_i) \;-\; eS_{N-2}^{\cdots\cancel{i}\cdots\cancel{j}\cdots}(p'-k_i, p+k_j). \tag{14}$$

Now let's use the recursive formula (12) in reverse, to go from the $\sum_{j\neq i} S_{N-2}$ to the S_{N-1}. Specifically,

$$\sum_{j\neq i} eS_{N-2}^{\cdots\cancel{i}\cdots\cancel{j}\cdots}(p'-k_i, p+k_j) \times \left(ie\gamma^{\mu_j}\right)\frac{i}{\not{p}-m} \;=\; eS_{N-1}^{\cdots\cancel{i}\cdots}(p'-k_i, p) \tag{15}$$

and likewise

$$\sum_{j\neq i} eS_{N-2}^{\cdots\cancel{i}\cdots\cancel{j}\cdots}(p', p+k_i+k_j) \times \left(ie\gamma^{\mu_j}\right)\frac{i}{\not{p}+\not{k}_i-m} \;=\; eS_{N-1}^{\cdots\cancel{i}\cdots}(p', p+k_i). \tag{16}$$

Note that in the last formula the incoming electron propagator has a different momentum from what we had in eq. (13) — $p+k_i$ instead of p, — but since this propagator is the same for all j, we can correct for it using an overall factor:

$$\frac{i}{\not{p}-m} \;=\; \frac{1}{\not{p}+\not{k}_i-m} \times \left(1+\not{k}_i\,\frac{1}{\not{p}-m}\right) \tag{17}$$

and hence

$$\sum_{j\neq i} eS_{N-2}^{\cdots\cancel{i}\cdots\cancel{j}\cdots}(p', p+k_i+k_j) \times \left(ie\gamma^{\mu_j}\right)\frac{i}{\not{p}-m} \;=\; eS_{N-1}^{\cdots\cancel{i}\cdots}(p', p_i) \times \left(1+\not{k}_i\,\frac{1}{\not{p}-m}\right). \tag{18}$$

Altogether, eqs. (14), (15), and (18) tell us that the sum on the first line of eq. (13) amounts to

$$\begin{aligned}\text{first line} \;&=\; \sum_{i\neq j}(k_i)_{\mu_i} \times S_{N-1}^{\cdots\cancel{j}\cdots}(p', p+k_j) \times \left(ie\gamma^{\mu_j}\right)\frac{i}{\not{p}-m}\\ &=\; eS_{N-1}^{\cdots\cancel{i}\cdots}(p', p+k_i) \times \left(1+\not{k}_i\,\frac{1}{\not{p}-m}\right) \;-\; eS_{N-1}^{\cdots\cancel{i}\cdots}(p'-k_i, p).\end{aligned} \tag{19}$$

As to the $j = i$ term on the second line of eq. (13), it does not need the induction hypotheses, we may simply add it as it is to eq. (19):

$$\begin{aligned}(k_i)_{\mu_i} \times S_N^{1,\ldots,N}(p',p) \;&=\; eS_{N-1}^{\ldots\not{i}\ldots}(p',p+k_i) \times \left(1 + \cancel{\not{k}_i \frac{1}{\not{p}-m}}\right) \;-\; eS_{N-1}^{\ldots\not{i}\ldots}(p'-k_i,p) \\ &\quad + S_{N-1}^{\ldots\not{i}\ldots}(p',p+k_i) \times \cancel{(ie\not{k}_i)\frac{i}{\not{p}-m}} \\ &=\; eS_{N-1}^{\ldots\not{i}\ldots}(p',p+k_i) \;-\; eS_{N-1}^{\ldots\not{i}\ldots}(p'-k_i,p),\end{aligned} \tag{20}$$

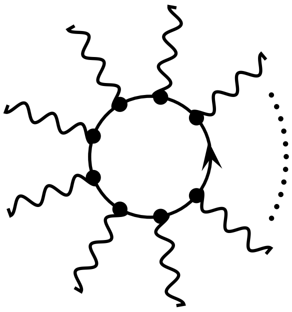
which proves the induction step and hence the whole Lemma 1.

(2) **Lemma 2:** *Ward–Takahashi identity* (1) *holds at the one-loop level.*

Now let's put the 2-electron S_N amplitudes aside for a moment and focus on the no-external-electrons amplitudes V_N. Since there are no tree diagrams for any of the V_N, our starting point is the one-loop level, hence the present Lemma.

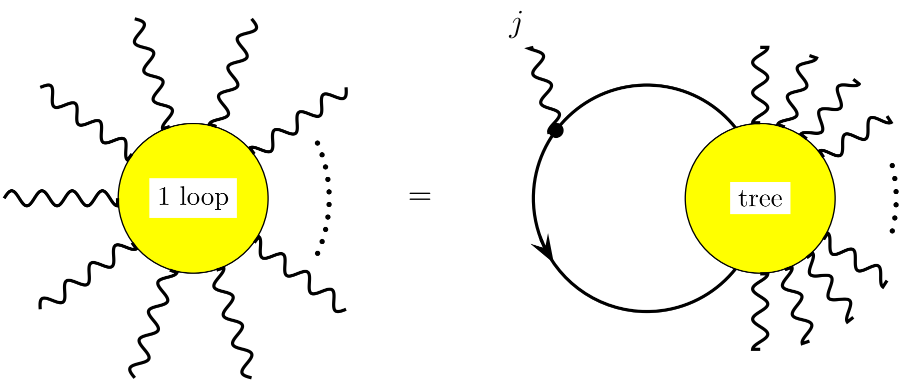
At the one-loop level, the V_N come from electron loops going through N photonic vertices,

$$iV_N^{1\,\text{loop}} \;=$$



$$+\;\text{photon permutations.} \tag{21}$$

Note that only the *cyclic order* of the photon vertices is relevant to the electron loop, so we may always keep one particular photon — say photon $\#j$ — at the beginning of the loop, and then we should sum over $(N-1)!$ permutations of the other $N-1$ photons. Schematically,



(22)

which translates to

$$i\;{}^{\text{1 loop}}V_N^{1,\ldots,N} \;=\; -\int\frac{d^4p}{(2\pi)^4}\,\mathrm{tr}\Big[\big(ie\gamma^{\mu_j}\big)\times{}^{\text{tree}}S_{N-1}^{\ldots\cancel{j}\ldots}(p,p+k_j)\Big], \qquad \text{same } \forall j. \tag{23}$$

Thanks to this relation, we may use Lemma 1 to prove the present Lemma 2. Indeed,

$$(k_i)_{\mu_i}\times iV_N^{1,\ldots,N} \;=\; -\int\frac{d^4p}{(2\pi)^4}\,\mathrm{tr}\Big[\big(ie\gamma^{\mu_j}\big)\times(k_i)_{\mu_i}\times S_{N-1}^{\ldots\cancel{j}\ldots}(p,p+k_j)\Big] \tag{24}$$

$\langle\!\langle$ for some $j\neq i$ $\rangle\!\rangle$

$$= -\int\frac{d^4p}{(2\pi)^4}\,\mathrm{tr}\left[\big(ie\gamma^{\mu_j}\big)\times\begin{pmatrix} eS_{N-2}^{\ldots\cancel{i}\ldots\cancel{j}\ldots}(p+k_i,p+k_j) \\ -\;eS_{N-2}^{\ldots\cancel{i}\ldots\cancel{j}\ldots}(p,p+k_j-k_i)\end{pmatrix}\right] \tag{25}$$

$$= -e\int\frac{d^4p}{(2\pi)^4}\,\mathrm{tr}\Big[\big(ie\gamma^{\mu_j}\big)\times eS_{N-2}^{\ldots\cancel{i}\ldots\cancel{j}\ldots}(p+k_i,p+k_j)\Big]$$
$$+\,e\int\frac{d^4p}{(2\pi)^4}\,\mathrm{tr}\Big[\big(ie\gamma^{\mu_j}\big)\times eS_{N-2}^{\ldots\cancel{i}\ldots\cancel{j}\ldots}(p,p+k_j-k_i)\Big] \tag{26}$$

$$= 0 \tag{27}$$

because the two integrals (26) are related by a constant shift of the integration variable, $p\to p-k_i$.

This argument appears to prove Lemma 2, but the caution signs in eqs. (26) and (27) warn of a loophole in the last two steps in our argument. Specifically, we have turned an integral of a difference into a difference of two integrals, and then we have shifted the integration variable in just one of these integrals. When all the integrals converge, such manipulations work fine, *but using them for divergent integrals is dangerous and may easily produce wrong results.*

In Quantum Field Theory, a divergent momentum integral is a short-hand notation for a long procedure: first, we impose a UV cutoff, then we re-calculate the integrand using the Feynman rules of the cut-off theory, then we take the integral, and finally we go back to the original theory by taking the $\Lambda\to\infty$ or the $D\to 4$ limit. For the problem at hand, we need a UV regulator that

- Renders all the integrals (26) finite (for a large but finite Λ, or for $D<4$);
- Allows shifting of the momentum integration variables;

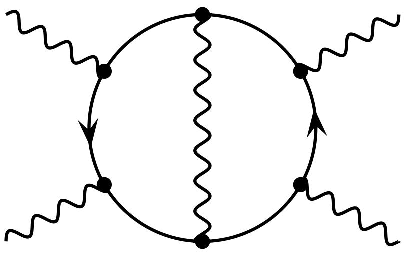
- Does not change the QED Feynman rules in a way that screws up the tree-level Ward–Takahashi identities (2).

Fortunately, QED does have UV regulators that satisfy all these criteria — for example, the dimensional regularization — so eqs. (26) and (27) work as written and the Ward–Takahashi identities (1) hold true.

Likewise, other gauge theories with true-vector currents $\overline{\Psi}\gamma^\mu\Psi$ obey Ward–Takahashi identities similar to the (1). However, the chiral gauge theories — in which the left-handed and the right-handed Weyl fermions may have different charges or belong to different multiplets — do not allow dimensional regularization or any other UV regulators that would make eqs. (26) and (27) work for $N = 3$ (or $N = 4$ for some non-abelian theories). Consequently, some of the WT identities suffer from the *anomalies* — I shall explain them later in class, probably in April — and if those anomalies do not cancel, the gauge theory fails as a quantum theory.

(3) Going Beyond One Loop

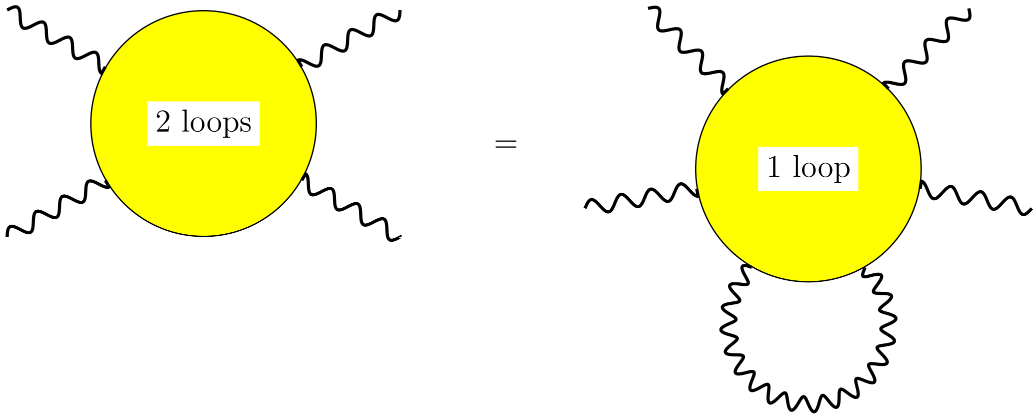
In §2 we have proved the Ward–Takahashi identities (1) at the one loop level, now let's extend the proof to the multi-loop diagrams. For starters, consider the two-loop diagrams with one electronic loop and one internal photon propagator (which makes for the second loop), for example



(28)

When evaluating such a diagram, let us integrate over the electron's momentum before we integrate over the momentum of the internal photon. The first stages of this evaluation — the Dirac traceology and integrating over the p_e — are exactly similar to working a one-loop diagram with $N + 2$ external photons instead of N. Also, totaling up similar diagrams with different cyclic orders of the photonic vertices on the electronic line — including the vertices belonging to the internal photon — works exactly similar to the one-loop diagrams.

Consequently



(29)

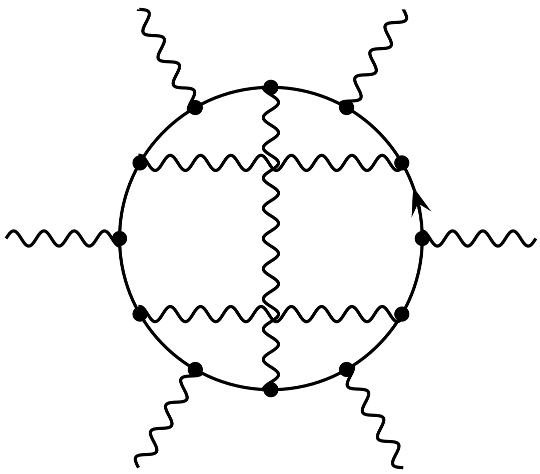
which means

$$
\begin{aligned}
{}^{2\,\mathrm{loops}}V_N^{\mu_1\ldots\mu_N}(k_1,\ldots,k_N) \;&= \int \frac{d^4\hat{k}}{(2\pi)^4}\,\frac{-i}{\hat{k}^2+i0}\left(g_{\nu\rho}+(\xi-1)\frac{\hat{k}_\nu\hat{k}_\rho}{\hat{k}^2}\right)\\
&\qquad\qquad\times\,{}^{1\,\mathrm{loop}}V_{N+2}^{\mu_1\ldots\mu_N,\nu\rho}(k_1,\ldots,k_N,+\hat{k},-\hat{k}).
\end{aligned}
\tag{30}
$$

Thanks to this relation, the Ward–Takahashi identity (1) for the one-loop V_{N+2} immediately implies a similar identity for the two-loop V_N,

$$
\begin{aligned}
(k_i)_{\mu_i}\times{}^{1\,\mathrm{loop}}V_{N+2}^{\mu_1\ldots\mu_N,\nu\rho}(k_1,\ldots,k_N,+\hat{k},-\hat{k}) \;&=\; 0\\
&\Downarrow\\
(k_i)_{\mu_i}\times{}^{2\,\mathrm{loops}}V_N^{\mu_1\ldots\mu_N}(k_1,\ldots,k_N) \;&=\; 0.
\end{aligned}
\tag{31}
$$

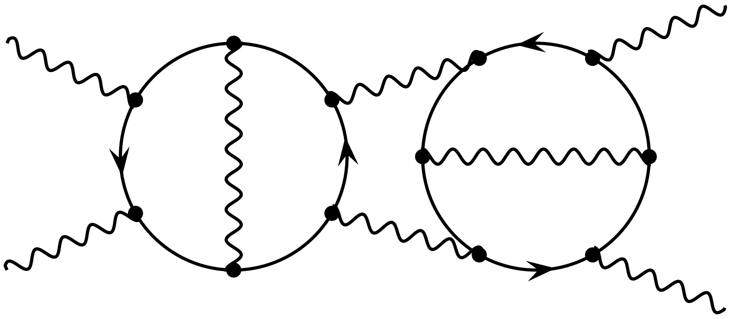
The same argument applies to the multi-loop diagrams that have one electron loop and several internal photon propagators, for example



(32)

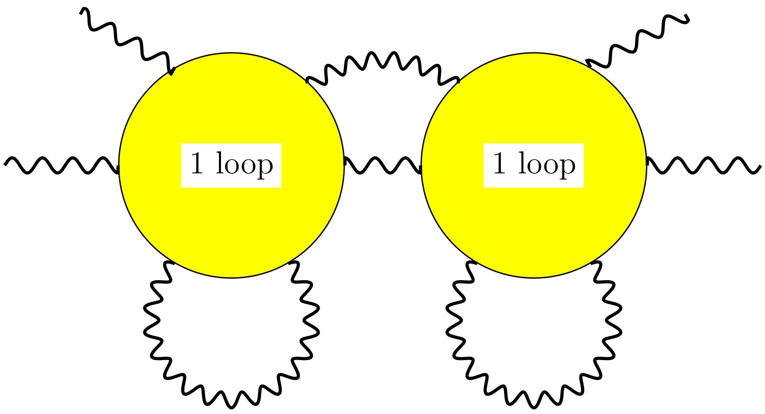
Again, once we total up the diagrams in which the photons — external or internal — attach to the electron line in all the cyclic orders, the net amplitude V_N becomes the integral of the one-loop amplitude V_{N+2m} times the internal photon propagators. Once multiplied by the k_μ of any external photon, the $k_\mu \times V_{N+2m}$ inside the integral vanishes by Lemma 2, which makes the whole integral vanish and hence $k_\mu \times V_N^{\text{multi loop}} = 0$.

Finally, consider diagrams with multiple electronic loops such as



(33)

Let's group such diagrams according to how many internal photons connect each pair of electronic loops (or one electronic loop to itself) or which loop is connected to which external photon; the diagrams in which the same photons are attached to the same electron lines — albeit in a different order — belong to the same group. For example, the diagram (33) belongs to a group of 1800 diagrams that can be summarized as



(34)

For each diagram, we do the Dirac traceology and integrals over the electron momenta before integrating over the photon momenta. We also total up all the diagram in the group before integrating over the photon momenta, which gives

$$V_N[\text{group}] \;=\; \int\limits_{\substack{\text{photon}\\\text{momenta}}} d^{4n}\hat{k} \prod \begin{pmatrix}\text{photon}\\\text{propagators}\end{pmatrix} \times \prod_{\substack{\text{electron}\\\text{loops}}} V_M^{1\,\text{loop}}\,. \tag{35}$$

Each external photon is attached to one of the electronic loops, and the corresponding $V_M^{1\,\text{loop}}$ factor carries that photon's index μ. Consequently,

$$k_\mu \times \left(V_M^{\text{that loop}}\right)^\mu \;=\; 0, \tag{36}$$

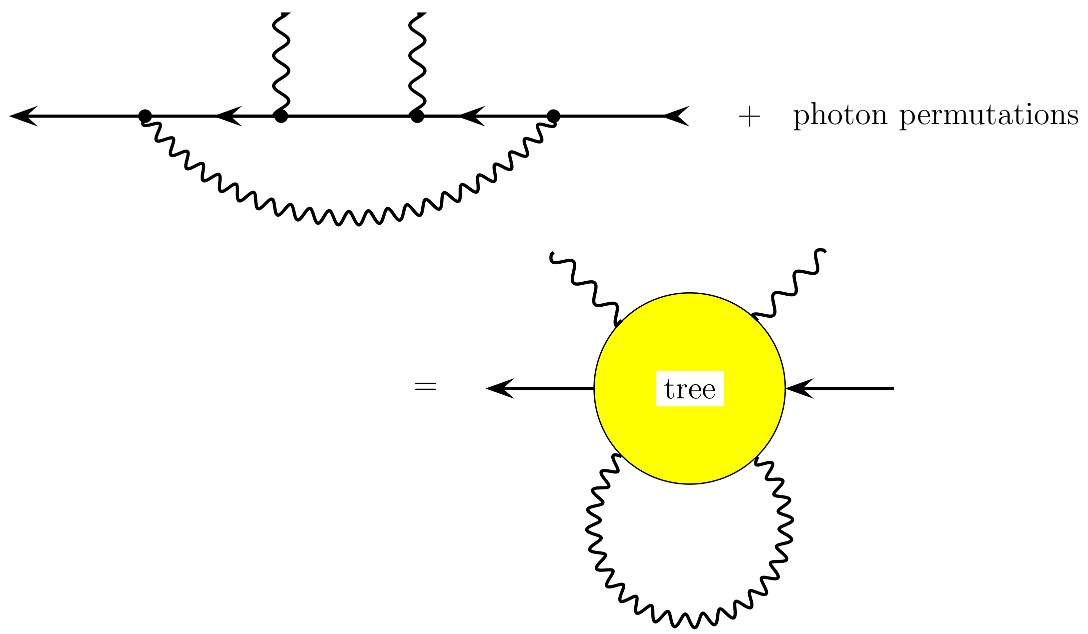
which makes the whole integral (35) vanish,

$$k_\mu \times \left(V_N^{\text{whole group}}\right)^\mu \;=\; 0. \tag{37}$$

Finally, combining all the diagram groups which contribute to an L-loop, N-photon amplitude, we prove the WT identity

$$k_\mu \times \left(V_N^{\text{net}}\right)^\mu \;=\; 0. \tag{1}$$

Now let's turn our attention back to the WT identities (2) for the two-electron, N-photon amplitudes S_N. Back in §1 we have proved those identities for the tree-level amplitudes, and now we are going to extend the proof to the loop amplitudes. Let's start with the one-loop amplitudes such as



(38)

Adding up all the photon permutations, we obtain

$$ {}^{1\,\mathrm{loop}}S_N^{\mu_1\ldots\mu_N}(p',p;\,k_1,\ldots,k_N) \;=\; \int\frac{d^4\hat{k}}{(2\pi)^4}\,\mathrm{prop}_{\nu\rho}(\hat{k})\times{}^{\mathrm{tree}}S_{N+2}^{\mu_1\ldots\mu_n\nu\rho}(p',p;\,k_1,\ldots,k_N,+\hat{k},-\hat{k}). \tag{39} $$

Consequently, when we multiply this amplitude by the k_μ of an external photon, the WT identity for the tree-level S_{N+2} immediately produces a similar identity for the one-loop-level S_N,

$$ \begin{aligned} k_\mu\times\left(S_{N+2}^{\mathrm{tree}}\right)^{\mu\cdots}(p',p;\cdots) \;&=\; e\left(S_{N+1}^{\mathrm{tree}}\right)^{\cdots}(p',p+k;\cdots) \;-\; e\left(S_{N+1}^{\mathrm{tree}}\right)^{\cdots}(p'-k,p;\cdots) \\ &\Downarrow \\ k_\mu\times\left(S_N^{1\,\mathrm{loop}}\right)^{\mu\cdots}(p',p;\cdots) \;&=\; e\left(S_{N-1}^{1\,\mathrm{loop}}\right)^{\cdots}(p',p+k;\cdots) \;-\; e\left(S_{N-1}^{1\,\mathrm{loop}}\right)^{\cdots}(p'-k,p;\cdots) \end{aligned} \tag{40} $$

Clearly, the same argument applies to the diagrams with more internal photon propagators, so all the multi-photon-loop amplitudes obey similar WT identities (2).

Finally, let's allow for all kinds of multi-loop diagrams with two external electrons and N external photons. All such diagrams have one open electronic line — which begins at the incoming electron line, goes through a few vertices and propagators, and ends at the outgoing electron line. In addition, there may be any number of closed electronic loops. All these electronic lines — open or closed — are connected to each other by some internal photon propagators; some internal photons may also connect an electron line to itself. Finally, each of the external photons is connected to one of the electron lines, open or closed.

As we did before, we should group such diagrams according to the numbers of electronic lines, the numbers of the internal photons connecting each pair of those lines (or a line to itself), and also according to which external photons attach to which line. Again, all diagrams related by permutations of the photon vertices on the same electron line — open or closed — belong in the same group, and we must add them all up to make the WT identities work. As usual, it's convenient to add them up after evaluating the electron lines and integrating over the electron momenta, but before integrating over the photon's momenta, thus

$$ S_N^{\mathrm{whole\,group}}(p',p) \;=\; \int\limits_{\substack{\mathrm{photon}\\ \mathrm{momenta}}} d^{4L'}\hat{k}\,\prod\binom{\text{photon}}{\text{propagators}}\times\prod_{\substack{\mathrm{electron}\\ \mathrm{loops}}}V_M^{1\,\mathrm{loop}}\times S_n^{\mathrm{tree}}(p',p). \tag{41} $$

This formula — plus the Lemmas 1 and 2 — tells us what happens when we multiply such a multi-loop amplitude by a k_μ of an external photon: it depends on whether that photon is connected to an open electron line or to the one of the closed electron loops. For a photon connected to a closed loop we have

$$k_\mu \times \left(V_M^{\text{that loop}}\right)^{\mu\cdots} \;=\; 0 \quad\Longrightarrow\quad k_\mu \times \left(S_N^{\text{whole group}}\right)^{\mu\cdots} \;=\; 0. \tag{42}$$

On the other hand, for an external photon attached to the open line we have

$$\begin{aligned} k_\mu \times \left(S_n^{\text{tree}}\right)^{\mu\cdots}(p',p;\cdots) \;&=\; e\left(S_n^{\text{tree}}\right)^{\cdots}(p',p+k;\cdots) \;-\; e\left(S_n^{\text{tree}}\right)^{\cdots}(p'-k,p;\cdots)\\ &\Downarrow\\ k_\mu \times \left(S_N^{\text{whole group}}\right)^{\mu\cdots}(p',p;\cdots) \;&=\; e\left(S_{N-1}^{\text{whole group}}\right)^{\cdots}(p',p+k;\cdots)\\ &\quad -\; e\left(S_{N-1}^{\text{whole group}}\right)^{\cdots}(p'-k,p;\cdots) \end{aligned} \tag{43}$$

because all the other factors in (41) do not depend on *that* external photon or on the external electron momenta p and p'.

To make the WT identities work for all the external photons, we need to combine the diagrams into bigger groups so that each photon can be attached to any of the electron lines, open or closed. Consequently, for any external photon $\#i$ we have

$$\begin{aligned} (k_i)_{\mu_i} \times S_N^{\text{big group}}(p',p) \;&=\; \sum_\ell^{\text{lines}} (k_i)_{\mu_i} \times S_N[i\to\ell](p',p)\\ \langle\!\langle \text{ in light of eqs. (42) and (43) } \rangle\!\rangle&\\ &=\; (k_i)_{\mu_i} \times S_N[i\to\text{open}](p',p) \;+\; 0\\ &=\; eS_{N-1}^{\text{big group}}(p',p+k_i) \;-\; eS_{N-1}^{\text{big group}}(p'-k_i,p). \end{aligned} \tag{44}$$

In other words, the WT (2) identities works for the bigger groups of diagrams, and once we total up all the diagrams (up to some maximal #loops), the identities work for the complete multi-loop amplitudes.

Quod erat demonstrandum.

(4) Ward–Takahashi Identities, Renormalization, and Counterterms.

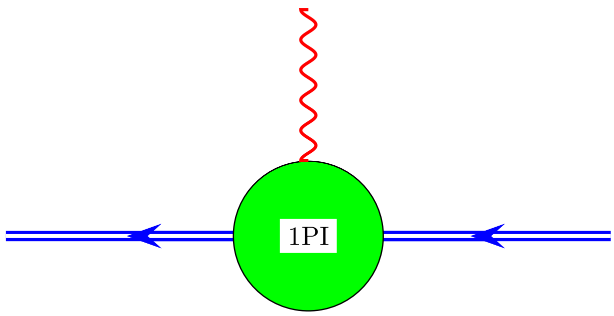
Thus far, we have worked in the bare perturbation theory. Indeed, we have proved the identities (1) and (2) for groups of tree or loop diagrams, but we have not considered the diagrams containing the counterterms vertices. Later in this section we shall take care of such vertices, — and hence of the Ward–Takahashi identities in the counterterm perturbation theory. But first, let me show how the identity (2) for $N = 1$ in the bare perturbation theory leads to <u>the</u> *Ward identity*

$$Z_1 = Z_2. \tag{45}$$

To see how this works, note that the $S_0(p' = p)$ amplitude is the two-point function $\mathcal{F}_2(p)$ for the electron, also known as the *dressed electron propagator*,

$$S_0(p' = p) = \mathcal{F}_2(p) =$$  $$= \frac{i}{\not{p} - m_b - \Sigma(\not{p}) + i\epsilon}. \tag{46}$$

The $S_1^\mu(p', p)$ is more complicated: it comprises a 1PI dressed electron-electron-photon vertex and two an-amputated dressed propagators for the electron external legs,

$$S_1^\mu(p', p) =$$  (47)

Let's call the 1PI dressed vertex $ie_b\Gamma^\mu(p', p)$; note that in the bare perturbation theory the electric charge e_b here is the bare charge rather than the physical charge e we would use in the counterterm perturbation theory. Consequently, eq. (47) becomes

$$S_1^\mu(p', p) = S_0(p') \times ie_b\Gamma^\mu(p', p) \times S_0(p). \tag{48}$$

Now, let's plug this formula into the Ward–Takahashi identity (2) for $N = 1$:

$$k_\mu \times S_1^\mu(p', p) = S_0(p') \times ie_b k_\mu \Gamma^\mu(p', p) \times S_0(p) = e_b S(p) - e_b S(p'); \tag{49}$$

note that in the bare perturbation theory the e factor on the RHS of eq. (2) becomes the bare charge e_b. Next, let's divide both sides of this equation by the $S_0(p')$ on the left and by the $S_0(p)$ on the right; this gives us

$$\begin{aligned} ie_b k_\mu \Gamma^\mu(p',p) &= \frac{e_b}{S_0(p)} - \frac{e_b}{S_0(p')} \\ &= -ie_b\big(\not{p} - m_b - \Sigma(\not{p})\big) + ie_b\big(\not{p}' - m_b - \Sigma(\not{p}')\big) \end{aligned} \tag{50}$$

and hence

$$k_\mu \Gamma^\mu(p',p) = \not{p}' - \not{p} - \Sigma(\not{p}') + \Sigma(\not{p}). \tag{51}$$

At this point, let's take the limit of a small photon momentum $k = p' - p \to 0$ while the incoming and outgoing electron momenta p and p' go to the mass shell, $p^2, p'^2 \to M^2_{\text{phys}}$. In the $k \to 0$ limit, eq. (51) becomes

$$k_\mu \Gamma^\mu(p',p) \to \not{k} - k_\mu \times \frac{\partial \Sigma(\not{p})}{\partial p_\mu} + O(k^2). \tag{52}$$

Moreover, in the bare perturbation theory in the on-shell limit

$$m_b + \Sigma(\not{p}) \to M_{\text{phys}}, \quad 1 - \frac{\partial \Sigma(\not{p})}{\partial \not{p}} \to \frac{1}{Z_2}, \tag{53}$$

and likewise for the outgoing momentum p', hence

$$k_\mu \Gamma^\mu(p',p) \to k_\mu \times \frac{\gamma^\mu}{Z_2} + O(k^2). \tag{54}$$

Therefore, in the limit of $k \to 0$, $p, p' \to$ mass shell,

$$\Gamma^\mu(p',p) \to \frac{\gamma^\mu}{Z_2} + O(k) \tag{55}$$

or in other words

$$\Gamma^\mu(\text{on shell}\, p' = p) = \frac{\gamma^\mu}{Z_2}. \tag{56}$$

To relate this formula to the electric charge renormalization, consider how we measure the physical electron's charge $-e_{\text{ph}}$ in terms of QED. Basically, we let an on-shell electron emit or absorb a zero-momentum photon and measure the amplitude

$$\mathcal{M} \;=\; e_{\text{phys}} \times \epsilon_\mu \bar{u}' \gamma^\mu u. \tag{57}$$

Thanks to the LSZ reduction formula (see my notes), the electron-electron-photon 'scattering' amplitude obtains from the bare perturbation theory as

$$i\mathcal{M} \;=\; Z_2\sqrt{Z_3} \times \left(\sum \begin{pmatrix}\text{amputated}\\ \text{diagrams}\end{pmatrix}\right) \times \begin{pmatrix}\text{spin/polarization}\\ \text{factors } \epsilon_\mu, \bar{u}', u\end{pmatrix}. \tag{58}$$

Since any amputated diagram with just 3 external legs is 1PI, the sum of amputated diagrams here amounts to the $ie_b\Gamma^\mu$ for the appropriate momenta: $k = 0$ and on-shell $p' = p$. Consequently, we have

$$e_{\text{phys}} \times \gamma^\mu \;=\; Z_2\sqrt{Z_3} \times e_{\text{bare}} \times \Gamma^\mu(\text{on shell}\, p' = p). \tag{59}$$

Moreover, by definition of the electric charge renormalization factor Z_1,

$$Z_2\sqrt{Z_3} \times e_{\text{bare}} \;=\; Z_1 \times e_{\text{phys}}\,, \tag{60}$$

hence eq. (59) becomes

$$e_{\text{phys}} \times \gamma^\mu \;=\; Z_1 \times e_{\text{phys}} \times \Gamma^\mu(\text{on shell}\, p' = p) \tag{61}$$

and therefore

$$\frac{\gamma^\mu}{Z_1} \;=\; \Gamma^\mu(\text{on shell}\, p' = p). \tag{62}$$

Comparing this formula to eq. (56), we immediately see that we must have

$$Z_1 \;=\; Z_2\,, \tag{45}$$

quod erat demonstrandum.

⋆ ⋆ ⋆

In terms of the electric charge renormalization, the Ward identity (45) reduces eq. (59) to simply

$$e_{\rm phys} \;=\; \sqrt{Z_3}\times e_{\rm bare}\,. \tag{63}$$

Thus, the electric charge renormalization in QED stems solely from the EM field renormalization, regardless to what happens to the electron field. Moreover, thanks to eq. (63),

$$e_{\rm phys}A^\mu_{\rm phys} \;=\; e_{\rm bare}A^\mu_{\rm bare}\,, \tag{64}$$

which means that the gauge-covariant derivative $D_\mu = \partial_\mu - ieA_\mu$ works in exactly the same way in terms of bare or physical fields and couplings. Thus, the gauge-covariant kinetic term for the electron field in the physical Lagrangian

$$\mathcal{L}_{\rm phys} \;\supset\; \overline{\Psi}(i\gamma^\mu D_\mu)\Psi \tag{65}$$

in the bare Lagrangian simply gets multiplied by the overall factor Z_2,

$$\mathcal{L}_{\rm bare} \;\supset\; Z_2\times\overline{\Psi}(i\gamma^\mu D_\mu)\Psi, \tag{66}$$

but the covariant derivative D_μ remains unchanged.

In QED with multiple charged fermions, each fermion species gets its own bare mass $m_{b,i}$, and its own field and coupling renormalization factors Z^1_i and Z^2_i, but for each species $Z^1_i = Z^2_i$. Consequently, the renormalized electric charges of all species remain exactly the same multiples of the physical charge unit:

$$\text{given } q_i^{\rm bare} \;=\; n_i\times e^{\rm bare},\quad \text{we get } q_i^{\rm phys} \;=\; n_i\times e^{\rm phys}\quad \text{for same } e_{\rm phys} \;=\; \sqrt{Z_3}\times e_{\rm bare}. \tag{67}$$

If we add a charged scalar field to QED, its renormalization is governed by similar Ward identities. The gauge-covariant kinetic term in the physical Lagrangian for such scalar is

$$\mathcal{L}_{\rm phys} \;\supset\; (D_\mu\Phi^*)(D_\mu\Phi) \;=\; (\partial_\mu\Phi^*)(\partial^\mu\Phi) \;+\; eA_\mu\times(-i\Phi^*\partial^\mu\Phi + i\Phi\partial^\mu\Phi^*) \;+\; e^2A_\mu A^\mu\times\Phi^*\Phi, \tag{68}$$

which in the bare Lagrangian becomes

$$\mathcal{L}_{\rm bare} \;\supset\; Z_2\times(\partial_\mu\Phi^*)(\partial^\mu\Phi) + Z_1^{1\gamma}\times eA_\mu\times(-i\Phi^*\partial^\mu\Phi + i\Phi\partial^\mu\Phi^*) + Z_1^{2\gamma}\times e^2A_\mu A^\mu\times\Phi^*\Phi. \tag{69}$$

A priori, we should have 3 field and coupling renormalization factors here, Z_2, $Z_1^{1\gamma}$, and $Z_1^{2\gamma}$, but the Ward identity for the scalar field makes them identically equal,

$$Z_2 \;=\; Z_1^{1\gamma} \;=\; Z_1^{2\gamma}. \tag{70}$$

Consequently, the 3 terms in the bare Lagrangian (69) can be reassembled into a gauge-invariant combination

$$\mathcal{L}_{\text{bare}} \;\supset\; Z_2 \times (D_\mu\Phi^*)(D^\mu\Phi). \tag{71}$$

⋆ ⋆ ⋆

For the counterterms, the Ward identity $Z_1 = Z_2$ of the bare perturbation theory means

$$\delta_1 \;=\; \delta_2\,. \tag{72}$$

This identity — as well as all the other Ward–Takahashi identities (1) and (2) — can be proved directly from the counterterm perturbation theory, without invoking the bare theory at all, but the proof is a bit more convoluted than what we had thus far in these notes. To save time and aggravation, let me skip the gory details of this proof and give you only the outline.

- ⋆ The proof works order by order in $\alpha = e^2/4\pi$ by induction in power of α. That is, given $\delta_1 = \delta_2$ to order α^L, we prove the identities (2) and (1) and hence $\delta_1 = \delta_2$ to order α^{L+1}.
- ⋆ The induction base is verifying $\delta_1 = \delta_2$ — including the finite parts of the counterterms — to order α. We shall do this later in class.
- ⋆ To prove the induction step, we *assume* $\delta_1 = \delta_2$ to order α^L and then consider the diagrams containing the counterterm vertices of all kinds as well as the physical vertices.
  - ∗ We start with the tree diagrams contributing to the S_N amplitudes (2 electrons, N photons), and show that IF $\delta_1 = \delta_2$ THEN the Ward–Takahashi identities (2) work for for such diagrams, or rather groups of diagrams related by photon permutations. This part of the proof proceeds by induction in N similarly to what we had in part (1) of these notes.

* Given the identities (2) for tree diagrams (but including the counterterm vertices), we proceed similar to part (2) and prove the photonic identities (1) for one-loop diagrams including the counterterms, and then we follow part (3) to extend the WT identities to the multi-loop diagrams.

- Since out induction assumptions is $\delta_1 = \delta_2$ only up to order α^L, the above arguments establish Ward–Takahashi identities (2) and (1) for complete amplitudes (all contributing diagrams) up to order α^L. At the next order α^{L+1}, the complete amplitudes involve $L+1$ loop diagrams without counterterms, L loop diagrams with a single $O(\alpha)$ counterterm, *etc.*, *etc.*, and ending up with tree diagrams involving a single counterterm of order α^{L+1}. By the induction assumption, all diagram types except the later obey the Ward–Takahashi identities.

- Consider the net order–α^{L+1} amplitudes

$$\Sigma_{\text{net}}^{\text{order}\,\alpha^{L+1}}(\not p) = \Sigma_{\text{loops}}^{\text{order}\,\alpha^{L+1}}(\not p) + \delta_m^{\text{order}\,\alpha^{L+1}} - \delta_2^{\text{order}\,\alpha^{L+1}}\not p, \tag{73}$$

$$(\Gamma^\mu)_{\text{net}}^{\text{order}\,\alpha^{L+1}}(p',p) = (\Gamma^\mu)_{\text{loops}}^{\text{order}\,\alpha^{L+1}}(p',p) + \delta_1\text{order}\,\alpha^{L+1}\gamma^\mu, \tag{74}$$

where Σ_{loops} and $\Gamma^\mu_{\text{loops}}$ include all the $O(\alpha^{L=1})$ diagrams — from $L+1$ loops with no counterterm vertices to a single loop with an $O(\alpha^L)$ counterterm — except a pure counterterm vertex without any loops at all. The order $\alpha^{L=1}$ counterterms follow from these loop amplitudes and the renormalization conditions (in the counterterm perturbation theory)

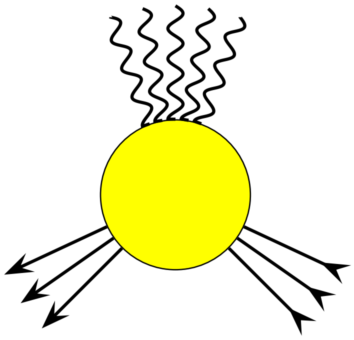
$$\text{for on-shell } p' = p, \quad \Gamma^\mu_{\text{net}} = 0, \quad \Sigma_{\text{net}} = 0, \quad \frac{d\Sigma}{d\not p} = 0. \tag{75}$$

- From to the WT identity for the $(S_1^\mu)_{\text{loops}}^{\text{order}\,\alpha^{L+1}}$ and $(S_0)_{\text{loops}}^{\text{order}\,\alpha^{L+1}}$, we obtain a relation between the on-shell $(\Gamma^\mu)_{\text{loops}}^{\text{order}\,\alpha^{L+1}}(p'=p)$ and $(d/d\not p)\Sigma_{\text{loops}}^{\text{order}\,\alpha^{L+1}}(\not p)$, which translates to $\delta_1^{\text{order}\,\alpha^{L+1}} = \delta_2^{\text{order}\,\alpha^{L+1}}$ for the counterterms.

⋆ And this completes the induction step and hence the proof.

(5) General Ward–Takahashi Identities

Besides the identities (1) and (2) for amplitudes involving zero or two external electron lines, there are similar Ward–Takahashi identities for amplitudes with any number of incoming and outgoing electrons. In general, we may have M incoming electron lines, same number of outgoing electron lines, and N external photons,



$$= \; S^{\mu_1\ldots\mu_N}_{MN}(p'_1,\ldots,p'_M;\,p_1,\ldots,p_M;\,k_1,\ldots,k_N) \tag{76}$$

(Dirac indices suppressed). Similar to the earlier amplitudes, all the external photonic lines are amputated but the incoming and outgoing electron lines are NOT amputated.

For all such amplitudes, the Ward–Takahashi identities relate an amplitude contracted with a k_μ of an external photon to amplitudes without that photon. Specifically,

$$\begin{aligned}
(k_i)_{\mu_i}\times & S^{\mu_1\ldots\mu_N}_{MN}(p'_1,\ldots,p'_M;\,p_1,\ldots p_M;\,k_1,\ldots,k_N) \\
&= \; e\sum_{j=1}^{M} S^{\cdots\cancel{\mu_i}\cdots}_{M,N-1}(p'_1,\ldots,p'_M;\,p_1,\ldots,p_j+k_i,\ldots,p_M;\,k_1,\ldots\cancel{k_i}\ldots,k_N) \\
&\quad - \; e\sum_{j=1}^{M} S^{\cdots\cancel{\mu_i}\cdots}_{M,N-1}(p'_1,\ldots,p'_j-k_i,\ldots,p'_M;\,p_1,\ldots p_M;\,k_1,\ldots\cancel{k_i}\ldots,k_N).
\end{aligned} \tag{77}$$

The proof of these identities works similarly to what we have in §3, so let me outline it without working through the details. A generic diagram contributing to the amplitude (76) has M open electronic lines, any number of closed electronic loops, a bunch of internal photons connecting all these lines to each other (or to themselves), and each external photon should be connected to one of the electronic lines, open or closed. Combining such diagrams in groups related by permutations of photons attached to the same electronic line, we relate the S[group] to the product of tree-level 2-electron S_N for the open lines and one-loop-level no-electron V_n for the closed loops. Consequently, contracting the S[group] with k_μ of an external photon gives us zero if that photon is attached to a closed line; if it's attached to an open line, we get two terms that look like S of a similar group but without the external photon in question. Finally, adding up the groups where the photon in question is attached to all possible electronic lines, we obtain the WT identity (77).

Versions of QED which include additional charged fields besides electrons have more Ward–Takahashi identities for amplitudes involving the extra fields. Most generally, consider an off-shell amplitude for N particles of any kinds — photons, electrons, other leptons, quarks, charged or neutral scalars, vectors such as $W^\pm$, whatever; let's call it $\mathcal{F}_N(p_1,\ldots,p_N)$, all indices suppressed and all momenta treated as incoming, $p_1+\cdots+p_N=0$. For simplicity, let's keep *all the external legs NOT amputated*, even the legs belonging to photons. Now, consider the $N+1$ particle amplitude $\mathcal{F}^\mu_{N+1}(p_1,\ldots,p_N\,;k)$ involving the same N particles as before, plus one extra photon (k,μ); *the external leg for the new photon is amputated, but the other N external legs are NOT amputated,* even the legs belonging to the other photons, if any. The general Ward–Takahashi identity relates the $\mathcal{F}^\mu_{N+1}$ contracted with k_μ of the extra photon to the amplitude $\mathcal{F}_N$ without that extra photon, specifically,

$$k_\mu\times\mathcal{F}^\mu_{N+1}(p_1,\ldots,p_N\,;k)\;=\;\sum_{j=1}^N \text{charge(particle}\#j)\times\mathcal{F}_N(p_1,\ldots,p_j+k,\ldots,p_N). \tag{78}$$

There are similar WT identities for the amplitudes with amputated external legs for all the photons. Indeed, if S_{nm} is the amplitude involving n photons and m particles of other kinds, with amputated photon legs but un-amputated legs for other particles, then the un-amputated

$$\mathcal{F}^{\mu_1,\ldots,\mu_n}_{n+m}(p_1,\ldots,p_m;k_1,\ldots,k_n)\;=\;S^{\nu_1,\ldots,\nu_n}_{nm}(p_1,\ldots,p_m;k_1,\ldots,k_n)\times\prod_{i=1}^N\begin{pmatrix}\text{dressed}\\ \text{propagator}\end{pmatrix}^{\mu_i}_{\nu_i}(k_i). \tag{79}$$

For the $\mathcal{F}_{N+1}$ amplitude with one more photon we have exactly similar decomposition into the amputated $S_{n+1,m}$ and n dressed photon propagators; note that the extra photon's leg is already amputated, so we do not have the $n+1^{\text{st}}$ external leg factor. Consequently, on both sides of eq. (78) we have identical products of n photonic external legs multiplying the amputated amplitudes $S_{n+1,m}$ and $S_{n,m}$. Throwing away those common factors, we obtain

$$\begin{aligned}(k_{n+1})_\mu \times S_{n+1,m}^{\nu_1,\ldots,\nu_n;\mu}&(p_1,\ldots,p_m;k_1,\ldots,k_n;k_{n+1})\\ &= \sum_{j=1}^{m} \mathrm{charge}_j \times S_{n,m}^{\mu_1,\ldots,\nu_n}((p_1,\ldots,p_j+k_{n+1},\ldots,p_m;k_1,\ldots,k_n).\end{aligned} \tag{80}$$

Note: in these identities, the photonic external legs are amputated, but the external legs for all the charged particles are NOT amputated; if any neutral particles besides the photons are involved, we may amputate them or leave then un-amputated, as long as we do the same on both sides of the identity.

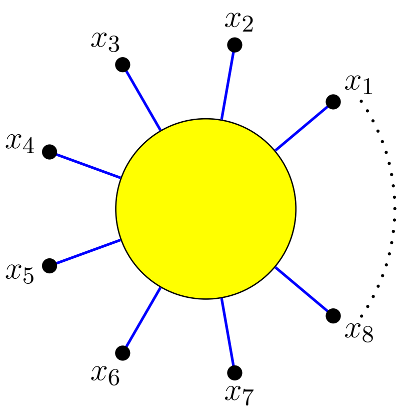
(6) Ward–Takahashi Identities and Electric Current Conservation

Physically, the Ward–Takahashi identities follow from the electric current conservation, $\partial_\mu J^\mu(x) = 0$. The best way see the connection is to recast the WT identities in terms of correlation functions of quantum fields,

$$\mathcal{F}_n(x_1,\ldots,x_n) \;=\; \langle\Omega|\,\mathbf{T}\,\hat\phi_1(x_1)\cdots\hat\phi_n(x_n)\,|\Omega\rangle^{\mathrm{connected}}. \tag{81}$$

Here the $\hat\phi_i$ stand for all species of quantum fields, including the $\hat\Psi$, $\hat{\overline{\Psi}}$, $\hat{A}^\mu$, as well as any other fields we may want to add to the basic QED. All fields are in the Heisenberg picture, so their time dependence is affected by the interactions. Earlier in class (see my notes) we saw that the correlation functions (81) are related to the un-amputated Feynman amplitudes; in the coordinate basis,

$$\mathcal{F}_N(x_1,\ldots,x_n) \;=\; \tag{82}$$

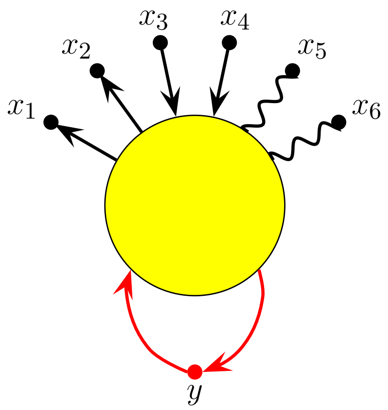


Now consider the correlation functions involving the electric current operator $\hat{J}^\mu(y)$ as well as other quantum fields,

$$\mathcal{F}^\mu_{n+1}(x_1,\ldots,x_n;y) \;=\; \langle\Omega|\,\mathbf{T}\,\hat{\phi}_1(x_1)\cdots\hat{\phi}_n(x_n)\,\hat{J}^\mu(y)\,|\Omega\rangle^{\text{connected}}. \tag{83}$$

In the basic QED (EM, electrons, and nothing else) $\hat{J}^\mu(y)=-e\hat{\overline{\Psi}}(y)\gamma^\mu\hat{\Psi}(y)$, so in the Feynman rules for the correlation functions, $\hat{J}^\mu(y)$ becomes an external vertex of valence $=2$ connected to 2 electron lines, one for the $\hat{\Psi}(y)$ and the other for the $\hat{\overline{\Psi}}(y)$. For example,

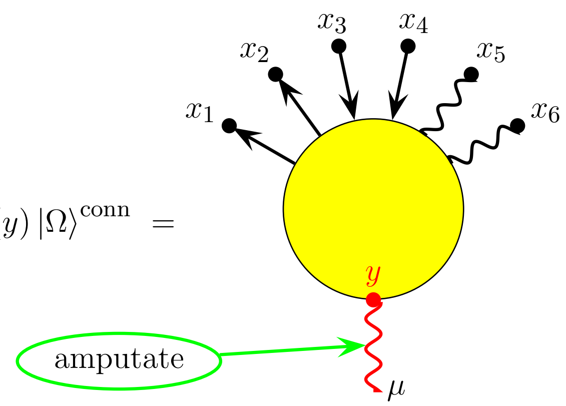
$$\langle\Omega|\,\mathbf{T}\,\hat{\overline{\Psi}}(x_1)\hat{\overline{\Psi}}(x_2)\hat{\Psi}(x_3)\hat{\Psi}(x_4)\hat{A}^\lambda(x_5)\hat{A}^\kappa(x_6)\,\hat{J}^\mu(y)\,|\Omega\rangle^{\text{conn}} \;=$$



(84)

The Dirac indexology of the bottom vertex at y is $(-e\gamma^\mu)_{\alpha\beta}$ — which is exactly similar to the photon's vertex $(ie\gamma^\mu)_{\alpha\beta}$ (up to a factor of i). So the diagram (84) is equivalent to a diagram with an external photon $\hat{A}^\mu(y)$ instead of the current operator $\hat{J}^\mu(y)$, except there no propagator for that photon. In other words,

$$-i\,\langle\Omega|\,\mathbf{T}\,\hat{\overline{\Psi}}(x_1)\hat{\overline{\Psi}}(x_2)\hat{\Psi}(x_3)\hat{\Psi}(x_4)\hat{A}^\lambda(x_5)\hat{A}^\kappa(x_6)\,\hat{J}^\mu(y)\,|\Omega\rangle^{\text{conn}} \;=$$



(85)

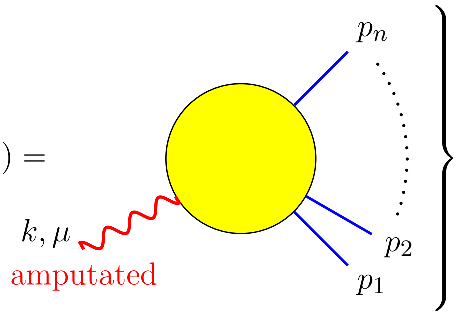
Generalizing to the other correlation functions (83) involving the electric current operator and

Fourier transforming to the momentum space,

$$\mathcal{F}^{\mu}_{n+1}(p_1,\ldots,p_n;k) \;=\; \int d^4x_1\,e^{ip_1x_1}\,\cdots\int d^4x_n\,e^{ip_nx_n}\int d^4y\,e^{iky}\times\mathcal{F}^{\mu}_{n+1}(x_1,\ldots,x_n;y), \qquad (86)$$

we arrive at the off-shell amplitudes involving n particles corresponding to the fields $\hat{\phi}_i$ plus one extra photon for the current $\hat{J}^\mu$; the external leg for that extra photon is amputated, but all the other external legs are NOT amputated,

$$-i\mathcal{F}^{\mu}_{n+1}(p_1,\ldots,p_n;k) =$$



NOT amputated (87)

Note that *these are precisely the amplitudes which appear on the LHS of the Ward–Takahashi identities* (78). But on the RSH of the same identities we have the completely un-amputated amplitudes $\mathcal{F}_n$, which correspond to the correlation functions of the $\hat{\phi}_i$ fields without the electric current operator.

So let us rephrase the Ward–Takahashi identities (78) in terms of the correlation functions. We begin by Fourier transforming those identities to the coordinate space. On the left hand side we have

$$\begin{aligned} k_\mu\times\mathcal{F}^{\mu}_{n+1}(p_1,\ldots,p_n;k) \;\xrightarrow[\text{Fourier}]{}\; & i\frac{\partial}{\partial y^\mu}\,\mathcal{F}^{\mu}_{n+1}(x_1,\ldots,x_n;y) \\ &= \frac{\partial}{\partial y^\mu}\,\langle\Omega|\,\mathbf{T}\,\hat{\phi}_1(x_1)\cdots\hat{\phi}_n(x_n)\,\hat{J}^\mu(y)\,|\Omega\rangle\,. \end{aligned} \qquad (88)$$

On the right hand side, for each term we have

$$\begin{aligned} \mathcal{F}_n(p_1,\ldots,p_j+k,\ldots,p_n) \;\xrightarrow[\text{Fourier}]{}\; & \mathcal{F}_n(x_1,\ldots,x_n)\times\delta^{(4)}(y-x_j) \\ &= \langle\Omega|\,\mathbf{T}\,\hat{\phi}_1(x_1)\cdots\hat{\phi}_n(x_n)\,|\Omega\rangle\times\delta^{(4)}(y-x_j) \end{aligned} \qquad (89)$$

where the δ-function follows from Fourier transforming a function of independent momenta p_j and k that depends on them only via their sum p_j+k. To see how that works, let's ignore the other $n-1$ momenta for a moment and work out the Fourier transform of a function of two variables that actually depends only on their sum, $F(p,k)=f(p+k)$:

$$
\begin{aligned}
F(p,k)\;=\;f(p+k)\;\xrightarrow[\text{Fourier}]{}\;F(x,y)\;&=\int\!\frac{d^4p\,d^4k}{(2\pi)^8}\,e^{-ipx-iky}\times f(k+p)\\
&=\int\!\frac{d^4p}{(2\pi)^4}\int\!\frac{d^4(q=p+k)}{(2\pi)^4}\,e^{-ipx-i(q-p)y}\times f(q)\\
&=\int\!\frac{d^4q}{(2\pi)^4}\,e^{-iqy}\times f(q)\times\int\!\frac{d^4p}{(2\pi)^4}\,e^{-ip(x-y)}\\
&=\;f(y)\times\delta^{(4)}(x-y)\;=\;f(x)\times\delta^{(4)}(x-y).
\end{aligned}
$$

Similarly, when the $\mathcal{F}_n(p_1,\ldots,p_j+k,\ldots,p_n)$ is Fourier transformed as a function of $n+1$ momenta $p_1,\ldots,p_n$ and k, we get a δ–function for the extra coordinate y, thus eq. (89).

Together, eqs. (88) and (89) let us Fourier transform the Ward–Takahashi identities (78) into coordinate-space relations for the correlation functions:

$$
\frac{\partial}{\partial y^\mu}\,\langle\Omega|\,\mathbf{T}\,\hat\phi_1(x_1)\cdots\hat\phi_n(x_n)\,\hat J^\mu(y)\,|\Omega\rangle\;=\;\langle\Omega|\,\mathbf{T}\,\hat\phi_1(x_1)\cdots\hat\phi_n(x_n)\,|\Omega\rangle\times\sum_{j=1}^n q_j\times\delta^{(4)}(x_j-y)
\tag{90}
$$

where q_j is the electric charge of the field ϕ_j, or rather of the particle annihilated by $\hat\phi_j$ and created by the $\hat\phi_j^\dagger$. (Thus, we use the electron's charge $-e$ for the $\hat\Psi$ field and the positron's charge $+e$ for the $\hat{\overline{\Psi}}$ field.)

Physically, the identities (90) — and hence all the other Ward–Takahashi identities — follow from the electric current conservation, $\partial_\mu\hat J^\mu(y)=0$. Let us see how that works. Naively, we would expect to get a zero on the right hand side of eq. (90):

$$
\frac{\partial}{\partial y^\mu}\,\langle\Omega|\,\mathbf{T}\,\hat\phi_1(x_1)\cdots\hat\phi_n(x_n)\,\hat J^\mu(y)\,|\Omega\rangle\;\stackrel{?}{=}\;\langle\Omega|\,\mathbf{T}\,\hat\phi_1(x_1)\cdots\hat\phi_n(x_n)\,\partial_\mu\hat J^\mu(y)\,|\Omega\rangle\;=\;0,
\tag{91}
$$

but there is a caveat: the time ordering $\mathbf{T}$ does not quite commute with the time derivatives such as $\partial/\partial y^0$. In general, for any two local operators $\hat A(x)$ and $\hat B(y)$ we have

$$
\frac{\partial}{\partial y^0}\,\mathbf{T}\Big(\hat A(x)\times\hat B(y)\Big)\;=\;\mathbf{T}\Big(\hat A(x)\times\frac{\partial}{\partial y^0}\hat B(y)\Big)\;+\;\delta(x^0-y^0)\times\big[\hat A(x),\hat B(y)\big].
\tag{92}
$$

In particular, for any quantum field $\hat{\phi}(x)$ and the electric current $\hat{J}^\mu(y)$ we have

$$\frac{\partial}{\partial y^\mu}\,\mathbf{T}\Big(\hat{\phi}(x)\times\hat{J}^\mu(y)\Big)\;=\;\mathbf{T}\Big(\hat{\phi}(x)\times\partial_\mu\hat{J}^\mu(y)\Big)+\;\delta(x^0-y^0)\times\big[\hat{\phi}(x),\hat{J}^0(y)\big] \tag{93}$$

The first term here vanishes by the electric current conservation, but the second term gives rise to a singularity when $x=y$. Indeed, *at equal times*

$$\big[\hat{\phi}(\mathbf{x},t),\hat{J}^0(\mathbf{y},t)\big]\;=\;\delta^{(3)}(\mathbf{x}-\mathbf{y})\times q\times\hat{\phi}(\mathbf{x},t) \tag{94}$$

where q is the electric charge of the field $\hat{\phi}$, or rather of the particle annihilated by the $\hat{\phi}$ and created by the $\hat{\phi}^\dagger$. For example, for the electron field $\hat{\Psi}(x)$, $q=-e$. Consequently,

$$\frac{\partial}{\partial y^\mu}\,\mathbf{T}\Big(\hat{\phi}(x)\times\hat{J}^\mu(y)\Big)\;=\;0\;+\;\delta^{(4)}(x-y)\times q\times\hat{\phi}(x). \tag{95}$$

Likewise, for multiple fields inside the times ordering $\mathbf{T}$, we have

$$\frac{\partial}{\partial y^\mu}\,\mathbf{T}\Big(\hat{\phi}_1(x_1)\cdots\hat{\phi}_n(x_n)\times\hat{J}^\mu(y)\Big)\;=\;0+\sum_{j=1}^{n}\delta^{(4)}(x_j-y)\times q_j\times\mathbf{T}\Big(\hat{\phi}_1(x_1)\cdots\hat{\phi}_n(x_n)\Big), \tag{96}$$

which immediately leads to the Ward–Takahashi identities

$$\frac{\partial}{\partial y^\mu}\,\langle\Omega|\,\mathbf{T}\,\hat{\phi}_1(x_1)\cdots\hat{\phi}_n(x_n)\,\hat{J}^\mu(y)\,|\Omega\rangle\;=\;\langle\Omega|\,\mathbf{T}\,\hat{\phi}_1(x_1)\cdots\hat{\phi}_n(x_n)\,|\Omega\rangle\times\sum_{j=1}^{n}q_j\times\delta^{(4)}(x_j-y) \tag{90}$$

Quod erat demonstrandum!