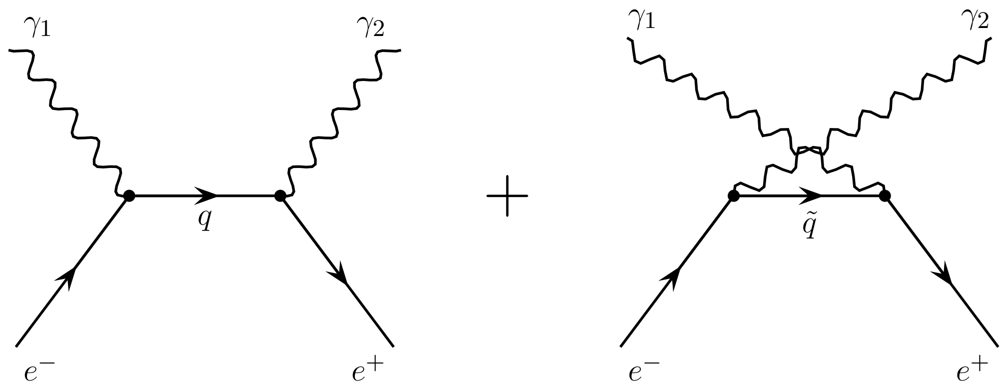
ANNIHILATION

In these notes I explain the $e^+e^- \to \gamma\gamma$ annihilation process. At the tree level of QED, there are two diagrams related by interchanging of the two photons in the final state:



(1)

The net amplitude due to these diagrams is

$$\begin{aligned}
\mathcal{M} &= \mathcal{E}^*_\mu(k_1,\lambda_1)\,\mathcal{E}^*_\nu(k_2,\lambda_2) \times \mathcal{M}^{\mu\nu},\\
\mathcal{M}^{\mu\nu} &= \mathcal{M}_1^{\mu\nu} + \mathcal{M}_2^{\mu\nu},\\
i\mathcal{M}_1^{\mu\nu} &= \bar{v}(e^+)(ie\gamma^\nu)\frac{i}{\not{q}-m}(ie\gamma^\mu)u(e^-),\\
i\mathcal{M}_2^{\mu\nu} &= \bar{v}(e^+)(ie\gamma^\mu)\frac{i}{\tilde{\not{q}}-m}(ie\gamma^\nu)u(e^-),
\end{aligned} \tag{2}$$

where $q = p_- - k_1 = k_2 - p_+$ and $\tilde{q} = p_- - k_2 = k_1 - p_+$. Note the opposite orders of the γ^μ and γ^ν vertices in the $\mathcal{M}_1$ and the $\mathcal{M}_2$ amplitudes since the two photons attach to the electron line in opposite order. Also note the bosonic symmetry between the two photons in the final state: exchanging the photons is equivalent to exchanging the two diagrams, thus

$$\mathcal{M}_1^{\mu\nu}(k_1,k_2;p_-,p_+) = \mathcal{M}_2^{\nu\mu}(k_1 \leftrightarrow k_2;p_-,p_+) \quad\Longrightarrow\quad \mathcal{M}_{\rm net}^{\mu\nu} = \mathcal{M}_{\rm net}^{\nu\mu}(k_1 \leftrightarrow k_2). \tag{3}$$

For calculation purposes, it is convenient to eliminate the matrix denominators from the amplitudes $\mathcal{M}_1$ and $\mathcal{M}_2$ using

$$\frac{1}{\not{q}-m} = \frac{\not{q}+m}{q^2-m^2} = \frac{\not{q}+m}{t-m^2} \quad\text{and}\quad \frac{1}{\tilde{\not{q}}-m} = \frac{\tilde{\not{q}}+m}{\tilde{q}^2-m^2} = \frac{\tilde{\not{q}}+m}{u-m^2}\,, \tag{4}$$

hence

$$\mathcal{M}_1^{\mu\nu} \;=\; \frac{-e^2}{t-m^2}\times\bar{v}\gamma^\nu(\not{q}+m)\gamma^\mu u \quad\text{and}\quad \mathcal{M}_2^{\mu\nu} \;=\; \frac{-e^2}{u-m^2}\times\bar{v}\gamma^\mu(\tilde{\not{q}}+m)\gamma^\nu u. \qquad (5)$$

Ward Identities

Before we go any further, lets check the Ward identities for the annihilation amplitude: For the first photon we should have $k_{1\mu}\mathcal{M}^{\mu\nu}=0$, and for the second photon $k_{2\nu}\mathcal{M}^{\mu\nu}=0$. Let's start with the first photon and the first diagram. Multiplying the second factor in the first eq. (5) by $k_{1\mu}$, we have

$$\begin{aligned}
\bar{v}\gamma^\nu(\not{q}+m)\gamma^\mu u\times k_{1\mu} \;&=\; \bar{v}\gamma^\nu(\not{p}_- - \not{k}_1+m)\,\not{k}_1 u\\
\langle\!\langle\text{ using } \not{k}_1\not{k}_1=k_1^2=0 \text{ and } \not{p}_-\not{k}_1=2(p_-k_1)-\not{k}_1\not{p}_-\,\rangle\!\rangle &\\
&=\; \bar{v}\gamma^\nu\Big(2(p_-k_1)-\not{k}_1(\not{p}_- - m)\Big)u \qquad (6)\\
\langle\!\langle\text{ using } (\not{p}_- - m)u=0\,\rangle\!\rangle &\\
&=\; 2(p_-k_1)\times\bar{v}\gamma^\nu u,
\end{aligned}$$

where $(2p_-k_1)=k_1^2+p_-^2-(p_- - k_1)^2=0+m^2-t$. This factor cancels the denominator in the amplitude $\mathcal{M}_1^{\mu\nu}$ (*cf.* eq. (5)) except for the overall sign, and we are left with

$$\mathcal{M}_1^{\mu\nu}\times k_{1\mu} \;=\; +e^2\times\bar{v}\gamma^\nu u\,. \qquad (7)$$

Note the non-zero right hand side — the first diagram does not obey the Ward identity all by itself. As for the second diagram, we have

$$\begin{aligned}
\bar{v}\gamma^\mu(\tilde{\not{q}}+m)\gamma^\nu u\times k_{1\mu} \;&=\; \bar{v}\,\not{k}_1(\not{k}_1-\not{p}_+ +m)\gamma^\nu u\\
\langle\!\langle\text{ using } \not{k}_1\not{k}_1=k_1^2=0 \text{ and } \not{k}_1\not{p}_+=2(k_1p_+)-\not{p}_+\not{k}_1\,\rangle\!\rangle &\\
&=\; \bar{v}\Big(-2(p_+k_1)\;+\;(\not{p}_+ +m)\,\not{k}_1\Big)\gamma^\nu u\\
\langle\!\langle\text{ using } \bar{v}(\not{p}_+ +m)=0\,\rangle\!\rangle &\\
&=\; -2(p_+k_1)\times\bar{v}\gamma^\nu u,
\end{aligned} \qquad (8)$$

where $-2(p_+k_1)=(p_+-k_1)^2-p_+^2-k_1^2=u-0-m^2$. Again, this factor cancels the denominator of the $\mathcal{M}_2^{\mu\nu}$ amplitude (*cf.* eq. (5)), and we are left with

$$\mathcal{M}_2^{\mu\nu} \times k_{1\mu} \;=\; -e^2 \times \bar{v}\gamma^\nu u\,. \tag{9}$$

Similar to the first diagram's amplitude $\mathcal{M}_1^{\mu\nu}$, the second's diagram amplitude $\mathcal{M}_2^{\mu\nu}$ also does not obey the Ward identity all by itself. However, the right hand sides of eqs. (7) and (9) cancel each other, so the *net amplitude* does obey the Ward identity,

$$\mathcal{M}_{\rm net}^{\mu\nu} \times k_{1\mu} \;=\; \mathcal{M}_1^{\mu\nu} \times k_{1\mu} \;+\; \mathcal{M}_2^{\mu\nu} \times k_{1\mu} \;=\; 0. \tag{10}$$

This is an example of a general rule: The Ward identities does not work diagram by diagram, but only for sums of all diagrams related by permutations of photonic vertices on the same fermionic line — or for bigger sums, such as complete amplitudes to N-loop order for $N = 0, 1, 2, \ldots$.

The Ward identity $\mathcal{M}^{\mu\nu} \times k_{2\nu} = 0$ for the second photon works similarly to the first. In fact, thanks to the Bose symmetry (3) between the two photons, the two Ward identities are equivalent to each other,

$$\mathcal{M}^{\mu\nu} \;=\; \mathcal{M}^{\nu\mu}(k_1 \leftrightarrow k_2) \quad\Longrightarrow\quad \Bigl(\mathcal{M}^{\mu\nu} \times k_{1\mu} \;=\; 0 \quad\Longleftrightarrow\quad \mathcal{M}^{\mu\nu} \times k_{2\nu} \;=\; 0\Bigr). \tag{11}$$

Thus, for the second photon

$$\mathcal{M}_1^{\mu\nu} \times k_{2\nu} \;=\; -e^2 \times \bar{v}\gamma^\mu u \;\neq\; 0, \quad \mathcal{M}_2^{\mu\nu} \times k_{2\nu} \;=\; +e^2 \times \bar{v}\gamma^\mu u \;\neq\; 0, \quad \text{but} \quad \mathcal{M}_{\rm net}^{\mu\nu} \times k_{2\nu} \;=\; 0. \tag{12}$$

Summing over the Spins and Polarizations

In a typical annihilation experiment, the initial electrons and positrons come from un-polarized beams where both spin states are equally likely. Likewise, the photon detector is sensitive to the outgoing photons' momenta but it does not care about their polarization states. To calculate the annihilation cross-section for such un-polarized process, we should sum the $|\mathcal{M}|^2$ over the final photon polarizations and average over the spins of the initial fermions.

Summing the $|\mathcal{M}|^2$ over the photon polarizations is explained in detail in my notes on Ward identities. Thanks to the Ward identities, we can do it in terms of the $\mathcal{M}^{\mu\nu}$ amplitude as

$$\sum_{\lambda_1,\lambda_2} |\mathcal{M}|^2 \;=\; +\mathcal{M}^{\mu\nu}\mathcal{M}^*_{\mu\nu}\,. \tag{13}$$

For the annihilation process at hand $\mathcal{M}^{\mu\nu} = \mathcal{M}_1^{\mu\nu} + \mathcal{M}_2^{\mu\nu}$, so

$$\sum_{\lambda_1,\lambda_2} |\mathcal{M}|^2 \;=\; +\mathcal{M}^{\mu\nu}\mathcal{M}^*_{\mu\nu} \;=\; \mathcal{M}_1^{\mu\nu}\mathcal{M}^*_{1\mu\nu} \;+\; \mathcal{M}_2^{\mu\nu}\mathcal{M}^*_{2\mu\nu} \;+\; 2\,\mathrm{Re}\,\mathcal{M}_1^{\mu\nu}\mathcal{M}^*_{2\mu\nu}\,. \tag{14}$$

Note that this formula does not need the $\mathcal{M}_1^{\mu\nu}$ and $\mathcal{M}_2^{\mu\nu}$ amplitudes to obey the Ward identities by themselves, it is enough that the net amplitude $\mathcal{M}_1^{\mu\nu} + \mathcal{M}_2^{\mu\nu}$ obeys the identities. Specifically, for the $\mathcal{M}_1^{\mu\nu}$ and $\mathcal{M}_2^{\mu\nu}$ as in eqs. (5), we have

$$\begin{aligned}
\sum_{\lambda_1,\lambda_2} |\mathcal{M}|^2 \;=\;& \frac{e^4}{(t-m^2)^2} \times \bar{v}\gamma^\nu(\not{q}+m)\gamma^\mu u \times \bar{u}\gamma_\mu(\not{q}+m)\gamma_\nu v \\
&+\; \frac{e^4}{(u-m^2)^2} \times \bar{v}\gamma^\mu(\tilde{\not{q}}+m)\gamma^\nu u \times \bar{u}\gamma_\nu(\tilde{\not{q}}+m)\gamma_\mu v \\
&+\; \frac{2e^4}{(t-m^2)(u-m^2)} \times \mathrm{Re}\Big(\bar{v}\gamma^\nu(\not{q}+m)\gamma^\mu u \times \bar{u}\gamma_\nu(\tilde{\not{q}}+m)\gamma_\mu v\Big).
\end{aligned} \tag{15}$$

This formula takes care of summing over the photon polarizations, and now we need to average the result over the initial fermions' spins. As explained in my notes on Dirac traces, in general

$$\sum_{s_1,s_2} \bar{v}(p_+,s_+)\Gamma u(p_-,s_-) \times \bar{u}(p_-,s_-)\Gamma' v(p_+,s_+) \;=\; \mathrm{Tr}\Big((\not{p}_+ - m)\Gamma(\not{p}_- + m)\Gamma'\Big). \tag{16}$$

Consequently, averaging eq. (15) over the electron's and positron's spins gives us

$$\begin{aligned}
\overline{|\mathcal{M}|^2} \;\equiv\;& \tfrac14 \sum_{s_-,s_+}\sum_{\lambda_1,\lambda_2} |\mathcal{M}|^2 \\
=\;& \frac{e^4}{(t-m^2)^2} \times A_{11} \;+\; \frac{e^4}{(u-m^2)^2} \times A_{22} \;+\; \frac{2e^4}{(t-m^2)(u-m^2)} \times \mathrm{Re}\,A_{12}\,,
\end{aligned} \tag{17}$$

where

$$\begin{aligned}
A_{11} &= \tfrac{1}{4}\sum_{s_-,s_+} \bar{v}(p_+,s_+)\gamma^\nu(\not{q}+m)\gamma^\mu u(p_-,s_-)\times\bar{u}(p_-,s_-)\gamma_\mu(\not{q}+m)\gamma_\nu v(p_+,s_+)\\
&= \tfrac{1}{4}\,\mathrm{Tr}\Big((\not{p}_+-m)\gamma^\nu(\not{q}+m)\gamma^\mu(\not{p}_-+m)\gamma_\mu(\not{q}+m)\gamma_\nu\Big),\\
A_{22} &= \tfrac{1}{4}\sum_{s_-,s_+} \bar{v}(p_+,s_+)\gamma^\mu(\tilde{\not{q}}+m)\gamma^\nu u(p_-,s_-)\times\bar{u}(p_-,s_-)\gamma_\nu(\tilde{\not{q}}+m)\gamma_\mu v(p_+,s_+)\\
&= \tfrac{1}{4}\,\mathrm{Tr}\Big((\not{p}_+-m)\gamma^\mu(\tilde{\not{q}}+m)\gamma^\nu(\not{p}_-+m)\gamma_\nu(\tilde{\not{q}}+m)\gamma_\mu\Big),\\
A_{12} &= \tfrac{1}{4}\sum_{s_-,s_+} \bar{v}(p_+,s_+)\gamma^\nu(\not{q}+m)\gamma^\mu u(p_-,s_-)\times\bar{u}(p_-,s_-)\gamma_\nu(\tilde{\not{q}}+m)\gamma_\mu v(p_+,s_+)\\
&= \tfrac{1}{4}\,\mathrm{Tr}\Big((\not{p}_+-m)\gamma^\nu(\not{q}+m)\gamma^\mu(\not{p}_-+m)\gamma_\nu(\tilde{\not{q}}+m)\gamma_\mu\Big).
\end{aligned}\tag{18}$$

And now we need to calculate these big traces. . .

Traceology 1

Let's start with the A_{11} trace. It looks rather formidable, but we may simplify it using formulae

$$\gamma^\mu\gamma_\mu = 4,\quad \gamma^\mu\not{a}\gamma_\mu = -2\,\not{a},\quad \gamma^\mu\not{a}\not{b}\gamma_\mu = 4(ab),\quad \gamma^\mu\not{a}\not{b}\not{c}\gamma_\mu = -2\,\not{c}\not{b}\not{a}\tag{19}$$

from the homework#6. Indeed, after a cyclic permutation of matrices inside the trace, we obtain

$$A_{11} = \tfrac{1}{4}\,\mathrm{Tr}\Big(\gamma_\nu(\not{p}_+-m)\gamma^\nu\times(\not{q}+m)\times\gamma^\mu(\not{p}_-+m)\gamma_\mu\times(\not{q}+m)\Big)\tag{20}$$

where

$$\gamma^\mu(\not{p}_-+m)\gamma_\mu = -2(\not{p}_--2m),\qquad \gamma_\nu(\not{p}_+-m)\gamma^\nu = -2(\not{p}_++2m)\tag{21}$$

thanks to eq. (19), hence

$$A_{11} = \mathrm{Tr}\Big((\not{p}_++2m)(\not{q}+m)(\not{p}_--2m)(\not{q}+m)\Big).\tag{22}$$

Next, we expand the parentheses inside this trace and throw away terms with odd numbers of momenta $\not{p}$ or $\not{q}$. This gives us

$$\begin{aligned}
A_{11} &= \mathrm{Tr}(\not{p}_+\not{q}\,\not{p}_-\not{q}) \;+\; m^2\times\mathrm{Tr}(\not{p}_+\not{p}_-) \;-\; 4m^2\times\mathrm{Tr}(\not{q}\,\not{q}) \\
&\quad +\; 2\times 2m^2\times\mathrm{Tr}(\not{p}_-\not{q}) \;-\; 2\times 2m^2\times\mathrm{Tr}(\not{p}_+\not{q}) \;-\; 4m^4\times\mathrm{Tr}(1) \\
&= 2\times 4(p_+q)(p_-q) \;-\; 4(p_+p_-)(q^2) \;+\; m^2\times 4(p_+p_-) \;-\; 4m^2\times 4(q^2) \\
&\quad +\; 4m^2\times 4(p_-q) \;-\; 4m^2\times 4(p_+q) \;-\; 4m^4\times 4. \\
&= 8(p_+q)(p_-q) \;-\; 4(p_+p_-)\times(q^2-m^2) \;-\; 16m^2\times\left(q^2 \;-\; (p_-q) \;+\; (p_+q) \;+\; m^2\right).
\end{aligned} \tag{23}$$

We may further simplify this formula by expressing all the momenta products in terms of the Mandelstam's variables s, t, and u. Using $p_-^2=p_+^2=m^2$ and $k_1^2=k_2^2=0$, we have

$$q^2 \;=\; (p_- - k_1)^2 \;=\; t, \tag{24}$$

$$2p_-p_+ \;=\; (p_- + P_+)^2 \;-\; p_+^2 \;-\; p_-^2 \;=\; s \;-\; 2m^2, \tag{25}$$

$$2k_1p_- \;=\; k_1^2 \;+\; p_-^2 \;-\; (k_1-p_-)^2 \;=\; 0 \;+\; m^2 \;-\; t, \tag{26}$$

$$2k_2p_+ \;=\; k_2^2 \;+\; p_+^2 \;-\; (k_2-p_+)^2 \;=\; 0 \;+\; m^2 \;-\; t, \tag{27}$$

and hence

$$qp_- \;=\; (p_- - k_1)p_- \;=\; p_-^2 \;-\; p_-k_1 \;=\; m^2 \;+\; \tfrac{1}{2}(t-m^2) \;=\; +\tfrac{1}{2}(m^2+t), \tag{28}$$

$$qp_+ \;=\; (k_2 - p_+)p_+ \;=\; p_+k_2 \;-\; p_+^2 \;=\; -\tfrac{1}{2}(t-m^2) \;-\; m^2 \;=\; -\tfrac{1}{2}(t+m^2). \tag{29}$$

Consequently, on the last line of eq. (23), the last term vanishes —

$$q^2 \;-\; (p_-q) \;+\; (p_+q) \;+\; m^2 \;=\; t \;-\; \tfrac{1}{2}(t+m^2) \;-\; \tfrac{1}{2}(t+m^2) \;+\; m^2 \;=\; 0 \tag{30}$$

— while the remaining terms add up to

$$\begin{aligned}
A_{11} &= 8(p_+q)(p_-q) \;-\; 4(q^2-m^2)\times(p_+p_-) \\
&= -2(t+m^2)^2 \;-\; 2(t-m^2)\times(s-2m^2 \;=\; -t-u) \\
&= -2t^2 \;-\; 4t\,m^2 \;-\; 2m^4 \;+\; 2t^2 \;+\; 2tu \;-\; 2t\,m^2 -\; 2u\,m^2 \\
&= 2tu \;-6tm^2 \;-\; 2um^2 \;-\; 2m^4 \\
&= 2(t-m^2)\,(u-3m^2) \;-\; 8m^4.
\end{aligned} \tag{31}$$

This completes our evaluation of the first trace.

As to the second trace A_{22}, we could work it out through a similar calculation, but fortunately there is a shortcut. The two diagrams (1) for the annihilation process are related to each other by a crossing symmetry, which exchanges $t \leftrightarrow u$ and also $A_{11} \leftrightarrow A_{22}$. Consequently, given eq. (31) for the first trace, the second trace follows as

$$A_{22}(t,u) \;=\; A_{11}(t\leftrightarrow u) \;=\; 2(u-m^2)\,(t-3m^2) \;-\; 8m^4. \tag{32}$$

Traceology 2

Now consider the the third trace

$$A_{12} \;=\; \tfrac{1}{4}\,\mathrm{Tr}\Big(\gamma^\nu(\not{q}+m)\gamma^\mu(\not{p}_- + m)\times\gamma_\nu(\tilde{\not{q}}+m)\gamma_\mu(\not{p}_+ - m)\Big) \tag{33}$$

which accounts for the interference between the two diagrams (1). Again, this is a rather formidable trace, but we may simplify it using the relations

$$\gamma^\mu\gamma_\mu \;=\; 4,\quad \gamma^\mu\not{a}\gamma_\mu \;=\; -2\,\not{a},\quad \gamma^\mu\not{a}\not{b}\gamma_\mu \;=\; 4(ab),\quad \gamma^\mu\not{a}\not{b}\not{c}\gamma_\mu \;=\; -2\,\not{c}\not{b}\not{a}. \tag{19}$$

Indeed, consider the first 5 factors inside the trace A_{12}, from the first γ^ν to the second γ_ν through everything in-between:

$$\begin{aligned}
\gamma^\nu\times(\not{q}+m)\gamma^\mu(\not{p}_-+m)\times\gamma_\nu \;&=\; m^2\times\gamma^\nu\gamma^\mu\gamma_\nu \;+\; m\times\gamma^\nu(\not{q}\gamma^\mu+\gamma^\mu\not{p}_-)\gamma_\nu \;+\; \gamma^\nu(\not{q}\gamma^\mu\not{p}_-)\gamma_\nu\\
&\qquad\langle\!\langle\,\text{in light of eqs. (19)}\,\rangle\!\rangle\\
&=\; -2m^2\gamma^\mu \;+\; 4m(q+p_-)^\mu \;-\; 2\,\not{p}_-\gamma^\mu\not{q}\,.
\end{aligned} \tag{34}$$

Plugging this formula into eq. (33) for the A_{12}, we obtain

$$\begin{aligned}
A_{12} \;&=\; \tfrac{1}{4}\,\mathrm{Tr}\Big(\gamma^\nu(\not{q}+m)\gamma^\mu(\not{p}_-+m)\gamma_\nu\times(\tilde{\not{q}}+m)\gamma_\mu(\not{p}_+-m)\Big)\\
&=\; \mathrm{Tr}\left(\Big[m(q+p_-)^\mu \;-\; \tfrac{1}{2}\big(m^2\gamma^\mu+\not{p}_-\gamma^\mu\not{q}\big)\Big]\times\Big[m\big(\gamma_\mu\not{p}_+-\tilde{\not{q}}\gamma_\mu\big)+\big(\tilde{\not{q}}\gamma_\mu\not{p}_+ \;-\; m^2\gamma_\mu\big)\Big]\right)\\
&\qquad\langle\!\langle\,\text{throwing away products of odd numbers of }\gamma\text{ matrices}\,\rangle\!\rangle\\
&=\; \mathrm{Tr}\Big(m(q+p_-)^\mu\times m\big(\gamma_\mu\,\not{p}_+-\tilde{\not{q}}\gamma_\mu\big)\Big) \;-\; \tfrac{1}{2}\,\mathrm{Tr}\Big(\big(m^2\gamma^\mu+\not{p}_-\gamma^\mu\not{q}\big)\times\big(\tilde{\not{q}}\gamma_\mu\not{p}_+ \;-\; m^2\gamma_\mu\big)\Big)
\end{aligned} \tag{35}$$

where the two traces on the bottom line evaluate to

$$\begin{aligned}\mathrm{Tr}\Big(m(q+p_-)^\mu \times m\big(\gamma_\mu \not{p}_+ - \not{\tilde{q}}\gamma_\mu\big)\Big) \;&=\; m^2(q+p_-)^\mu \times \mathrm{Tr}\big((p_+ - \tilde{q})\gamma_\mu\big)\\ &=\; m^2(q+p_-)^\mu \times 4(p_+ - \tilde{q})_\mu\\ &=\; 4m^2\Big(-(q\tilde{q}) \,+\, (qp_+) \,-\, (\tilde{q}p_-) \,+\, (p_-p_+)\Big)\end{aligned} \tag{36}$$

and

$$\begin{aligned}&\mathrm{Tr}\Big(\big(m^2\gamma^\mu + \not{p}_-\gamma^\mu\not{q}\big) \times \big(\not{\tilde{q}}\gamma_\mu\not{p}_+ \,-\, m^2\gamma_\mu\big)\Big) \;=\\ &=\; \mathrm{Tr}\big(\not{p}_-\gamma^\mu\not{q}\not{\tilde{q}}\gamma_\mu\not{p}_+\big) \;+\; m^2\,\mathrm{Tr}\big(\gamma^\mu\not{\tilde{q}}\gamma_\mu\not{p}_+\big) \;-\; m^2\,\mathrm{Tr}\big(\not{p}_-\gamma^\mu\not{q}\gamma_\mu\big) \;-m^4\,\mathrm{Tr}\big(\gamma^\mu\gamma_\mu\big)\\ &\qquad \langle\!\langle\,\text{using } \gamma^\mu\not{q}\not{\tilde{q}}\gamma_\mu \;=\; 4(q\tilde{q}),\; \gamma^\mu\not{\tilde{q}}\gamma_\mu \;=\; -2\,\not{\tilde{q}},\; \gamma^\mu\not{q}\gamma_\mu \;=\; -2\,\not{q},\; \text{and } \gamma^\mu\gamma_\mu \;=\; 4\,\rangle\!\rangle\\ &=\; 4(q\tilde{q}) \times \mathrm{Tr}(\not{p}_-\,\not{p}_+) \;-\; 2m^2 \times \mathrm{Tr}(\not{\tilde{q}}\,\not{p}_+) \;+\; 2m^2 \times \mathrm{Tr}(\not{p}_-\,\not{q}) \;-4m^4 \times \mathrm{Tr}(1)\\ &=\; 16(q\tilde{q})(p_-p_+) -\; 8m^2(\tilde{q}p_+) \;+\; 8m^2(qp_-) \;-\; 16m^4.\end{aligned} \tag{37}$$

Combining the two traces, we arrive at

$$A_{12} \;=\; -8(q\tilde{q})(p_-p_+) + 4m^2\big(-(q\tilde{q}) + (qp_+) - (qp_-) + (\tilde{q}p_+) - (\tilde{q}p_-) + (p_-p_+)\big) + 8m^4. \tag{38}$$

We may simplify this rather messy formula by expressing all the momenta products in terms of the Mandelstam variables s, t, u. Back in eqs. (24) through (29) we saw that

$$q^2 \;=\; t, \quad (qp_-) \;=\; +\tfrac{1}{2}(t+m^2), \quad (qp_+) \;=\; -\tfrac{1}{2}(t+m^2), \quad \text{and} \quad (p_-p_+) \;=\; \tfrac{1}{2}(s-2m^2), \tag{39}$$

and now we also need

$$\tilde{q}^2 \;=\; (k_1 - p_+)^2 \;=\; u, \tag{40}$$

$$2k_1p_+ \;=\; k_1^2 \;+\; p_+^2 \;-\; (k_1 - p_+)^2 \;=\; 0 \;+\; m^2 \;-\; u, \tag{41}$$

$$2k_2p_- \;=\; k_2^2 \;+\; p_-^2 \;-\; (k_2 - p_-)^2 \;=\; 0 \;+\; m^2 \;-\; u, \tag{42}$$

and hence

$$\tilde{q}p_- \;=\; (p_- - k_2)p_- \;=\; p_-^2 \;-\; k_2p_- \;=\; m^2 \;-\; \tfrac{1}{2}(m^2 - u) \;=\; +\tfrac{1}{2}(u + m^2), \tag{43}$$

$$
\begin{aligned}
\tilde{q}p_+ \;&=\; (k_1-p_+)p_+ \;=\; k_1p_+ \,-\, p_+^2 \;=\; \tfrac{1}{2}(m^2-u) \,-\, m^2 \;=\; -\tfrac{1}{2}(u+m^2), && (44)\\
\tilde{q}q \;&=\; (p_- - k_2)(p_- - k_1) \;=\; p_-^2 \,-\, p_-(k_1+k_2=p_-+p_+) \,+\, k_1k_2 \\
&=\; k_1k_2 \,-\, p_-p_+ \;=\; \tfrac{1}{2}s \,-\tfrac{1}{2}(s-2m^2) \;=\; m^2. && (45)
\end{aligned}
$$

Plugging all these formulae into eq. (38), we finally arrive at

$$
\begin{aligned}
A_{12} \;&=\; -8m^2\times(\tfrac{1}{2}s-m^2) \,+\, 8m^4 \\
&\quad +\, 4m^2\times\Big(-m^2 \,-\, \tfrac{1}{2}(t+m^2)\times 2 \,-\tfrac{1}{2}(u+m^2)\times 2 \,+\, (\tfrac{1}{2}s-m^2)\Big) \\
&=\; -4m^2s \,+\, 16m^4 \,+\, 4m^2\times\Big(-4m^2 \,-\, t \,-\, u \,+\, \tfrac{1}{2}s\Big) \\
&=\; -2m^2\times(2t+2u+s) \\
&=\; -2m^2\times(t+u+2m^2) \\
&=\; -2m^2(t-m^2) \,-\, 2m^2(u-m^2) \,-\, 8m^4.
\end{aligned} \tag{46}
$$

Annihilation Summary

Having worked out the big traces, let's plug them back into eq. (17):

$$
\begin{aligned}
\overline{|\mathcal{M}|^2} \;&=\; \frac{e^4}{(t-m^2)^2}\times\Big(2(t-m^2)(u-3m^2) \,-\, 8m^4\Big) \\
&\quad +\, \frac{e^4}{(u-m^2)^2}\times\Big(2(u-m^2)(t-3m^2) \,-\, 8m^4\Big) \\
&\quad +\, \frac{2e^4}{(t-m^2)(u-m^2)}\times\Big(-2m^2(t-m^2) \,-\, 2m^2(u-m^2) \,-\, 8m^4\Big) \\
&=\; 2e^4\left[\begin{aligned}&\frac{u-3m^2}{t-m^2} + \frac{t-3m^2}{u-m^2} - \frac{2m^2}{u-m^2} - \frac{2m^2}{t-m^2} \\ &\qquad - \frac{4m^4}{(t-m^2)^2} - \frac{4m^4}{(u-m^2)^2} - \frac{8m^4}{(t-m^2)(u-m^2)}\end{aligned}\right] \\
&=\; 2e^4\left[\begin{aligned}&\frac{u-m^2}{t-m^2} + \frac{t-m^2}{u-m^2} - 4m^2\left(\frac{1}{t-m^2}+\frac{1}{u-m^2}\right) \\ &\qquad - 4m^4\left(\frac{1}{t-m^2}+\frac{1}{u-m^2}\right)^2\end{aligned}\right],
\end{aligned} \tag{47}
$$

or more compactly

$$\overline{|\mathcal{M}|^2} \;=\; 2e^4 \left[\frac{u-m^2}{t-m^2} \,+\, \frac{t-m^2}{u-m^2} \,+\, 1 \;-\; \left(1 \,+\, \frac{2m^2}{t-m^2} \,+\, \frac{2m^2}{u-m^2}\right)^2\right]. \tag{48}$$

This is our final result; the rest is kinematics.

Annihilation Kinematics

In the center of mass frame, $p^\mu_\mp = (E, \pm\mathbf{p})$ where $E = +\sqrt{\mathbf{p}^2 + m^2}$, and $k^\mu_{1,2} = (\omega, \pm\mathbf{k})$ where $\omega = |\mathbf{k}| = E$. Consequently,

$$\begin{aligned}
s &= 4E^2, \\
t &= -(\mathbf{p}-\mathbf{k})^2 \;=\; -\mathbf{p}^2 \,-\, E^2 \,+\, 2|\mathbf{p}|E\,\cos\theta, \\
u &= -(\mathbf{p}+\mathbf{k})^2 \;=\; -\mathbf{p}^2 \,-\, E^2 \,-\, 2|\mathbf{p}|E\,\cos\theta, \\
t-m^2 &= -2E(E-|\mathbf{p}|\cos\theta), \\
u-m^2 &= -2E(E+|\mathbf{p}|\cos\theta),
\end{aligned} \tag{49}$$

and therefore

$$\begin{aligned}
\frac{u-m^2}{t-m^2} \,+\, \frac{t-m^2}{u-m^2} \,+\, 1 &= \frac{E+|\mathbf{p}|\cos\theta}{E-|\mathbf{p}|\cos\theta} \,+\, \frac{E-|\mathbf{p}|\cos\theta}{E+|\mathbf{p}|\cos\theta} \,+\, 1 \\
&= \frac{3E^2+\mathbf{p}^2\cos^2\theta}{E^2-\mathbf{p}^2\cos^2\theta} \\
&= \frac{3m^2+\mathbf{p}^2(3+\cos^2\theta)}{m^2+\mathbf{p}^2\sin^2\theta}\,, \\
\frac{1}{t-m^2} \,+\, \frac{1}{u-m^2} &= \frac{-1}{2E}\left(\frac{1}{E-|\mathbf{p}|\cos\theta} \,+\, \frac{1}{E+|\mathbf{p}|\cos\theta}\right) \\
&= \frac{-1}{2E} \times \frac{2E}{E^2-\mathbf{p}^2\cos^2\theta} \;=\; \frac{-1}{m^2+\mathbf{p}^2\sin^2\theta}\,, \\
1 \,+\, \frac{2m^2}{t-m^2} \,+\, \frac{2m^2}{u-m^2} &= \frac{\mathbf{p}^2\sin^2\theta-m^2}{\mathbf{p}^2\sin^2\theta+m^2}\,.
\end{aligned} \tag{50}$$

Altogether

$$\overline{|\mathcal{M}|^2} \;=\; 2e^4\left[\frac{3m^2+\mathbf{p}^2(3+\cos^2\theta)}{m^2+\mathbf{p}^2\sin^2\theta}\;-\;\left(\frac{\mathbf{p}^2\sin^2\theta-m^2}{\mathbf{p}^2\sin^2\theta+m^2}\right)^2\right], \tag{51}$$

and hence the partial cross section of annihilation

$$\frac{d\sigma(e^+e^-\to\gamma\gamma)}{d\Omega_{\text{c.m.}}} \;=\; \frac{|\mathbf{k}|}{|\mathbf{p}|}\frac{\overline{|\mathcal{M}|^2}}{64\pi^2 s} \;=\; \frac{\alpha^2}{8E|\mathbf{p}|}\times\left[\frac{3m^2+\mathbf{p}^2(3+\cos^2\theta)}{m^2+\mathbf{p}^2\sin^2\theta}\;-\;\left(\frac{\mathbf{p}^2\sin^2\theta-m^2}{\mathbf{p}^2\sin^2\theta+m^2}\right)^2\right]. \tag{52}$$

For the non-relativistic electron and positron with $|\mathbf{p}|\ll m$, the expression in the square brackets becomes $3-(-1)^2=2$, hence *isotropic* partial cross section

$$\frac{d\sigma(\text{slow } e^+e^-\to\gamma\gamma)}{d\Omega_{\text{c.m.}}} \;=\; \frac{\alpha^2}{4m|\mathbf{p}|}\,. \tag{53}$$

And the total cross section in this limit is

$$\sigma_{\text{tot}}(\text{slow } e^+e^-\to\gamma\gamma) \;=\; \frac{4\pi}{2}\times\frac{\alpha^2}{4m|\mathbf{p}|} \;=\; \frac{\pi\alpha^2}{2m|\mathbf{p}|}\,, \tag{54}$$

where the total solid angle is $4\pi/2$ because of 2 identical photons in the final state.

In the opposite limit of ultra-relativistic e^- and e^+ with $|\mathbf{p}|\approx E\gg m$, we have

$$\left[\cdots\right] \;\approx\; \frac{3+\cos^2\theta}{\sin^2\theta}\;-\;1 \;=\; \frac{2(1+\cos^2\theta)}{\sin^2\theta} \tag{55}$$

and hence highly anisotropic cross section

$$\frac{d\sigma(\text{fast } e^+e^-\to\gamma\gamma)}{d\Omega_{\text{c.m.}}} \;\approx\; \frac{\alpha^2}{4E^2}\times\frac{1+\cos^2\theta}{\sin^2\theta}\,. \tag{56}$$

Note how this cross-section is strongly peaked in the forward direction $\theta=0$ where one photon continues the electron's motion while the other continues the positron's motion.

According to eq. (56), the total annihilation cross-section

$$\sigma_{\text{tot}}(\text{fast } e^+e^- \to \gamma\gamma) \;=\; 2\pi \int_0^{\pi/2} d\theta\, \sin\theta\, \frac{d\sigma}{d\Omega_{\text{cm}}} \tag{57}$$

diverges at small angles, but that's an artefact of the approximation (55) becoming inaccurate at small angles where $\mathbf{p}^2 \sin^2\theta \lesssim m^2$. Instead, for small angles we have

$$\Big[\cdots\Big] \;=\; \frac{4\mathbf{p}^2}{m^2 + \mathbf{p}^2\theta^2} \;+\; O(1) \tag{58}$$

and consequently

$$\frac{d\sigma(\text{fast } e^+e^- \to \gamma\gamma)}{d\Omega_{\text{c.m.}}} \;\approx\; \frac{\alpha^2}{2E^2} \times \left(\frac{\mathbf{p}^2}{m^2 + \mathbf{p}^2\theta^2} \;+\; O(1) \right). \tag{59}$$

This cross-section is strongly peaked in the forward direction, but it does not diverge; instead, it integrates to

$$\sigma_{\text{tot}}(\text{fast } e^+e^- \to \gamma\gamma) \;=\; \frac{\pi\alpha^2}{E^2} \times \left(\log\frac{E}{m} \;+\; O(1) \right). \tag{60}$$

More accurately, using an approximation valid at both small and large angles,

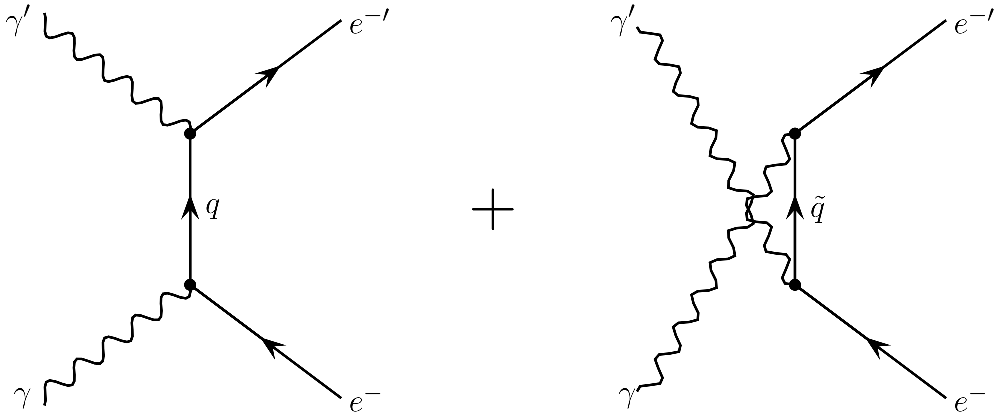
$$\frac{d\sigma(\text{fast } e^+e^- \to \gamma\gamma)}{d\Omega_{\text{c.m.}}} \;\approx\; \frac{\alpha^2}{2E^2} \left(\frac{2}{(m/E)^2 + \sin^2\theta} \;-\; 1 \right), \tag{61}$$

we get the total cross-section

$$\sigma_{\text{tot}}(\text{fast } e^+e^- \to \gamma\gamma) \;=\; \frac{\pi\alpha^2}{E^2} \times \left(\log\frac{2E}{m} \;-\; \frac{1}{2} \;+ O(m^2/E^2) \right). \tag{62}$$

Compton Scattering

Compton scattering of an electron and a photon $e^-\gamma \to e^-\gamma$ is related by crossing symmetry to the $e^-e^+ \to \gamma\gamma$ annihilation. Indeed, at the tree level there are two diagrams



(63)

which are obviously related by the $s \leftrightarrow t$ crossing to the annihilation diagrams (1). Hence, given eq. (48) for the annihilation, we may immediately write down a similar formula for the Compton scattering without doing any work. All we need is to exchange $s \leftrightarrow t$ in eq. (48) and change the overall sign because we cross one fermion, thus

$$\overline{|\mathcal{M}^{\text{Compton}}|^2} \;=\; 2e^4\left[-\frac{u-m^2}{s-m^2} - \frac{s-m^2}{u-m^2} - 1 + \left(1+\frac{2m^2}{s-m^2}+\frac{2m^2}{u-m^2}\right)^2\right]. \tag{64}$$

This is it, except for the kinematics.

The Compton scattering is usually studied in the lab frame where the initial electron is at rest, $p^\mu = (m, \mathbf{0})$. In this frame, the initial and the final photon energies ω and ω' are related to photon's scattering angle θ via the Compton's formula

$$\frac{1}{\omega'} \;=\; \frac{1}{\omega} + \frac{1-\cos\theta}{m_e}\,, \tag{65}$$

originally written by Arthur Compton in terms of the photon's wavelengths as

$$\lambda' - \lambda \;=\; \frac{2\pi\hbar}{m_e c}\times(1-\cos\theta). \tag{66}$$

According to this formula, there is an upper limit on the energy of the final photon for any *fixed* $\theta \neq 0$: regardless of the initial energy ω, the final energy ω' can never exceed $m_e/(1-\cos\theta)$.

The Compton's formula follows from the energy–momentum conservation $p' = p + k - k'$, which leads to

$$p'^2 \;=\; (p + k - k')^2 \;=\; p^2 \;+\; k^2 \;+\; k'^2 \;+\; 2pk \;-\; 2pk' \;-\; 2kk'. \tag{67}$$

In light of the mass-shell considtions $p'^2 = p^2 = m^2$ and $k'^2 = k^2 = 0$, this means

$$2pk \;-\; 2pk' \;-\; 2kk \;=\; 0. \tag{68}$$

In the lab frame $pk = m\omega$, $pk' = m\omega'$, while $kk' = \omega\omega' - \mathbf{k}\cdot\mathbf{k}' = \omega\omega'(1-\cos\theta)$, so eq. (68) becomes

$$2m\omega \;-\; 2m\omega' \;-\; 2\omega\omega'(1-\cos\theta) \;=\; 0, \tag{69}$$

and after dividing every term by $2\omega\omega' m$ we get the Compton formula

$$\frac{1}{\omega'} \;-\; \frac{1}{\omega} \;-\; \frac{1-\cos\theta}{m} \;=\; 0. \tag{65}$$

The Mandelstam variables s and u in the lab frame are

$$\begin{aligned} s \;&\equiv\; (k+p)^2 \;=\; (\omega+m)^2 \;-\; (\mathbf{k}+\mathbf{0})^2 \;=\; 2\omega m \;+\; m^2, \\ u \;&\equiv\; (k'-p)^2 \;=\; (\omega'-m)^2 \;-\; (\mathbf{k}'-\mathbf{0})^2 \;=\; -2\omega' m \;+\; m^2, \end{aligned} \tag{70}$$

and hence

$$s \,-\, m^2 \;=\; +2m\omega, \qquad u \,-\, m^2 \;=\; -2m\omega'. \tag{71}$$

Plugging these values into eq. (64), we have

$$-\frac{u-m^2}{s-m^2} - \frac{s-m^2}{u-m^2} \;=\; +\frac{\omega'}{\omega} + \frac{\omega}{\omega'}\,, \tag{72}$$

$$\frac{2m^2}{s-m^2} + \frac{2m^2}{u-m^2} = \frac{m}{\omega} - \frac{m}{\omega'}$$

$$\langle\!\langle \text{ by the Compton formula } \rangle\!\rangle$$

$$= -(1-\cos\theta), \tag{73}$$

$$-1 + \left(1 + \frac{2m^2}{s-m^2} + \frac{2m^2}{u-m^2}\right)^2 = -1 + \big(1 - 1 + \cos\theta\big)^2$$

$$= -1 + \cos^2\theta = -\sin^2\theta, \tag{74}$$

and therefore

$$\overline{|\mathcal{M}^{\text{Compton}}|^2} = 2e^4 \times \left(\frac{\omega'}{\omega} + \frac{\omega}{\omega'} - \sin^2\theta\right). \tag{75}$$

Finally, we need the phase space factor for the lab frame. For a generic $2 \to 2$ scattering process,

$$\begin{aligned} d\sigma &= \overline{|\mathcal{M}|^2} \times d\mathcal{P}, \quad \text{where} \\ d\mathcal{P} &= \frac{1}{2E_1 \times 2E_2 \times \Delta v} \times \frac{d^3\mathbf{p}'_1}{(2\pi)^3\, 2E'_1} \times \frac{d^3\mathbf{p}'_3}{(2\pi)^2\, 2E'_2} \times (2\pi)^4 \delta^{(4)}(p'_1 + p'_2 - p_1 - p_2) \\ &= \frac{1}{64\pi^2\, E_1 E_2 E'_1 E'_2\, \Delta v} \times d^3\mathbf{p}'_1\ \delta(E'_1 + E'_2 - E_1 - E_2)\big|_{\mathbf{p}'_2 = \mathbf{p}_1 + \mathbf{p}_2 - \mathbf{p}'_1} \\ &= \frac{d\Omega'_1}{64\pi^2} \times \frac{\mathbf{p}'^2_1}{E_1 E_2 E'_1 E'_2\, \Delta v} \times \left(\frac{d(E'_1 + E'_2)}{d|\mathbf{p}'_1|}\bigg|_{\mathbf{p}'_2 = \mathbf{p}_1 + \mathbf{p}_2 - \mathbf{p}'_1}\right)^{-1}. \end{aligned} \tag{76}$$

Specializing to the Compton scattering and the lab frame for initial electron, we immediately obtain

$$d\mathcal{P} = \frac{d\Omega_\gamma}{64\pi^2} \times \frac{\omega'^2}{\omega m \omega' E'} \times \left(1 + \frac{dE'}{d\omega'}\bigg|_{\mathbf{p}' = \mathbf{k} - \mathbf{k}'}\right)^{-1}. \tag{77}$$

The only non-trivial issue here is the derivative in the parentheses. This derivative should be taken for a fixed photon angle θ and before applying the energy conservation rule $E'_e = \omega + m - \omega'$. Instead, we use the momentum conservation $\mathbf{p}' = \mathbf{k} - \mathbf{k}'$, hence

$$\begin{aligned} \mathbf{p}'^2 &= \mathbf{k}^2 + \mathbf{k}'^2 - 2\mathbf{k}\cdot\mathbf{k}' = \omega^2 + \omega'^2 - 2\omega\omega'\cos\theta, \\ E'^2 &= \mathbf{p}'^2 + m^2 = m^2 + \omega^2 + \omega'^2 - 2\omega\omega'\cos\theta, \end{aligned} \tag{78}$$

Taking the derivative WRT ω' at fixed ω and θ, we get

$$2E' \times dE' \;=\; 2(\omega' - \omega\cos\theta) \times d\omega' \quad\Longrightarrow\quad \frac{dE'}{d\omega'} \;=\; \frac{\omega' - \omega\cos\theta}{E'}\,. \tag{79}$$

Once we have taken this derivative, me may use the energy conservation, thus

$$1 \;+\; \frac{dE'}{d\omega'} \;=\; \frac{E' + \omega' - \omega\cos\theta}{E'} \;=\; \frac{m + \omega - \omega\cos\theta}{E'} \;=\; \frac{\omega m}{\omega' E'}\,, \tag{80}$$

where the last equality follows from the Compton formula (65). Plugging the derivative (80) into eq. (77), we arrive at

$$d\mathcal{P} \;=\; \frac{d\Omega_\gamma}{64\pi^2} \times \frac{\omega'^2}{m^2\omega^2} \tag{81}$$

and hence the *Klein–Nishina formula* for the partial cross-section:

$$\frac{d\sigma^{\text{Compton}}}{d\Omega_{\text{lab}}} \;=\; \frac{\alpha^2}{2m_e^2} \times \frac{\omega'^2}{\omega^2} \times \left(\frac{\omega'}{\omega} \,+\, \frac{\omega}{\omega'} \,-\, \sin^2\theta\right) \tag{82}$$

where ω' is given by the Compton formula (65).

For low photon energies $\omega \ll m_e$, the Compton's formula gives $\omega' \approx \omega$, and the Klein–Nishina cross-section (82) becomes the good old Thompson cross-section

$$\frac{d\sigma^{\text{Compton}}}{d\Omega_{\text{lab}}} \;\to\; \frac{d\sigma^{\text{Thompson}}}{d\Omega_{\text{lab}}} \;=\; \frac{\alpha^2}{2m_e^2} \times (2 - \sin^2\theta \;=\; 1 + \cos^2\theta), \tag{83}$$

and the total cross-section is

$$\sigma_{\text{total}}^{\text{Thompson}} \;=\; \frac{8\pi}{3}\,\frac{\alpha^2}{m_e^2} \;\approx\; 0.663\,\text{barn}. \tag{84}$$

On the other hand, for very high photon energies $\omega \gg m_e$ and $\theta \not\approx 0$, we have

$$\omega' \;\ll\; \omega \quad\Longrightarrow\quad \frac{\omega'}{\omega} \,+\, \frac{\omega}{\omega'} \,-\, \sin^2\theta \;\approx\; \frac{\omega}{\omega'}\,, \tag{85}$$

and the Klein–Nishina formula becomes

$$\frac{d\sigma^{\text{Compton}}}{d\Omega_{\text{lab}}} \;\approx\; \frac{\alpha^2}{2m_e^2} \times \frac{\omega'}{\omega} \;\approx\; \frac{\alpha^2}{2m_e \times \omega} \times \frac{1}{1 - \cos\theta}\,. \tag{86}$$

This approximation is not accurate at small angles $\theta \lesssim \sqrt{2m_e/\omega}$ for which $\omega' \not\ll \omega$, so the cross section does not really diverge for $\theta \to 0$. Instead, at small angles we have large but finite partial cross-section

$$\frac{d\sigma^{\text{Compton}}}{d\Omega_{\text{lab}}} \approx \frac{\alpha^2}{m_e \times \omega} \times \frac{\theta^4 - 2\theta^2(2m_e/\omega) + 2(2m_e/\omega)^2}{(\theta^2 + (2m_e/\omega))^3} \not\to \infty \tag{87}$$

and hence finite total cross-section

$$\sigma^{\text{Compton}}_{\text{total}} \approx \frac{\pi\alpha^2}{m_e \times \omega} \times \left(\log\frac{2\omega}{m_e} + \frac{1}{2}\right). \tag{88}$$