Poisson Brackets and Commutator Brackets

Both classical mechanics and quantum mechanics use bi-linear brackets of variables with similar algebraic properties. In classical mechanics the variables are functions of the canonical coordinates and momenta, and the Poisson bracket of two such variables $A(q,p)$ and $B(q,p)$ are defined as

$$[A,B]_P \stackrel{\text{def}}{=} \sum_i \left(\frac{\partial A}{\partial q_i} \frac{\partial B}{\partial p_i} - \frac{\partial A}{\partial p_i} \frac{\partial B}{\partial q_i} \right). \tag{1}$$

In quantum mechanics the variables are linear operators in some Hilbert space, and the commutator bracket of two operators is

$$[A,B]_C \stackrel{\text{def}}{=} AB - BA. \tag{2}$$

Both types of brackets have similar algebraic properties:

1. Linearity: $[\alpha_1 A_1 + \alpha_2 A_2, B] = \alpha_1 [A_1, B] + \alpha_2 [A_2, B]$ and $[A, \beta_1 B_1 + \beta_2 B_2] = \beta_1 [A, B_1] + \beta_2 [A, B_2]$.
2. Antisymmetry: $[A,B] = -[B,A]$.
3. Leibniz rules: $[AB, C] = A[B,C] + [A.C]B$ and $[A, BC] = B[A,C] + [A,B]C$.
4. Jacobi Identity: $[A,[B,C]] + [B,[C,A]] + [C,[A,B]] = 0$.

Also, both types of brackets involving the Hamiltonian can be used to describe the time dependence of the classical/quantum variables. In classical mechanics,

$$\begin{aligned} \frac{d}{dt} A(q,p) &= \sum_i \left(\frac{\partial A}{\partial q_i} \frac{dq_i}{dt} + \frac{\partial A}{\partial p_i} \frac{dp_i}{dt} \right) \\ &\qquad \langle\!\langle \text{ by the Hamilton equations } \rangle\!\rangle \\ &= \sum_i \left(\frac{\partial A}{\partial q_i} \frac{\partial H}{\partial p_i} - \frac{\partial A}{\partial p_i} \frac{\partial H}{\partial q_i} \right) \\ &\equiv [A,H]_P\,, \end{aligned} \tag{3}$$

while in quantum mechanics we have

$$i\hbar \frac{d}{dt} \langle\psi|\,\hat{A}\,|\psi\rangle = \langle\psi|\,[\hat{A},\hat{H}]_C\,|\psi\rangle \tag{4}$$

(the Heisenberg–Dirac equation); in particular, in the Heisenberg picture of QM

$$i\hbar\frac{d}{dt}\hat{A} \;=\; [\hat{A},\hat{H}]_C\,. \tag{5}$$

The similarity between the classical Poisson brackets and the quantum commutator brackets stems from the following theorem: Once we generalize the Poisson brackets to the non-commuting variables of quantum mechanics, they become proportional to the commutator brackets,

$$[\hat{A},\hat{B}]_P \;=\; \frac{\hat{A}\hat{B}-\hat{B}\hat{A}}{i\hbar}\,. \tag{6}$$

Mathematically speaking: *for any non-commutative but associative variables, any bracket* $[A,B]$ *with the algebraic properties 1–4 is proportional to the commutator bracket:*

$$[A,B] \;=\; c(AB-BA) \tag{7}$$

for a universal constant c (same c for all variables); in Physics $c=1/i\hbar$.

Proof: Take any 4 variables A,B,U,V and calculate $[AU,BV]$ using the Leibniz rules, first for the AU and then for the BV:

$$\begin{aligned} [AU,BV] \;&=\; A[U,BV] \;+\; [A,BV]U \\ &=\; AB[U,V] \;+\; A[U,B]V \;+\; B[A,V]U \;+\; [A,B]VU. \end{aligned} \tag{8}$$

OOH, if we use the two Leibniz rules in the opposite order we get a different expression

$$\begin{aligned} [AU,BV] \;&=\; B[AU,V] \;+\; [AU,B]V \\ &=\; BA[U,V] \;+\; B[A,V]U \;+\; A[U,B]V \;+\; [A,B]UV. \end{aligned} \tag{9}$$

To make sure the two expressions are equal to each other we need

$$\begin{gathered} AB[U,V] \;+\; [A,B]VU = \; BA[U,V] \;+\; [A,B]UV \\ \Big\Downarrow \\ (AB-BA)[U,V] \;=\; [A,B](UV-VU) \qquad (10) \\ \Big\Downarrow \\ [U,V](UV-VU)^{-1} \;=\; (AB-BA)^{-1}[A,B] \end{gathered}$$

On the last line here, the LHS depends only on the U and V while the RHS depends only on the A and B, and the only way a relation like that can work for any *unrelated* variables is if the ratios on both sides of equations are equal to the same universal constant c, thus

$$[A, B] \;=\; c(AB - BA) \quad \text{and} \quad [U, V] \;=\; c(UV - VU). \tag{11}$$

Quod erat demonstrandum.

Thanks to this theorem, we may quantize a classical theory described in terms of non-canonical variables $\xi_1, \ldots, \xi_{2N}$ (instead of the canonical $q_1, \ldots, q_N$ and $p_1, \ldots, p_N$) as long as we have a consistent algebra of Poisson brackets. (Their definition would be different from eqs. (1), but they have to obey the algebraic rules 1–4.) Given the classical Poisson algebra, the quantization maps it to the commutator algebra of operators in some Hilbert space. That is, if classically $[A, B]_P = C$, then the corresponding operators in quantum mechanics should obey $[\hat{A}, \hat{B}] = i\hbar\hat{C}$.

In particular, if we do have classical canonical variables q_i and p_i, then

$$[q_i, q_j]_P \;=\; 0, \quad [p_i, p_j]_P \;=\; 0, \quad , [q_i, p_j]_P \;=\; \delta_{ij}\,, \tag{12}$$

so the corresponding quantum operators should obey the *canonical commutation relations*

$$[\hat{q}_i, \hat{q}_j]_C \;=\; 0, \quad [\hat{p}_i, \hat{p}_j]_C \;=\; 0, \quad [\hat{q}_i, \hat{p}_j]_C \;=\; i\hbar\delta_{ij}\,. \tag{13}$$