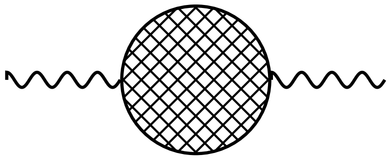
EM Field renormalization: the $\Sigma^{\mu\nu}(k)$ at One Loop

In QED, the two-photon 1PI bubble

$$i\Sigma^{\mu\nu}(k) \;=\; \tag{1}$$



should satisfy Ward identities

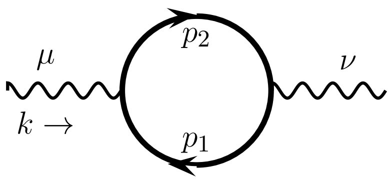
$$k_\mu \times \Sigma^{\mu\nu}(k) \;=\; 0, \quad k_\nu \times \Sigma^{\mu\nu}(k) \;=\; 0. \tag{2}$$

Between these identities and the Lorentz symmetry, the k–dependence of the $\Sigma^{\mu\nu}(k)$ should have form

$$\Sigma^{\mu\nu}(k) \;=\; \Pi(k^2) \times \left(g^{\mu\nu} \times k^2 \;-\; k^\mu k^\nu\right). \tag{3}$$

In this section, we calculate the $\Sigma^{\mu\nu}(k)$ to one-loop order and verify that it indeed has this form.

In pure QED, there is only one 1–loop diagram with 2 photonic external legs, namely



$$\tag{4}$$

which gives us

$$i\Sigma^{\mu\nu}_{1\,\text{loop}}(k) \;=\; -\int\frac{d^4p_1}{(2\pi)^4}\,\text{tr}\left[(ie\gamma^\nu)\,\frac{i}{\not{p}_1 - m + i0}\,(ie\gamma^\mu)\,\frac{i}{\not{p}_2 - m + i0}\right] \tag{5}$$

where $p_2 \equiv p_1 + k$ and the overall minus sign comes from the closed electron loop. To evaluate the trace here, we use

$$\frac{1}{\not{p} - m + i0} \;=\; \frac{\not{p} + m}{p^2 - m^2 + i0}\,, \tag{6}$$

hence

$$\text{tr}[\cdots] \;=\; e^2 \times \text{tr}\left[\gamma^\nu \times \frac{\not{p}_1 + m}{p_1^2 - m^2 + i0} \times \gamma^\mu \times \frac{\not{p}_1 + m}{p_2^2 - m^2 + i0}\right] \;=\; \frac{e^2\mathcal{N}^{\mu\nu}}{\mathcal{D}} \tag{7}$$

where

$$\mathcal{D} \;=\; (p_1^2 - m^2 + i0) \times (p_2^2 - m^2 + i0) \tag{8}$$

and

$$\begin{aligned}
\mathcal{N}^{\mu\nu} \;&=\; \mathrm{tr}\Big(\gamma^\nu(\not{p}_2 + m)\gamma^\mu(\not{p}_1 + m)\Big) \\
&=\; \mathrm{tr}\big(\gamma^\nu\not{p}_2\gamma^\mu\not{p}_1\big) \;+\; m^2\,\mathrm{tr}\big(\gamma^\nu\gamma^\mu\big) \\
&=\; 4p_1^\mu p_2^\nu \;+\; 4p_1^\nu p_2^\mu \;-\; 4(p_1p_2)g^{\mu\nu} \;+\; 4m^2 g^{\mu\nu}.
\end{aligned} \tag{9}$$

In the denominator $\mathcal{D}$, we may combine the two factors using the Feynman's parameter trick, thus

$$\begin{aligned}
\frac{1}{\mathcal{D}} \;&=\; \int_0^1 dx\,\frac{1}{[(1-x)\times(p_1^2 - m^2 + i0) + x\times(p_2^2 - m^2 + i0)]^2} \\
\text{where } [\cdots] \;&=\; (1-x)\times p_1^2 \;+\; x\times(p_2 \equiv p_1 + k)^2 \;-\; m^2 \;+\; i0 \\
&=\; p_1^2 \;+\; 2x\times(p_1k) \;+\; x\times k^2 \;-\; m^2 \;+\; i0 \\
&=\; (p_1 + xk)^2 \;+\; (x - x^2)\times k^2 \;-\; m^2 \;+\; i0 \\
&\equiv\; p^2 \;-\; \Delta(x) \;+\; i0
\end{aligned} \tag{10}$$

in terms of $p \stackrel{\text{def}}{=} p_1 + xk$ and

$$\Delta(x) \;\stackrel{\text{def}}{=}\; m^2 \;-\; x(1-x)\times k^2. \tag{11}$$

Altogether,

$$\text{the trace } = \int_0^1 dx\,\frac{e^2\mathcal{N}^{\mu\nu}}{[p^2 - \Delta(x) + i0]^2} \tag{12}$$

and therefore

$$\Sigma^{\mu\nu}_{1\,\text{loop}}(k) \;=\; ie^2\int\frac{d^4p_1}{(2\pi)^4}\int_0^1 dx\,\frac{\mathcal{N}^{\mu\nu}}{[(p_1 + xk)^2 - \Delta + i0]^2} \;=\; ie^2\int_0^1 dx\int\frac{d^4p}{(2\pi)^4}\,\frac{\mathcal{N}^{\mu\nu}}{[p^2 - \Delta + i0]^2}\,. \tag{13}$$

The second equality here obtains from changing the order of integration over the loop momentum p_1 and over the Feynman parameter x, followed by shifting the loop momentum variable from p_1 to $p = p_1 + xk$. To make full use of this shift, we need to re-express the numerator $\mathcal{N}^{\mu\nu}$ in terms of p, k, and x. Thus, using

$$p_1 \;=\; p \;-\; x \times k, \quad p_2 \;=\; p_1 \;+\; k \;=\; p \;+\; (1-x) \times k, \tag{14}$$

we obtain

$$\begin{aligned} p_1^\mu p_2^\nu \;+\; p_2^\mu p_1^\nu \;&=\; 2p^\mu p^\nu \;+\; (1-2x) \times \big(p^\mu k^\nu \;+\; k^\mu p^\nu\big) \;-\; 2x(1-x) \times k^\mu k^\nu, \\ (p_1 p_2) \;&=\; p^2 \;+\; (1-2x) \times (pk) \;-\; x(1-x) \times k^2, \end{aligned} \tag{15}$$

and hence

$$\begin{aligned} \mathcal{N}^{\mu\nu} \;=\; & 8p^\mu p^\nu \;+\; 4(1-2x) \times \big(p^\mu k^\nu \;+\; k^\mu p^\nu\big) \;-\; 8x(1-x) \times k^\mu k^\nu \\ & +\; 4g^{\mu\nu} \times \Big(m^2 \;-\; p^2 \;-\; (1-2x) \times (pk) \;+\; x(1-x) \times k^2\Big). \end{aligned} \tag{16}$$

There are many terms in this expression, and it is convenient to re-organize them into 3 groups: *the good, the bad, and the odd,* thus

$$\mathcal{N}^{\mu\nu} \;=\; \mathcal{N}^{\mu\nu}_{\text{good}} \;+\; \mathcal{N}^{\mu\nu}_{\text{bad}} \;+\; \mathcal{N}^{\mu\nu}_{\text{odd}}\,, \tag{17}$$

where

$$\mathcal{N}^{\mu\nu}_{\text{good}} \;=\; 8x(1-x) \times \Big(g^{\mu\nu} \times k^2 \;-\; k^\mu k^\nu\Big), \tag{18}$$

$$\mathcal{N}^{\mu\nu}_{\text{bad}} \;=\; 8p^\mu p^\nu \;+\; 4g^{\mu\nu} \times \Big(m^2 \;-\; p^2 \;-\; x(1-x) \times k^2 \;=\; \Delta(x) \;-\; p^2\Big), \tag{19}$$

$$\mathcal{N}^{\mu\nu}_{\text{odd}} \;=\; 4(1-2x) \times \Big(p^\mu k^\nu \;+\; k^\mu p^\nu \;-\; g^{\mu\nu} \times (pk)\Big). \tag{20}$$

The purpose of this re-organization is to extract the *good* terms (18) — which clearly have the desirable form (3) — while the remaining *bad* and *odd* terms should not contribute to the momentum integral.

Indeed, consider the *odd* terms (20) which comprise all the odd powers of the independent momentum variable p^α. Consequently, under the symmetry $p^\alpha \to -p^\alpha$ (for all 4 components of the p^α) the $\mathcal{N}^{\mu\nu}_{\rm odd}$ changes its sign. One the other hand, the $\int d^4p$ (over the whole momentum space) is invariant under this symmetry, and so is the denominator $[p^2-\Delta+i0]^2$. Consequently,

$$\int\!\frac{d^4p}{(2\pi)^4}\,\frac{\mathcal{N}^{\mu\nu}_{\rm odd}}{[p^2-\Delta+i0]^2}\ \xrightarrow[p\to -p]{}\ -\,\text{itself}\quad\Longrightarrow\quad \int\!\frac{d^4p}{(2\pi)^4}\,\frac{\mathcal{N}^{\mu\nu}_{\rm odd}}{[p^2-\Delta+i0]^2}\ =\ 0. \tag{21}$$

The momentum integral over the *bad* terms (19) also vanishes, but proving that takes more effort. First, let's Wick rotate the momentum integral from the Minkowski to the Euclidean space,

$$p^0\ \to\ ip^4,\quad d^4p\ \to\ id^4p_E\,,\quad p^2\ \to\ -p_E^2\,, \tag{22}$$

and hence

$$\begin{aligned}(\text{bad})\ \equiv\ i\int\!\frac{d^4p}{(2\pi)^4}\,&\frac{\mathcal{N}^{\mu\nu}_{\rm odd}\ =\ 8p^\mu p^\nu\ +\ 4g^{\mu\nu}\times(\Delta-p^2)}{[p^2-\Delta+i0]^2}\\ =\ -\int\!\frac{d^4p_E}{(2\pi)^4}\,&\frac{8p_E^\mu p_E^\nu\ +\ 4g^{\mu\nu}\times(\Delta+p_E^2)}{[\Delta+p_E^2]^2}\,.\end{aligned} \tag{23}$$

Note: even in the Euclidean momentum space, μ and ν remain Minkowski-signature vector indices, so $g^{\mu\nu}$ remains the Minkowski metric tensor, and p_E^μ and p_E^ν should be understood as (ip^4,p^1,p^2,p^3).

Second, let's use the $SO(4)$ symmetry of the Euclidean momentum space. Thanks to this symmetry,

$$\int\!\frac{d^3\Omega_p}{2\pi^2}\,p_E^ip_E^j\ =\ \delta^{ij}\times\frac{p_E^2}{4} \tag{24}$$

and hence for any spherically symmetric function $f(p_E^2)$

$$\int\!d^4p_E\,f(p_E^2)\times p_E^ip_E^j\ =\ \delta^{ij}\times\int\!d^4p_E\,f(p_E^2)\times\frac{p_E^2}{4}\,. \tag{25}$$

Or in terms of the Minkowski indices μ and ν,

$$\int\!d^4p_E\,f(p_E^2)\times p_E^\mu p_E^\nu\ =\ -g^{\mu\nu}\times\int\!d^4p_E\,f(p_E^2)\times\frac{p_E^2}{4}\,. \tag{26}$$

More generally, in D spacetime dimensions,

$$\int\!d^Dp_E\,f(p_E^2)\times p_E^\mu p_E^\nu\ =\ -g^{\mu\nu}\times\int\!d^Dp_E\,f(p_E^2)\times\frac{p_E^2}{D}\,. \tag{27}$$

Applying this formula to the Euclidean momentum integral in eq (23) gives us

$$\begin{aligned}
(\text{bad}) \ &= \ -\int\!\frac{d^4p_E}{(2\pi)^4}\,\frac{-g^{\mu\nu}\times(8p_E^2/D)\ +\ 4g^{\mu\nu}\times(\Delta+p_E^2)}{[\Delta+p_E^2]^2} \\
&= \ -g^{\mu\nu}\times\int\!\frac{d^4p_E}{(2\pi)^4}\,\frac{4\Delta\ +\ (4-\frac{8}{D})\times p_E^2}{[\Delta+p_E^2]^2}\,.
\end{aligned} \tag{28}$$

For the moment $D=4$, but we keep the dimension D as explicit parameter in order to allow for the dimensional regularization of the momentum integral. Indeed, this integral badly needs DR — or some other UV regulator — because it's quadratically divergent in $D=4$. Thus, we let

$$(\text{bad}) \ = \ -g^{\mu\nu}\times\mu^{4-D}\int\!\frac{d^Dp_E}{(2\pi)^D}\,\frac{4\Delta\ +\ (4-\frac{8}{D})\times p_E^2}{[\Delta+p_E^2]^2}\,, \tag{29}$$

for generic D, evaluate the integral for $D<2$ (which regulates the UV divergence), and then analytically continue the result back to $D=4-2\epsilon$. As usual, to evaluate the integral for non-integer D we relate the integrand to an exponential $\exp(-tp_E^2)$. Specifically, we let

$$\begin{aligned}
\frac{4\Delta\ +\ (4-\frac{8}{D})\times p_E^2}{[\Delta+p_E^2]^2} \ &= \ \frac{(4-\frac{8}{D})}{\Delta+p_E^2}\ +\ \frac{\frac{8}{D}\Delta}{[\Delta+p_E^2]^2} \\
&= \ \int\limits_0^\infty\!dt\,\left(\left(4-\frac{8}{D}\right)\ +\ \frac{8}{D}\Delta\times t\right)\times\exp\bigl(-t(\Delta+p_E^2)\bigr),
\end{aligned} \tag{30}$$

which leads to

$$\begin{aligned}
(\text{bad}) \ &= \ -g^{\mu\nu}\times\mu^{4-D}\int\!\frac{d^Dp_E}{(2\pi)^D}\int\limits_0^\infty\!dt\,\left(\left(4-\frac{8}{D}\right)\ +\ \frac{8}{D}\Delta\times t\right)\times\exp\bigl(-t(\Delta+p_E^2)\bigr), \\
&= \ -g^{\mu\nu}\times\int\limits_0^\infty\!dt\,\left(\left(4-\frac{8}{D}\right)\ +\ \frac{8}{D}\Delta\times t\right)e^{-t\Delta}\times\mu^{4-D}\int\!\frac{d^Dp_E}{(2\pi)^D}\,e^{-tp_E^2} \\
&= \ -g^{\mu\nu}\times\int\limits_0^\infty\!dt\,\left(\left(4-\frac{8}{D}\right)\ +\ \frac{8}{D}\Delta\times t\right)e^{-t\Delta}\times\mu^{4-D}\,(4\pi t)^{-D/2}.
\end{aligned} \tag{31}$$

The remaining integral over t here converges for $D<2$, and — moracle of miracles — it happens to vanish identically for any $D<2$. Indeed, up to the overall constant factor $-g^{\mu\nu}\mu^{4-D}(4\pi)^{-D/2}$,

$$
\begin{aligned}
\text{(bad)} \;&\propto \int\limits_0^\infty dt\, e^{-\Delta\times t}\left(\left(4-\frac{8}{D}\right)\times t^{-D/2} \;+\; \frac{8}{D}\,\Delta\times t^{1-(D/2)}\right)\\
&=\; \left(4-\frac{8}{D}\right)\times \Delta^{(D/2)-1}\Gamma\left(1-\frac{D}{2}\right) \;+\frac{8}{D}\,\Delta\times\Delta^{(D/2)-2}\Gamma\left(2-\frac{D}{2}\right)\\
&=\; \frac{8}{D}\times\Delta^{(D/2)-1}\times\left[\left(\frac{D}{2}-1\right)\times\Gamma\left(1-\frac{D}{2}\right)\;+\;\Gamma\left(2-\frac{D}{2}\right)\right]\\
&=\; \frac{8}{D}\times\Delta^{(D/2)-1}\times\left[-y\Gamma(y)\;+\;\Gamma(y+1)\quad\text{for } y \;=\; 1-\frac{D}{2}\right]\\
&=\; 0
\end{aligned}
\tag{32}
$$

because $\Gamma(y+1)-y\Gamma(y)=0$ for any y. Consequently, analytically continuing from $D<2$ to $D=4-2\epsilon$, we find that the dimensionally-regulated integral (28) vanishes and hence

$$
\int\limits_{\mathrm{DR}}\frac{d^4p}{(2\pi)^4}\,\frac{\mathcal{N}^{\mu\nu}_{\text{bad}}}{[p^2-\Delta+i0]^2} \;=\; 0. \tag{33}
$$

In other words, the *bad* terms in the numerator $\mathcal{N}^{\mu\nu}$ do not contribute to the photon's $\Sigma^{\mu\nu}_{1\,\text{loop}}(k)$.

At this point, the only terms in the the numerator $\mathcal{N}^{\mu\nu}$ that do contribute to the integral (13) are the *good* terms (18), thus

$$
\begin{aligned}
\Sigma^{\mu\nu}_{1\,\text{loop}}(k) \;&=\; ie^2\int\limits_0^1 dx\int\frac{d^4p}{(2\pi)^4}\,\frac{\mathcal{N}^{\mu\nu}_{\text{good}} \;=\; 8x(1-x)\times\left(g^{\mu\nu}k^2-k^\mu k^\nu\right)}{[p^2-\Delta+i0]^2}\\
&=\; 8e^2\left(g^{\mu\nu}k^2-k^\mu k^\nu\right)\times\int\limits_0^1 dx\,x(1-x)\times\int\frac{d^4p}{(2\pi)^4}\,\frac{i}{[p^2-\Delta+i0]^2}\,.
\end{aligned}
\tag{34}
$$

In other words, the one-loop $\Sigma^{\mu\nu}(k)$ does have the requisite form

$$
\Sigma^{\mu\nu}_{1\,\text{loop}}(k) \;=\; \left(g^{\mu\nu}k^2-k^\mu k^\nu\right)\times\Pi_{1\,\text{loop}}(k^2) \tag{35}
$$

where

$$
\Pi_{1\,\text{loop}}(k^2) \;=\; 8e^2\int\limits_0^1 dx\,x(1-x)\times\int\frac{d^4p}{(2\pi)^4}\,\frac{i}{[p^2-\Delta+i0]^2}\,. \tag{36}
$$

It remains to evaluate the momentum integral in the last formula and then integrate over x. The momentum integral has the form we have seen before in this class, so we evaluate it in the usual way: Wick rotate p to the Euclidean momentum space, and then dimensionally regularize the logarithmic divergence. Thus,

$$\begin{aligned}
\int\frac{d^4p}{(2\pi)^4}\,\frac{i}{[p^2-\Delta+i0]^2}\ &=\ \int\frac{d^4p_E}{(2\pi)^4}\,\frac{-1}{[p_E^2+\Delta]^2} \\
&\longrightarrow\ -\mu^{4-D}\int\frac{d^Dp_E}{(2\pi)^D}\,\frac{1}{[p_E^2+\Delta]^2} \\
&=\ -\mu^{4-D}\int\frac{d^Dp_E}{(2\pi)^D}\int\limits_0^\infty dt\,t\,e^{-t(\Delta+p_E^2)} \\
&=\ -\int\limits_0^\infty dt\,t\,e^{-\Delta\times t}\times\mu^{4-D}\int\frac{d^Dp_E}{(2\pi)^D}\,e^{-t\times P_E^2} \\
&=\ -\int\limits_0^\infty dt\,t\,e^{-\Delta\times t}\times\mu^{4-D}(4\pi t)^{-D/2} \\
&=\ -\frac{(4\pi\mu^2)^\epsilon}{16\pi^2}\times\int\limits_0^\infty dt\,t^{1-(D/2)=\epsilon-1}\,e^{-\Delta\times t} \\
&=\ -\frac{(4\pi\mu^2)^\epsilon}{16\pi^2}\times\Gamma(\epsilon)\Delta^{-\epsilon} \\
&=\ -\frac{1}{16\pi^2}\times\left(\frac{1}{\epsilon}\,-\,\gamma_E\,+\,\log\frac{4\pi\mu^2}{\Delta}\,+\,O(\epsilon)\right)
\end{aligned}\tag{37}$$

and consequently

$$\Pi_{1\,\mathrm{loop}}(k^2)\ =\ -\frac{8e^2}{16\pi^2}\int\limits_0^1 dx\,x(1-x)\times\left(\frac{1}{\epsilon}\,-\,\gamma_E\,+\,\log\frac{4\pi\mu^2}{\Delta(x)}\right).\tag{38}$$

Finally, using

$$\int\limits_0^1 dx\,x(1-x)\ =\ \frac{1}{6}\tag{39}$$

we reduce eq. (38) to

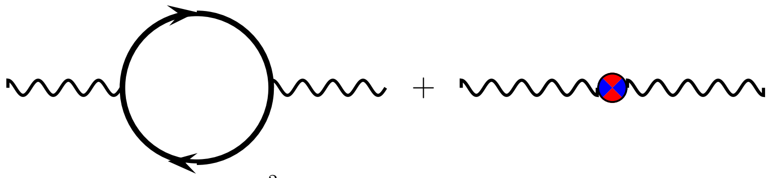
$$\Pi_{1\,\mathrm{loop}}(k^2)\ =\ -\frac{e^2}{12\pi^2}\times\left(\frac{1}{\epsilon}\,-\,\gamma_E\,+\,\log\frac{4\pi\mu^2}{m^2}\,+\,I(k^2/m^2)\right)\tag{40}$$

where

$$I(k^2/m^2) \;=\; \int\limits_0^1 dx\, 6x(1-x) \times \log \frac{m^2}{\Delta \;=\; m^2 - x(1-x) \times k^2} \tag{41}$$

Thus far, we have focused on the one-loop diagram (4) but ignored the counterterms. Adding the δ_3 counterterm to the picture gives us

$$i\Sigma^{\mu\nu}_{\mathrm{order}\,e^2}(k) \;=$$



$$\begin{aligned} &= \; i\Sigma^{\mu\nu}_{1\,\mathrm{loop}}(k) \,-\, i\delta_3^{\mathrm{order}\,e^2} \times \big(g^{\mu\nu}k^2 - k^\mu k^\nu\big), \end{aligned} \tag{42}$$

or in terms of $\Pi(k^2)$,

$$\Pi^{\mathrm{order}\,e^2}(k^2) \;=\; \Pi^{1\,\mathrm{loop}}(k^2) \,-\, \delta_3^{\mathrm{order}\,e^2}. \tag{43}$$

Consequently, by setting

$$\delta_3^{\mathrm{order}\,e^2} \;=\; -\frac{e^2}{12\pi^2} \times \left(\frac{1}{\epsilon} \;+\; \text{a finite constant}\right) \tag{44}$$

we may cancel the ultraviolet divergence of the electron loop.

The finite part of the δ_3 counterterm follows from the requirement

$$\Pi^{\mathrm{net}}(k^2 = 0) \;=\; 0. \tag{45}$$

Since $I(0) = 0$, this means we need

$$\delta_3^{\mathrm{order}\,e^2} \;=\; -\frac{e^2}{12\pi^2} \times \left(\frac{1}{\epsilon} \,-\, \gamma_E \,+\, \log\frac{4\pi\mu^2}{m_e^2}\right) \tag{46}$$

which leads to

$$\Pi^{\mathrm{order}\,e^2}(k^2) \;=\; -\frac{e^2}{12\pi^2} \times I(k^2/m_2^2). \tag{47}$$