Expansion of Free Relativistic Fields into Creation and Annihilation Operators

All kinds of free relativistic quantum fields can be expanded into annihilation and creation operators multiplied by the plane-wave solutions. To see how this works, let's start with the real (hermitian) scalar field $\hat{\Phi}(x)$. Earlier in class — *cf.* my notes on the Fock space — we have expanded the Schrödinger-picture $\hat{\Phi}(\mathbf{x})$ into plane waves in a box $\hat{\varphi}_{\mathbf{k}}$ and hence into annihilation and creation operators as

$$\hat{\Phi}(\mathbf{x}) \;=\; \sum_{\mathbf{k}} L^{-3/2} e^{i\mathbf{k}\cdot\mathbf{x}} \left(\hat{\varphi}_{\mathbf{k}} \;=\; \frac{\hat{a}_{\mathbf{k}} + \hat{a}^{\dagger}_{-\mathbf{k}}}{\sqrt{2\omega_{\mathbf{k}}}}\right) \tag{1}$$

where $\omega_{\mathbf{k}} = \omega_{-\mathbf{k}} = +\sqrt{\mathbf{k}^2 + m^2}$ is the energy of a free relativistic particle with 3–momentum $\mathbf{k}$. In the the infinite space, eq. (1) becomes

$$\begin{aligned}\hat{\Phi}(\mathbf{x}) \;&=\; \int\frac{d^3\mathbf{k}}{(2\pi)^3}\, e^{i\mathbf{k}\cdot\mathbf{x}} \left(\frac{\hat{a}_{\mathbf{k}} + \hat{a}^{\dagger}_{-\mathbf{k}}}{\sqrt{2\omega_{\mathbf{k}}}}\right)_{\text{non.rel}} \\ &=\; \int\frac{d^3\mathbf{k}}{(2\pi)^3}\, e^{i\mathbf{k}\cdot\mathbf{x}} \left(\frac{\hat{a}_{\mathbf{k}} + \hat{a}^{\dagger}_{-\mathbf{k}}}{2\omega_{\mathbf{k}}}\right)_{\text{rel}}\end{aligned} \tag{2}$$

where 'non.rel' vs. 'rel' denotes non-relativistic vs. relativistic normalization of the annihilation and creation operators,

$$\left(\hat{a}_{\mathbf{k}}\right)_{\text{rel}} \;=\; \sqrt{2\omega_{\mathbf{k}}}\left(\hat{a}_{\mathbf{k}}\right)_{\text{non.rel}}, \qquad \left(\hat{a}^{\dagger}_{\mathbf{k}}\right)_{\text{rel}} \;=\; \sqrt{2\omega_{\mathbf{k}}}\left(\hat{a}^{\dagger}_{\mathbf{k}}\right)_{\text{non.rel}}. \tag{3}$$

Next, let's separate the integral (2) into integrals over the annihilation operators and the integrals over the creation operators, and for the creation operators only change the integration variable from $\mathbf{k}$ to $-\mathbf{k}$. Thus,

$$\begin{aligned}\hat{\Phi}(\mathbf{x}) \;&=\; \int\frac{d^3\mathbf{k}}{(2\pi)^3}\, e^{i\mathbf{k}\cdot\mathbf{x}}\, \frac{\hat{a}_{\mathbf{k}}}{2\omega_{\mathbf{k}}} \;+\; \int\frac{d^3\mathbf{k}}{(2\pi)^3}\, e^{i\mathbf{k}\cdot\mathbf{x}}\, \frac{\hat{a}^{\dagger}_{-\mathbf{k}}}{2\omega_{\mathbf{k}}} \\ &=\; \int\frac{d^3\mathbf{k}}{(2\pi)^3}\, e^{i\mathbf{k}\cdot\mathbf{x}}\, \frac{\hat{a}_{\mathbf{k}}}{2\omega_{\mathbf{k}}} \;+\; \int\frac{d^3\mathbf{k}}{(2\pi)^3}\, e^{-i\mathbf{k}\cdot\mathbf{x}}\, \frac{\hat{a}^{\dagger}_{\mathbf{k}}}{2\omega_{-\mathbf{k}}} \\ &=\; \int\frac{d^3\mathbf{k}}{(2\pi)^3}\, \frac{1}{2\omega_{\mathbf{k}}} \left(e^{+i\mathbf{k}\cdot\mathbf{x}}\hat{a}_{\mathbf{k}} \;+\; e^{-i\mathbf{k}\cdot\mathbf{x}}\hat{a}^{\dagger}_{\mathbf{k}}\right).\end{aligned} \tag{4}$$

Now, let's change the QM picture from Schrödinger to Heisenberg, so all the operators evolve with with time. By linearity of eq. (4), we get

$$\hat{\Phi}(\mathbf{x},t) \;=\; \int \frac{d^3\mathbf{k}}{(2\pi)^3}\,\frac{1}{2\omega_{\mathbf{k}}}\left(e^{+i\mathbf{k}\cdot\mathbf{x}}\hat{a}_{\mathbf{k}}(t)\;+\;e^{-i\mathbf{k}\cdot\mathbf{x}}\hat{a}^{\dagger}_{\mathbf{k}}(t)\right), \tag{5}$$

where the time-dependence of the annihilation and creation operators follows from the Heisenberg equations

$$\frac{d}{dt}\hat{a}_{\mathbf{k}}(t) \;=\; -i[\hat{a}_{\mathbf{k}}(t),\hat{H}], \quad \frac{d}{dt}\hat{a}^{\dagger}_{\mathbf{k}}(t) \;=\; -i[\hat{a}^{\dagger}_{\mathbf{k}}(t),\hat{H}]. \tag{6}$$

In the relativistic normalization, the equal-time-bosonic commutation relations are

$$\left[\hat{a}_{\mathbf{k}},\hat{a}_{\mathbf{k}'}\right] \;=\; \left[\hat{a}^{\dagger}_{\mathbf{k}},\hat{a}^{\dagger}_{\mathbf{k}'}\right] \;=\; 0, \quad \left[\hat{a}_{\mathbf{k}},\hat{a}^{\dagger}_{\mathbf{k}'}\right] \;=\; 2\omega_{\mathbf{k}}\,(2\pi)^3\delta^{(3)}(\mathbf{k}-\mathbf{k}'), \tag{7}$$

while the Hamiltonian for the free scalar field is

$$\hat{H} \;=\; \int \frac{d^3\mathbf{k}}{(2\pi)^3}\,\frac{1}{2\omega_{\mathbf{k}}}\,\omega_{\mathbf{k}}\hat{a}^{\dagger}_{\mathbf{k}}\hat{a}_{\mathbf{k}}\,. \tag{8}$$

Consequently,

$$\left[\hat{a}_{\mathbf{k}},\hat{H}\right] \;=\; \frac{1}{2}\int \frac{d^3\mathbf{k}'}{(2\pi)^3}\,\left[\hat{a}_{\mathbf{k}},\hat{a}^{\dagger}_{\mathbf{k}'}\hat{a}_{\mathbf{k}'}\right] \tag{9}$$

where

$$\left[\hat{a}_{\mathbf{k}},\hat{a}^{\dagger}_{\mathbf{k}'}\hat{a}_{\mathbf{k}'}\right] \;=\; \left[\hat{a}_{\mathbf{k}},\hat{a}^{\dagger}_{\mathbf{k}'}\right]\hat{a}_{\mathbf{k}'}\;+\;0 \;=\; 2\omega_{\mathbf{k}}\,(2\pi)^3\delta^{(3)}(\mathbf{k}-\mathbf{k}')\hat{a}_{\mathbf{k}'} \tag{10}$$

and hence

$$\left[\hat{a}_{\mathbf{k}},\hat{H}\right] \;=\; \int \frac{d^3\mathbf{k}}{(2\pi)^3}\,\omega_{\mathbf{k}}\,(2\pi)^3\delta^{(3)}(\mathbf{k}-\mathbf{k}')\hat{a}_{\mathbf{k}'} \;=\; \omega_{\mathbf{k}}\hat{a}_{\mathbf{k}}. \tag{11}$$

Therefore

$$\frac{d}{dt}\hat{a}_{\mathbf{k}}(t) \;=\; -i\omega_{\mathbf{k}}\hat{a}_{\mathbf{k}}(t) \quad\Longrightarrow\quad \hat{a}_{\mathbf{k}}(t) \;=\; \exp(-i\omega_{\mathbf{k}}t)\times\hat{a}_{\mathbf{k}}(0), \tag{12}$$

and in a similar way

$$\frac{d}{dt}\hat{a}^{\dagger}_{\mathbf{k}}(t) \;=\; +i\omega_{\mathbf{k}}\hat{a}^{\dagger}_{\mathbf{k}}(t) \quad\Longrightarrow\quad \hat{a}^{\dagger}_{\mathbf{k}}(t) \;=\; \exp(+i\omega_{\mathbf{k}}t)\times\hat{a}^{\dagger}_{\mathbf{k}}(0). \tag{13}$$

Finally, plugging these time-dependent annihilation and creation operators into eq. (5) for the fields, we arrive at

$$\hat{\Phi}(\mathbf{x},t) \;=\; \int \frac{d^3\mathbf{k}}{(2\pi)^3}\,\frac{1}{2\omega_{\mathbf{k}}}\left(\exp(+i\mathbf{k}\cdot\mathbf{x}-i\omega_{\mathbf{k}}t)\times\hat{a}_{\mathbf{k}}(0)\;+\;\exp(-i\mathbf{k}\cdot\mathbf{x}+i\omega_{\mathbf{k}}t)\times\hat{a}^\dagger_{\mathbf{k}}(0)\right). \tag{14}$$

It remains to rewrite this formula in a manifestly relativistic form. Note that both exponents in eq. (14) are Lorentz-invariant

$$+i\mathbf{k}\cdot\mathbf{x}\,-\,i\omega_{\mathbf{k}}t \;=\; -ik_\mu x^\mu, \quad -i\mathbf{k}\cdot\mathbf{x}\,+\,i\omega_{\mathbf{k}}t \;=\; +ik_\mu x^\mu, \tag{15}$$

provided in both cases we identify $k^0 = +\omega_{\mathbf{k}} = +\sqrt{\mathbf{k}^2+m^2}$. Thus, eq. (14) becomes

$$\hat{\Phi}(x) \;=\; \int \frac{d^3\mathbf{k}}{(2\pi)^3}\,\frac{1}{2\omega_{\mathbf{k}}}\left(e^{-ikx}\hat{a}_{\mathbf{k}}(0)\;+\;e^{+ikx}\hat{a}^\dagger_{\mathbf{k}}(0)\right)^{k^0=+\omega_{\mathbf{k}}}. \tag{16}$$

Please note that each coefficient e^{-ikx} or e^{+ikx} of any annihilation or creation operators here is the plane-wave solution of the Klein–Gordon equation $(\partial^2+m^2)\phi=0$ for the free scalar field. Moreover, all the positive-frequency solutions e^{-ikx} accompany the annihilation operators $\hat{a}_{\mathbf{k}}$ while the negative-frequency plane waves e^{+ikx} accompany the creation operators. And both of these features apply to all kinds of free relativistic fields — real or complex, scalar vector, or spinor, or whatever.

For example, consider the real massive vector field $A^\mu(x)$. In homework set#1 (problem 1) you saw the free-field equation for this field,

$$\begin{aligned}(\partial^2+m^2)A^\mu \;-\; \partial^\mu\partial_\nu A^\nu \;&=\; 0, \quad \text{or equivalently}\\ (\partial^2+m^2)A^\mu \;=\; 0 \quad &\textbf{and} \quad \partial_\nu A^\nu \;=\; 0.\end{aligned} \tag{17}$$

The positive-frequency plane wave solutions of these equations have form

$$A^\mu(x) \;=\; e^{-ikx}f^\mu(\mathbf{k},\lambda) \tag{18}$$

where $k^0 = +\omega_{\mathbf{k}}$ while the polarization vector $f^\mu(\mathbf{k},\lambda)$ obeys $k_\mu f^\mu(\mathbf{k},\lambda)=0$ — and that's why there are 3 independent polarizations $\lambda=-1,0,+1$ for each wave vector $\mathbf{k}$. Similarly, the negative-frequency plane waves have form

$$A^\mu(x) \;=\; e^{+ikx} f^{\mu *}(\mathbf{k},\lambda). \tag{19}$$

Consequently, by analogy with eq. (16) for the scalar field, we expect to expand the quantum massive vector field into annihilation and creation operators according to

$$\hat{A}^\mu(x) \;=\; \int\frac{d^3\mathbf{k}}{(2\pi)^3}\,\frac{1}{2\omega_\mathbf{k}}\sum_\lambda\Bigl(e^{-ikx} f^\mu(\mathbf{k},\lambda)\times\hat{a}_{\mathbf{k},\lambda}(0)\;+\;e^{+ikx} f^{*\mu}(\mathbf{k},\lambda)\times\hat{a}^\dagger_{\mathbf{k},\lambda}(0)\Bigr)^{k^0=+\omega_\mathbf{k}}. \tag{20}$$

Deriving this decomposition is left out from this notes as an exercise for the students — specifically, problem 2 of the current homework set#3.

Next, consider the complex scalar field $\Phi(x)\neq\Phi^*(x)$ with free Lagrangian density

$$\mathcal{L} \;=\; \partial^\mu\Phi^*\partial_\mu\Phi \;-\; m^2\Phi^*\Phi. \tag{21}$$

The complex $\Phi(x)$ is equivalent to 2 independent real scalar fields $\phi_1(x)$ and $\phi_2(x)$ according to

$$\Phi(x) \;=\; \frac{\phi_1(x)\,+\,i\phi_2(x)}{\sqrt{2}}\,,\qquad \Phi^*(x) \;=\; \frac{\phi_1(x)\,-\,i\phi_2(x)}{\sqrt{2}}\,, \tag{22}$$

$$\mathcal{L} \;=\; \frac{1}{2}\partial_\mu\phi_1\partial^\mu\phi_1 \;-\; \frac{m^2}{2}\,\phi_1^2 \;+\; \frac{1}{2}\partial_\mu\phi_2\partial^\mu\phi_2 \;-\; \frac{m^2}{2}\,\phi_2^2\,. \tag{23}$$

Quantizing the two free real scalar fields independently from each other, we get 2 separate sets of annihilation and creation operators $\hat{a}_{\mathbf{k},i}$ and $\hat{a}^\dagger_{\mathbf{k},i}$ ($i=1,2$) with equal-time bosonic commutation relations

$$\bigl[\text{any}\,\hat{a},\text{any}\,\hat{a}\bigr] \;=\; 0,\quad \bigl[\text{any}\,\hat{a}^\dagger,\text{any}\,\hat{a}^\dagger\bigr] \;=\; 0,\quad \bigl[\hat{a}_{\mathbf{k},i},\hat{a}^\dagger_{\mathbf{k}',j}\bigr] \;=\; \delta_{ij}\times(2\omega_\mathbf{k})(2\pi)^3\delta^{(3)}(\mathbf{k}-\mathbf{k}'). \tag{24}$$

Physically, these operators generate the Fock space of two species of identical bosons, but since the two species have exactly the same mass, it's convenient to change the species basis to eigenstates of some charge operator such as electric charge, or baryon number, or lepton number, or whatever. I'll describe such charge operators in a later class or homework, but for the moment let me simply summarize the eigenstates as *the charged particle and the anti-particle:*

$$|\text{particle}(\mathbf{k})\rangle \;=\; \frac{|\mathbf{k},1\rangle - i\,|\mathbf{k},2\rangle}{\sqrt{2}}\,, \quad |\text{antiparticle}(\mathbf{k})\rangle \;=\; \frac{|\mathbf{k},1\rangle + i\,|\mathbf{k},2\rangle}{\sqrt{2}}\,. \tag{25}$$

In terms of creation and annihilation operators,

$$\begin{aligned}
\text{particle creation operator}\quad \hat{a}^\dagger_{\mathbf{k}} \;&=\; \frac{\hat{a}^\dagger_{\mathbf{k},1} - i\hat{a}^\dagger_{\mathbf{k},2}}{\sqrt{2}}\,,\\
\text{antiparticle creation operator}\quad \hat{b}^\dagger_{\mathbf{k}} \;&=\; \frac{\hat{a}^\dagger_{\mathbf{k},1} + i\hat{a}^\dagger_{\mathbf{k},2}}{\sqrt{2}}\,,\\
\text{particle annihilation operator}\quad \hat{a}_{\mathbf{k}} \;&=\; \frac{\hat{a}_{\mathbf{k},1} + i\hat{a}_{\mathbf{k},2}}{\sqrt{2}}\,,\\
\text{antiparticle annihilation operator}\quad \hat{b}_{\mathbf{k}} \;&=\; \frac{\hat{a}_{\mathbf{k},1} - i\hat{a}_{\mathbf{k},2}}{\sqrt{2}}\,,
\end{aligned} \tag{26}$$

hence equal-time bosonic commutation relations

$$\left[\text{any}\,\hat{a},\text{any}\,\hat{a}\right] \;=\; \left[\text{any}\,\hat{b},\text{any}\,\hat{b}\right] \;=\; \left[\text{any}\,\hat{a}^\dagger,\text{any}\,\hat{a}^\dagger\right] \;=\; \left[\text{any}\,\hat{b}^\dagger,\text{any}\,\hat{b}^\dagger\right] \;=\; 0, \tag{27}$$

$$\left[\text{any}\,\hat{b},\text{any}\,\hat{a}^\dagger\right] \;=\; \left[\text{any}\,\hat{a},\text{any}\,\hat{b}^\dagger\right] \;=\; 0, \tag{28}$$

$$\left[\hat{a}_{\mathbf{k}},\hat{a}^\dagger_{\mathbf{k}'}\right] \;=\; \left[\hat{b}_{\mathbf{k}},\hat{a}^\dagger_{\mathbf{k}'}\right] \;=\; (2\omega_{\mathbf{k}})(2\pi)^3\delta^{(3)}(\mathbf{k}-\mathbf{k}'). \tag{29}$$

Now let's expand the quantum fields into these creation and annihilation operators. Similar to a single real scalar field (16), the two real components of the complex scalar expand to

$$\begin{aligned}
\hat{\phi}_1(x) \;&=\; \int\frac{d^3\mathbf{k}}{(2\pi)^3}\,\frac{1}{2\omega_{\mathbf{k}}}\left(e^{-ikx}\hat{a}_{\mathbf{k},1}(0) \,+\, e^{+ikx}\hat{a}^\dagger_{\mathbf{k},1}(0)\right)^{k^0=+\omega_{\mathbf{k}}}\,,\\
\hat{\phi}_2(x) \;&=\; \int\frac{d^3\mathbf{k}}{(2\pi)^3}\,\frac{1}{2\omega_{\mathbf{k}}}\left(e^{-ikx}\hat{a}_{\mathbf{k},2}(0) \,+\, e^{+ikx}\hat{a}^\dagger_{\mathbf{k},2}(0)\right)^{k^0=+\omega_{\mathbf{k}}}\,.
\end{aligned} \tag{30}$$

Consequently,

$$\hat{\Phi}(x) \;=\; \frac{\hat{\phi}_1(x)+i\hat{\phi}_2(x)}{\sqrt{2}} \;=\; \int\frac{d^3\mathbf{k}}{(2\pi)^3}\,\frac{1}{2\omega_{\mathbf{k}}}\left(e^{-ikx}\times\hat{a}_{\mathbf{k}}(0) \;+\; e^{+ikx}\times\hat{b}^\dagger_{\mathbf{k}}(0)\right)^{k^0=+\omega_{\mathbf{k}}}, \tag{31}$$

$$\hat{\Phi}^\dagger(x) \;=\; \frac{\hat{\phi}_1(x) - i\hat{\phi}_2(x)}{\sqrt{2}} \;=\int \frac{d^3\mathbf{k}}{(2\pi)^3}\,\frac{1}{2\omega_\mathbf{k}} \left(e^{-ikx} \times \hat{b}_\mathbf{k}(0) \;+\; e^{+ikx} \times \hat{a}^\dagger_\mathbf{k}(0)\right)^{k^0=+\omega_\mathbf{k}}. \quad (32)$$

Note that the $\hat{\Phi}$ field comprises the particle annihilation and antiparticle creation operators, all of which change the

$$\text{net charge} \;=\; \text{net\#particles} \;-\; \text{net\#antiparticles} \qquad (33)$$

by -1, while the $\hat{\Phi}^\dagger$ field comprises the particle creation operators and the antiparticle annihilation operators, all of which change the net charge by $+1$. Thus, we can ascribe definite charges to the quantum fields themselves — which are hence often called *the charged fields* — and if all the interaction terms in the Hamiltonian happen to be neutral, then the net charge is conserved. Although the separate particle and antiparticle numbers are generally not conserved in any non-free theory. For example, consider the following interaction Hamiltonian for the charged scalar field:

$$\hat{H}_{\text{int}} \;=\; \frac{\lambda}{2}\int d^3\mathbf{x}\,\hat{\Phi}^\dagger\hat{\Phi}^\dagger\hat{\Phi}\hat{\Phi}. \qquad (34)$$

In terms of creation and annihilation operators, this interaction comprises terms of the form

$$\begin{aligned} \hat{H}_{\text{int}} \;\supset\; & \hat{a}^\dagger\hat{a}^\dagger\hat{a}\hat{a} \;+\; \hat{a}^\dagger\hat{b}^\dagger\hat{a}\hat{b} \;+\; \hat{b}^\dagger\hat{b}^\dagger\hat{b}\hat{b} \\ & +\; \hat{a}^\dagger\hat{a}^\dagger\hat{b}^\dagger\hat{a} \;+\; \hat{a}^\dagger\hat{b}^\dagger\hat{b}^\dagger\hat{b} \;+\; \hat{a}^\dagger\hat{a}^\dagger\hat{b}^\dagger\hat{b}^\dagger \\ & +\; \hat{a}^\dagger\hat{a}\hat{a}\hat{b} \;+\; \hat{b}^\dagger\hat{a}\hat{b}\hat{b} \;+\; \hat{a}\hat{a}\hat{b}\hat{b}; \end{aligned} \qquad (35)$$

the terms on the last two lines here change the separate numbers of particles and antiparticles, but they all preserve the net charge.

By comparison, for a single real field we have only one type of particle number, and for a field interacting with itself or other fields, this particle number is not conserved. For example,

$$\hat{H}_{\text{int}} \;=\; \frac{\lambda}{24}\int d^3\mathbf{x}\,\hat{\Phi}^4 \;\supset\; \hat{a}^\dagger\hat{a}^\dagger\hat{a}^\dagger\hat{a}^\dagger \;+\; \hat{a}^\dagger\hat{a}^\dagger\hat{a}^\dagger\hat{a} \;+\; \hat{a}^\dagger\hat{a}^\dagger\hat{a}\hat{a} \;+\; \hat{a}^\dagger\hat{a}\hat{a}\hat{a} + \;\hat{a}\hat{a}\hat{a}\hat{a} \qquad (36)$$

where most terms change the net particle number by ± 2 or ± 4. Consequently, the real fields — be they scalar, vector, or whatever, — are called *neutral* because their quanta do not have any kinds of conserved charges. Such particles are called *completely neutral* (as opposed to merely electrically neutral), and their antiparticles are indistinguishable from the particles themselves. For example, the antiphoton is identical to the photon.

General Free Field

To conclude, let me generalize the above examples of expanding free quantum fields into annihilation and creation operators to any type of a relativistic field scalar, vector, tensor, spinor, whatever. Let's label the components of such a field by $\widehat{\Psi}_\aleph(x)$ where $\aleph$ stands for a vector, tensor, or spinor index or multi-index. Then a neutral (real) field $\widehat{\Psi}_\aleph(x)$ expands to

$$\widehat{\Psi}_\aleph(x) \;=\; \int\frac{d^3\mathbf{k}}{(2\pi)^3}\,\frac{1}{2\omega_\mathbf{k}}\sum_\lambda\left(e^{-ikx}U_\aleph(\mathbf{k},\lambda)\,\hat{a}_{\mathbf{k},\lambda} \;+\; e^{+ikx}V_\aleph(\mathbf{k},\lambda)\,\hat{a}^\dagger_{\mathbf{k},\lambda}\right)^{k^0=+\omega_\mathbf{k}} \tag{37}$$

while a charged (complex) field and its conjugate expand to

$$\begin{aligned}\widehat{\Psi}_\aleph(x) \;&=\; \int\frac{d^3\mathbf{k}}{(2\pi)^3}\,\frac{1}{2\omega_\mathbf{k}}\sum_\lambda\left(e^{-ikx}U_\aleph(\mathbf{k},\lambda)\,\hat{a}_{\mathbf{k},\lambda} \;+\; e^{+ikx}V_\aleph(\mathbf{k},\lambda)\,\hat{b}^\dagger_{\mathbf{k},\lambda}\right)^{k^0=+\omega_\mathbf{k}}, \\ \widehat{\Psi}^\dagger_\aleph(x) \;&=\; \int\frac{d^3\mathbf{k}}{(2\pi)^3}\,\frac{1}{2\omega_\mathbf{k}}\sum_\lambda\left(e^{-ikx}V^*_\aleph(\mathbf{k},\lambda)\,\hat{b}_{\mathbf{k},\lambda} \;+\; e^{+ikx}U^*_\aleph(\mathbf{k},\lambda)\,\hat{a}^\dagger_{\mathbf{k},\lambda}\right)^{k^0=+\omega_\mathbf{k}}\end{aligned} \tag{38}$$

In all these formulae:

- The quantum fields are in the Heisenberg picture of QM $\Longrightarrow$ time-dependent, but the annihilation / creation operators $\hat{a}_{\mathbf{k},\lambda}$, $\hat{a}^\dagger_{\mathbf{k},\lambda}$, *etc.*, are in the Schrödinger picture (same as $\hat{a}_{\mathbf{k},\lambda}(t=0)$, *etc.*, in the Heisenberg picture.)
- $kx \equiv k_\mu x^\mu = \omega_k t - \mathbf{k}\cdot\mathbf{x}$ for $\omega_k = +\sqrt{\mathbf{k}^2+m^2}$.
- The $U_\aleph(\mathbf{k},\lambda)$ and $V_\aleph(\mathbf{k},\lambda)$ are the coefficients of the plane-wave solutions of the classical field equations,

$$\Psi_\aleph(x) \;=\; e^{-ikx}\times U_\aleph(\mathbf{k},\lambda) \quad\text{and}\quad \Psi_\aleph(x) \;=\; e^{+ikx}\times V_\aleph(\mathbf{k},\lambda) \quad\text{for } k^0=+\omega_\mathbf{k}\,, \tag{39}$$

where λ labels the *polarizations* — *i.e.*, independent solutions for the same k^μ.

- For the bosonic fields

$$\left[\hat{a}_{\mathbf{k},\lambda}, \hat{a}^\dagger_{\mathbf{k}',\lambda'}\right] = \left[\hat{b}_{\mathbf{k},\lambda}, \hat{b}^\dagger_{\mathbf{k}',\lambda'}\right] = \delta_{\lambda\lambda'} \times 2\omega_{\mathbf{k}}(2\pi)^3\delta^{(3)}(\mathbf{k} - \mathbf{k}') \tag{40}$$

  while all other pairs of creation or annihilation operators commute with each other. For the fermionic fields

$$\left\{\hat{a}_{\mathbf{k},\lambda}, \hat{a}^\dagger_{\mathbf{k}',\lambda'}\right\} = \left\{\hat{b}_{\mathbf{k},\lambda}, \hat{b}^\dagger_{\mathbf{k}',\lambda'}\right\} = \delta_{\lambda\lambda'} \times 2\omega_{\mathbf{k}}(2\pi)^3\delta^{(3)}(\mathbf{k} - \mathbf{k}') \tag{41}$$

  while all other pairs of creation or annihilation operators anti-commute with each other.