Functional Quantization of the Electromagnetic Field

In the functional quantization (path integral) formalism, the propagator of a free field follows from coupling a classical field to a source and then taking the functional integral for the partition function. But for the gauge fields — like the electromagnetic field $A^\mu(x)$ — this procedure must be modified to factor out the local redundancy due to gauge transforms $A_\mu(x) \to A_\mu(x) - \partial_\mu \Lambda(x)$. Otherwise, the naive functional integral

$$Z^{\text{naive}}[J] \;=\; \iiint \mathcal{D}[A_\mu(x)] \, \exp\left(- \int d^4x_e \big(\tfrac{1}{4} F_{\mu\nu}^2 \;-\; J_\mu A_\mu \big) \right) \tag{1}$$

does not converge in any sense unless the source $J_\mu(x)$ happens to be a conserved current, $\partial_\mu J_\mu = 0$. Indeed, the naive functional integral (1) over the vector field $A_\mu(x)$ includes integration over both the physical and the gauge-redundant degrees of freedom,

$$\iiint \mathcal{D}[A_\mu(x)] \;=\; \iiint \mathcal{D}[\text{physically distinct } A_\mu(x)] \iiint \mathcal{D}[\Lambda(x)]. \tag{2}$$

At the same time, for $\partial_\mu J_\mu \neq 0$ the Euclidean action is not gauge invariant but varies by

$$\delta S_E[A_\mu, J_\mu] \;=\; \int d^4x_E \Lambda(x) \times \partial_\mu J_\mu(x), \tag{3}$$

which makes the integral

$$\iiint \mathcal{D}[\Lambda(x)] \, \exp\big(-S_E[A_\mu - \partial_\mu \Lambda, J_m] \big) \tag{4}$$

badly divergent, and consequently the naive functional integral (1) is also badly divergent.

Moreover, in the integral (1) we integrate over both physical and unphysical degrees of freedom. Note that the gauge transforms of the EM potentials $A_\mu(x)$ are *redundancies* rather than local symmetries: Instead of relating similar but distinct field configurations, they relate different parametrizations of the same physics. Consequently, in the properly defined functional integral for the EM fields we should integrate *only* over the physically distinct potentials $A_\mu(x)$. That is, once we integrate over some potential $A_\mu(x)$, we should not integrate over any of the gauge-equivalent potentials $A_\mu(x) - \partial_\mu \Lambda(x)$.

Mathematically, we need to somehow factorize the functional integral over all the $A_\mu(x)$ into an integral over the gauge-inequivalent potentials only and the integral over the redundant degrees of freedom parametrized by the $\Lambda(x)$. In these notes, I explain the Fadde'ev–Popov formalism for achieving such a factorization

My starting point is gauge-fixing: Start with a generic EM field $A_\mu(x)$, and replace it with a physically equivalent field $A^\Lambda(x) = A_\mu(x) - \partial_\mu\Lambda(x)$ which obeys some gauge-fixing condition $G(A^\Lambda_\mu(x)) = 0$, for example:

- The Coulomb gauge $\nabla\cdot\mathbf{A}^\Lambda(x) = 0$ at all x.
- The axial gauge $A^\Lambda_3(x) = 0$ at all x.
- The Landau gauge $\partial_\mu A^\Lambda_\mu(x) = 0$ at all x.

In general, for any $A_\mu(x)$ there is a unique (or as good as unique) physically equivalent field $A^\Lambda_\mu(x)$ which obeys $G(A^\Lambda_\mu) \equiv 0$, so naively we may replace the functional integral over the original EM field $A_\mu(x)$ with

$$\iiint\mathcal{D}[A_\mu(x)] \;\longrightarrow\; \iiint\mathcal{D}[A_\mu(x)]\iiint\mathcal{D}[\Lambda(x)]\,\Delta[G(A^\Lambda_\mu)], \tag{5}$$

where Δ is the functional equivalent of the δ-function — the $\Delta[G(A^\Lambda_\mu)]$ vanishes unless $G(A^\Lambda_\mu) = 0$ at all x.

However, one must be careful about properly normalizing a δ-function of a function. Indeed, even in a calculus of a single variable α,

$$\int d\alpha\,\delta(g(\alpha))\times f(\alpha) \;=\; \frac{f(\alpha_0)}{g'(\alpha_0)} \quad \text{where } g(\alpha_0) = 0. \tag{6}$$

In other words, if we want to evaluate a function $f(\alpha)$ at the point α_0 where another function $g(\alpha)$ happens to vanish, we cannot simply take an integral of $f(\alpha)\times\delta(g(\alpha))$ as on the LHS above but we must accompany the δ-function with a derivative of g,

$$f\big(\alpha_0 \text{ where } g(\alpha) = 0\big) \;=\; \int d\alpha\, f(\alpha)\times\delta(g(\alpha))g'(\alpha). \tag{7}$$

Similarly, to fix N variables $(\alpha_1,\ldots,\alpha_N)$ using N independent constraints $g_i(\alpha_1,\ldots,\alpha_N) = 0$, and then to evaluate a function $f(\vec\alpha)$ at a point $\vec\alpha_0$ where all the constraints are satisfied, we use

$$f(\vec{\alpha}_0) \;=\; \int d^N\vec{\alpha}\, f(\vec{\alpha}) \times \prod_i \delta(g_i(\vec{\alpha})) \times \det\left(\frac{\partial g_i}{\partial \alpha_j}\right) \tag{8}$$

— the N-dimensional δ-function must be accompanied by the Jacobian $\det(\partial g_i/\partial\alpha_j)$.

Likewise, the functional δ-function $\Delta[G(\Lambda)]$ must be accompanied by the functional determinant $\det(\delta G/\delta\Lambda)$. In particular, in the context of the functional integral (5) over the EM fields $A_\mu(x)$, the proper integral over the gauge variables $\Lambda(x)$ is

$$\iiint \mathcal{D}[A_\mu(x)] \;\longrightarrow\; \iiint \mathcal{D}[A_\mu(x)] \iiint \mathcal{D}[\Lambda(x)]\, \Delta[G(A_\mu^\Lambda)] \times \mathrm{Det}\left[\frac{\delta G(A_\mu^\Lambda(x))}{\delta\Lambda(y)}\right]. \tag{9}$$

The functional determinant here

$$D_{FP} \;=\; \mathrm{Det}\left[\frac{\delta G(A_\mu^\Lambda(x))}{\delta\Lambda(y)}\right] \tag{10}$$

is called the *Fadde'ev–Popov determinant* after Ludvig Fadde'ev and Victor Popov. In the Landau gauge

$$G(A_\mu^\Lambda) \;=\; \partial_\mu A_\mu^\Lambda \;=\; \partial_\mu A_\mu \;-\; \partial^2\Lambda, \tag{11}$$

so the Fadde'ev–Popov determinant is

$$D_{FP} \;=\; \mathrm{Det}\left[\frac{\delta G(A_\mu^\Lambda)}{\delta\Lambda}\right] \;=\; \mathrm{Det}[-\partial^2]. \tag{12}$$

Although this determinant is badly divergent, it does not depend on the EM fields $A_\mu(x)$ or on the source $J_\mu(s)$, so we may treat it as a constant factor in the normalization of the functional integral.

Now that we have properly defined the gauge-fixing procedure in the context of the functional integral, let's use it to calculate the source-dependent partition function $Z[J]$.

Staring with the naive functional integral (1), we have

$$\begin{aligned} Z^{\text{naive}}[J] &= \iiint \mathcal{D}[A_\mu(x)] \exp\bigl(-S_E[A_\mu, J_\mu]\bigr) \\ &= \iiint \mathcal{D}[A_\mu(x)] \iiint \mathcal{D}[\Lambda(x)]\, \Delta[\partial_\mu A_\mu^\Lambda] \times \operatorname{Det}[-\partial^2] \times \exp\bigl(-S_E[A_\mu, J_\mu]\bigr). \end{aligned} \tag{13}$$

For a non-conserved source $J_\mu(x)$, the net action is not quite gauge invariant; instead

$$S[A_\mu^\Lambda, J_\mu] \;=\; S[A_\mu, J_\mu] \;-\; \int d^4x\, \Lambda(x) \times \partial_\mu J_\mu(x). \tag{14}$$

Consequently, re-expressing the action in eq. (13) in terms of the gauge-fixed EM field $A_\mu^\Lambda(x)$, we have

$$\begin{aligned} Z^{\text{naive}}[J] &= \iiint \mathcal{D}[A_\mu(x)] \iiint \mathcal{D}[\Lambda(x)]\, \Delta[\partial_\mu A_\mu^\Lambda] \times \operatorname{Det}[-\partial^2] \times \exp\bigl(-S_E[A_\mu^\Lambda, J_\mu]\bigr) \\ &\qquad \times \exp\left(+\int d^4x\, \Lambda\, \partial_\mu J_\mu\right). \end{aligned} \tag{15}$$

Now let's change the order of the functional integrals and integrate over the $A_\mu(x)$ before integrating over the $\Lambda(x)$. Thus, let

$$\text{let} \quad \widehat{Z}[J, \Lambda] \stackrel{\text{def}}{=} \iiint \mathcal{D}[A_\mu(x)]\Delta[\partial_\mu A_\mu^\Lambda] \times \operatorname{Det}[-\partial^2] \times \exp\bigl(-S_E[A_\mu^\Lambda, J_\mu]\bigr), \tag{16}$$

$$\text{then} \quad Z^{\text{naive}}[J] = \iiint \mathcal{D}[\Lambda(x)] \exp\left(\int d^4x\, \Lambda\, \partial_\mu J_\mu\right) \times \widehat{Z}[J, \Lambda]. \tag{17}$$

Note that the inner integral (16) is evaluated for a fixed $\Lambda(x)$, which means that the functional map from the original $A_\mu(x)$ to the $A_\mu^\Lambda(x) = A_\mu(x) - \partial_\mu \Lambda(x)$ does not change the measure of the functional integral, $\mathcal{D}[A_\mu^\Lambda] = \mathcal{D}[A_\mu]$ for a fixed $\Lambda(x)$. Consequently,

$$\begin{aligned} \widehat{Z}[J, \Lambda] &= \iiint \mathcal{D}[A_\mu(x)]\Delta[\partial_\mu A_\mu^\Lambda] \times \operatorname{Det}[-\partial^2] \times \exp\bigl(-S_E[A_\mu^\Lambda, J_\mu]\bigr) \\ &= \iiint \mathcal{D}[A_\mu^\Lambda(x)]\Delta[\partial_\mu A_\mu^\Lambda] \times \operatorname{Det}[-\partial^2] \times \exp\bigl(-S_E[A_\mu^\Lambda, J_\mu]\bigr) \\ &= \iiint \mathcal{D}[A_\mu(x)]\Delta[\partial_\mu A_\mu] \times \operatorname{Det}[-\partial^2] \times \exp\bigl(-S_E[A_\mu, J_\mu]\bigr) \end{aligned} \tag{18}$$

where on the last line I have simply *renamed* the integration variable from $A_\mu^\Lambda(x)$ to $A_\mu(x)$. However, thanks to the $\Delta[\partial_\mu A_\mu]$ factor, the $A_\mu(x)$ field on the last line is constrained to obey the Landau gauge condition $\partial_\mu A_\mu = 0$.

Note: Nothing on the last line of eq. (18) depends on the $\Lambda(x)$, which makes the $\widehat{Z}[J,\not\Lambda]$ completely Λ-independent. Consequently, eq. (17) becomes

$$\begin{aligned} Z^{\text{naive}}[J] &= \iiint \mathcal{D}[\Lambda(x)] \exp\left(\int d^4x\,\Lambda\,\partial_\mu J_\mu\right) \times \widehat{Z}[J\,\text{only}] \\ &= \widehat{Z}[J\,\text{only}] \times \iiint \mathcal{D}[\Lambda(x)] \exp\left(\int d^4x\,\Lambda\,\partial_\mu J_\mu\right). \end{aligned} \tag{19}$$

On the second line here, we have factorized the naive functional integral over all the EM potentials $A_\mu(x)$ into an integral $\widehat{Z}$ over the physically-distinct (*i.e.*, gauge-inequivalent) potential and an integral over the gauge redundancies $\Lambda(x)$. The second factor here is unphysical, and we should get rid of it. Thus, *we redefine the electromagnetic partition function* as a properly-normalized integral over the EM fields in the Landau gauge,

$$Z[J] \stackrel{\text{def}}{=} \widehat{Z}[J] = \iiint \mathcal{D}[A_\mu(x)]\Delta[\partial_\mu A_\mu] \times \text{Det}[-\partial^2] \times \exp\bigl(-S_E[A_\mu, J_\mu]\bigr). \tag{20}$$

Before we proceed to use this partition function, I would like to comment on the Fadde'ev–Popov determinant $\text{Det}[-\partial^2]$ as a factor inside the integral (20). Since this determinant does not depend on the EM field or the source, we may pull it outside the integral, and even absorb it into the overall normalization factor of the integral. However, I prefer to keep this factor explicit in eq. (20) for two reasons:

1. In the analogue of eq. (20) for the non-abelian gauge fields (see my notes for the details), the Fadde'ev–Popov determinant becomes field-dependent, and we really need to keep it inside the functional integral. In an abelian gauge theory like QED this is not necessary, but I would like you to get used to this factor before we get to the non-abelian theories.

2. While the determinant $\text{Det}[-\partial^2]$ does not depend on any fields, it does depend on the geometry of the Euclidean spacetime. In particular, at a finite temperature $T = 1/\beta$ the Euclidean time x_4 becomes periodic with period β, so the determinant $\text{Det}[-partial^2]$ should be taken over the Hilbert space of peridic functions $\psi(\mathbf{x}, x+\beta) = \psi(\mathbf{x}, x_4)$. Consequently, the Fadde'ev-Popov determinant depends on the temperature, which affects the thermal free energy $\mathcal{F} = -T\log Z(T)$ of the electromagnetic field. To see how this works in detail, see homework set#20, problem 1(f).

The Photon Propagator.

Now let's use the functional integral (20) to generate the photon propagator in the Landau gauge. Similar to the free scalar case, we start with the Euclidean action and split into a source-less term for a shifted field plus a term involving only the source,

$$\begin{aligned} S_E[A,J] &= \int d^4x \Big(\tfrac{1}{2}(\partial_\nu A_\mu)^2 \,-\, \tfrac{1}{2}(\partial_\mu A_\mu)^2 \,-\, A_\mu J_\mu \Big) \\ &= S_E[A',J] \,-\, \frac{1}{2}\int d^4x \int d^4y \, J_\mu(x) G_{\mu\nu}(x-y) J_\nu(y) \end{aligned} \tag{21}$$

where $G_{\mu\nu}(x-y)$ is some kernel while $A'_\mu(x) = A_\mu(x) +$ some J-dependent shift. However, due to the $\Delta[\partial_\mu A_\mu]$ factor in the functional integral (20), both the original A_μ field and the shifted field A'_μ must obey the Landau gauge condition $\partial_\mu A_\mu = \partial_\mu A'_\mu = 0$.

In the momentum space, the action becomes

$$S_E = \int \frac{d^4k}{(2\pi)^4} \left(\frac{k^2}{2} A_\mu(-k) A_\mu(k) \,-\, A_\mu(-k) J_\mu(k) \right). \tag{22}$$

To split the action as in eq. (21), we let

$$A'_\mu(k) = A_\mu(k) \,-\, \frac{1}{k^2} \Pi^\perp_{\mu\nu}(k) J_\nu(k) \tag{23}$$

where the

$$\Pi^\perp_{\mu\nu}(k) = \delta_{\mu\nu} \,-\, \frac{k_\mu k_\nu}{k^2} \tag{24}$$

factor assures that we preserve the Landau gauge condition $k_\mu A'_\mu(x) = k_\mu A_\mu(k) = 0$. Consequently,

$$\begin{aligned} \frac{k^2}{2} A'_\mu(-k) A'_\mu(k) &= \frac{k^2}{2} A_\mu(-k) A_\mu(k) \,-\, A_\mu(-k) \Pi^\perp_{\mu\nu}(k) J_\nu(k) \,+\, \frac{1}{2k^2} J_\mu(-k) \Pi^\perp_{\mu\nu}(k) J_\nu(k) \\ & \langle\!\langle \text{ using } A_\mu(-k) k_\mu = 0 \text{ and hence } A_\mu(-k)\Pi^\perp_{\mu\nu}(k) = A_\nu(-k) \,\rangle\!\rangle \\ &= \frac{k^2}{2} A_\mu(-k) A_\mu(k) \,-\, A_\nu(-k) J_\nu(k) \,+\, \frac{1}{2k^2} J_\mu(-k) \Pi^\perp_{\mu\nu}(k) J_\nu(k), \end{aligned} \tag{25}$$

and therefore

$$S_E[A,J] \;=\; S_E[A',\cancel{J}] \;-\; \frac{1}{2}\int\frac{d^4k}{(2\pi)^4}\,J_\mu(-k)\,\frac{\Pi^\perp_{\mu\nu}(k)}{k^2}\,J_\nu(k). \tag{26}$$

In the Euclidean coordinate space, this formula becomes eq. (21) for

$$G_{\mu\nu}(x-y) \;=\int\frac{d^4k}{(2\pi)^4}\,e^{-ik(x-y)}\times\frac{\Pi^\perp_{\mu\nu}(k)}{k^2} \;=\; \frac{1}{8\pi^2}\left(\frac{1}{|x-y|^2}\;+\;\frac{2(x-y)^\mu(x-y)^\nu}{|x-y|^4}\right). \tag{27}$$

Once we have proved eq. (21), the photon propagator in the Landau gauge emerges from the partition function (20) in the usual way. Indeed,

$$\begin{aligned}
Z[J] \;&=\iiint\mathcal{D}[A_\mu(x)]\,\Delta[\partial_\mu A_\mu(x)]\,\mathrm{Det}[-\partial^2]\times\exp\bigl(-S_E[A,J]\bigr)\\
&=\iiint\mathcal{D}[A'_\mu(x)]\,\Delta[\partial_\mu A'_\mu(x)]\,\mathrm{Det}[-\partial^2]\times\\
&\qquad\qquad\times\exp\left(S_E[A',\cancel{J}]\;-\;\frac{1}{2}\int\!d^4x\int\!d^4y J_\mu(x)G_{\mu\nu}(x-y)J_\nu(y)\right)\\
&=\;\exp\left(-\frac{1}{2}\int\!d^4x\int\!d^4y J_\mu(x)G_{\mu\nu}(x-y)J_\nu(y)\right)\times\\
&\quad\;\times\iiint\mathcal{D}[A'_\mu(x)]\,\Delta[\partial_\mu A'_\mu(x)]\,\mathrm{Det}[-\partial^2]\times\exp\bigl(-S_E[A',\cancel{J}]\bigr)\\
&=\;\exp\left(-\frac{1}{2}\int\!d^4x\int\!d^4y J_\mu(x)G_{\mu\nu}(x-y)J_\nu(y)\right)\times Z[0],
\end{aligned}\tag{28}$$

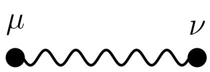
so the generation functional $F[J]=-\log Z[J]$ for the correlation functions of the EM fields is simply

$$F[J] \;=\; F[0] \;+\; \frac{1}{2}\int\!d^4x\int\!d^4y J_\mu(x)G_{\mu\nu}(x-y)J_\nu(y). \tag{29}$$

Therefore, the photon's propagator in the Euclidean coordinate space is

$$\langle A_\mu(x)A_\nu(y)\rangle \;=\; \frac{\delta^2F[J]}{\delta J_\mu(x)\,\delta J_\nu(y)} \;=\; G_{\mu\nu}(x-y), \tag{30}$$

precisely as in eq. (27). In the momentum space, this Landau-gauge photon propagator is



$$\begin{aligned}
&= \frac{\Pi^\perp_{\mu\nu}(k)}{k^2} \;=\; \frac{1}{k^2}\left(\delta_{\mu\nu}\;-\;\frac{k_\mu k_\nu}{k^2}\right) &&\text{(Euclidean)}\\
&\to\;\frac{-i}{k^2+i0}\left(g_{\mu\nu}\;-\;\frac{k_\mu k_\nu}{k^2+i0}\right) &&\text{(Minkowski)}
\end{aligned}\tag{31}$$

The Feynman Gauge

The functional integral (20) and the photon propagator (31) are for the EM field in the Landau gauge $\partial_\mu A_\mu(x) \equiv 0$. In the Feynman gauge, there is no simple gauge-fixing condition like this; instead, we use a two-step procedure.

1. First, pick an arbitrary scalar function $\omega(x)$ and replace the Landau Gauge condition with $\partial_\mu A_\mu(x) \equiv \omega(x)$ for the fixed $\omega(x)$. In terms of the functional integral, this means

$$Z[J,\omega] \;=\; \iiiint \mathcal{D}[A_\mu(x)]\,\Delta[\partial_\mu A_\mu - \omega] \times D_{FP} \times \exp\bigl(-S_E[A,J]\bigr) \tag{32}$$

where D_{FP} is the Fadde'ev–Popov determinant for the modified gauge condition,

$$D_{FP} \;=\; \operatorname{Det}\left[\frac{\delta\bigl(\partial_\mu A_\mu^\Lambda(x) - \omega(x)\bigr)}{\delta\Lambda(y)}\right]. \tag{33}$$

Actually, this determinant is independent of the $\omega(x)$; indeed,

$$\partial_\mu A_\mu^\Lambda(x) \;-\; \omega(x) \;=\; \partial_\mu A_\mu(x) \;-\partial^2\Lambda(x) \;-\; \omega(x) \;=\; [\Lambda\text{-independent}] \;-\; \partial^2\Lambda, \tag{34}$$

hence

$$\frac{\delta\bigl(\partial_\mu A_\mu^\Lambda - \omega\bigr)}{\delta\Lambda} \;=\; -\partial^2 \quad\Longrightarrow\quad D_{FP} \;=\; \operatorname{Det}[-\partial^2], \tag{35}$$

exactly as in the Landau gauge. Consequently, the functional integral

$$Z[J,\omega] \;=\; \iiiint \mathcal{D}[A_\mu(x)]\,\Delta[\partial_\mu A_\mu - \omega] \times \operatorname{Det}[-\partial^2] \times \exp\bigl(-S_E[A,J]\bigr) \tag{36}$$

does not depend on the $\omega(x)$ function; any $\omega(x)$ gives the same $Z[J,\not\omega]$ as in the Landau gauge.

2. Since the $\omega(x)$ does not make any difference, let's *average* the partition function (36) over all possible $\omega(x)$ functions with a Gaussian weight

$$\exp\left(-\frac{1}{2\xi}\int d^4x\,\omega^2(x)\right). \tag{37}$$

Up to a constant overall factor $(\operatorname{Det}[\xi])^{-1/2}$, this means redefining the partition function as

$$Z[J] = \iiint \mathcal{D}[\omega(x)] \exp\left(-\frac{1}{2\xi}\int d^4x\, \omega^2(x)\right) \times Z[J,\omega]. \tag{38}$$

If we combine the exponential here with the exponential in eq. (36), we may rewrite this gauge-averaged partition function as

$$Z[J] = \iiint \mathcal{D}[\omega(x)] \iiint \mathcal{D}[A_\mu(x)]\, \Delta[\partial_\mu A_\mu - \omega] \times \mathrm{Det}[-\partial^2] \times \exp\left(-\int d^4x \big(\mathcal{L}^{\mathrm{net}} - J_\mu A_\mu\big)\right) \tag{39}$$

where

$$\mathcal{L}^{\mathrm{net}} = \tfrac{1}{4}F_{\mu\nu}^2 + \frac{1}{2\xi}\omega^2. \tag{40}$$

Now in the context of the integral (39) over both $\omega(x)$ and $A_\mu(x)$, we may reinterpret the gauge-fixing factor $\Delta[\partial_\mu A_\mu - \omega]$ as a constraint on the $\omega(x)$ functions rather than a constraint on the EM potentials $A_\mu(x)$. Consequently, integrating over the $\omega(x)$ before integrating over the $A_\mu(x)$, we have

$$\iiint \mathcal{D}[\omega(x)]\, \Delta[\partial_\mu A_\mu - \omega] = 1, \tag{41}$$

but everywhere else in the integral (39) we should let $\omega(x) = \partial_\mu A_\mu(x)$. In particular, the net Lagrangian (40) becomes

$$\mathcal{L}^{\mathrm{net}} = \tfrac{1}{4}F_{\mu\nu}^2 + \frac{1}{2\xi}(\partial_\mu A_\mu)^2. \tag{42}$$

The second term here is usually called *the gauge-fixing term.*

Altogether, taking the integral over ω in eq. (39) leaves us with the Feynman-gauge partition function

$$Z[J] = \iiint \mathcal{D}[A_\mu(x)] \exp\left(-\int d^4x \big(\mathcal{L}^{\mathrm{net}} - J_\mu A_\mu\big)\right) \times \mathrm{Det}[-\partial^2]. \tag{43}$$

Note: this functional integral is over all possible potentials $A_\mu(x)$, including both the physically distinct and the gauge-equivalent potentials; there is no gauge-fixing Δ-functional in this integral. Instead, we have effectively modified the Lagrangian for the EM field by adding a gauge-fixing term. Consequently, we do not factor out the integral over the gauge-equivalent potentials; instead, the gauge-fixing term in the Lagrangian turns this unphysical integral into a convergent overall constant, which is OK.

Since in the Feynman gauge there are no constraints, the photon propagators obtains directly from the net Lagrangian (42). Indeed, integrating by parts the net Euclidean action, we obtain

$$S_E \;=\; \int\! d^4x \left(\tfrac{1}{4}F_{\mu\nu}^2 \;+\; \tfrac{1}{2\xi}\,(\partial_\mu A_\mu)^2 \;-\; J_\mu A_\mu\right) \;=\; \int\! d^4x \Big(\tfrac{1}{2} A_\mu D_{\mu\nu} A_\nu \;-\; J_\nu A_\nu\Big) \tag{44}$$

for the differential operator

$$D_{\mu\nu} \;=\; -\partial^2 \times \delta_{\mu\nu} \;+\; \partial_\mu\partial_\nu \;-\; \frac{1}{\xi}\,\partial_\mu\partial_\nu\,. \tag{45}$$

Then proceeding exactly as we did for the scalar and Dirac fields in class — calculating the Gaussian partition function by shifting the A field, then using $F = -\log Z[J]$ as the generating functional, — we find that the coordinate-space photon propagator is simply the inverse of the $D_{\mu\nu}$ operator. In the Euclidean momentum space, this operator becomes

$$D_{\mu\nu}(k) \;=\; k^2 \times \delta_{\mu\nu} \;+\; \left(\xi^{-1} \,-\, 1\right) \times k_\mu k_\nu \;=\; k^2 \times \Pi^{\perp}_{\mu\nu}(k) \;+\; \frac{k^2}{\xi} \times \Pi^{\|}_{\mu\nu}(k) \tag{46}$$

where $\Pi^{\perp}$ and $\Pi^{\|}$ are the projection matrices onto directions $\perp$ and $\|$ to the momentum k. Consequently, the inverse operator — and hence the photon propagator in the momentum space — obtains as the inverse of the matrix (46) for each k, thus

$$D^{-1}(k) \;=\; \frac{1}{k^2} \times \Pi^{\perp} \;+\; \frac{\xi}{k^2} \times \Pi^{\|}, \tag{47}$$

or in explicit index notations

$$\begin{aligned} &= \frac{1}{k^2}\left(\delta_{\mu\nu} \,+\, (\xi - 1) \times \frac{k_\mu k_\nu}{k^2}\right) && \text{(Euclidean)} \\ &\to \frac{-i}{k^2 + i0}\left(g_{\mu\nu} \,+\, (\xi - 1) \times \frac{k_\mu k_\nu}{k^2 + i0}\right) && \text{(Minkowski)} \end{aligned} \tag{48}$$

PS: Strictly speaking, eq. (48) gives a whole family of propagators for the Feynman-like gauges. The Feynman gauge proper obtains for $\xi = 1$.