1. As a warm-up exercise, consider a charged non-relativistic particle subject to given electric and magnetic fields. The classical Lagrangian for this particle (in $c = \hbar = 1$ units) is

$$L(\mathbf{x},\mathbf{v}) \;=\; \frac{m}{2}\,\mathbf{v}^2 \;-\; q\Phi(\mathbf{x}) \;+\; q\mathbf{v}\cdot\mathbf{A}(\mathbf{x}). \tag{1}$$

   (a) Derive the Hamiltonian formalism for the particle's motion. Show that the canonical momentum of the particle is *not* $m\mathbf{v}$ but rather

$$\mathbf{p} \;=\; m\mathbf{v} \;+\; q\mathbf{A}(\mathbf{x}) \tag{2}$$

   while the Hamiltonian is

$$H(\mathbf{x},\mathbf{p}) \;=\; \frac{1}{2m}\left(\mathbf{p} \;-\; q\mathbf{A}(\mathbf{x})\right)^2 +\; q\Phi(\mathbf{x}). \tag{3}$$

   (b) Write the Hamilton equations of motion for the particle and show that they lead to

$$m\dot{\mathbf{v}} \;=\; q\big(\mathbf{E} \;+\; \mathbf{v}\times\mathbf{B}\big). \tag{4}$$

For the quantum charged particle, the position $\mathbf{x}(t)$ and the canonical momentum $\mathbf{p}(t)$ become operators $\hat{\mathbf{x}}$ and $\hat{\mathbf{p}}$ obeying canonical equal-time commutation relations

$$\big[\hat{x}_i,\hat{x}_j\big] \;=\; 0, \quad \big[\hat{p}_i,\hat{p}_j\big] \;=\; 0, \quad \big[\hat{x}_i,\hat{p}_j\big] \;=\; i\delta_{ij} \tag{5}$$

while the Hamiltonian operator follows from the classical Hamiltonian (3),

$$\hat{H} \;=\; \frac{1}{2m}\,\vec{\hat{\pi}}^{2} \;+\; q\Phi(\hat{\mathbf{x}}) \tag{6}$$

$$\text{where}\quad \vec{\hat{\pi}} \;\stackrel{\text{def}}{=}\; \hat{\mathbf{p}} \;-\; q\mathbf{A}(\hat{\mathbf{x}}). \tag{7}$$

Physically, $\vec{\hat{\pi}}$ is the kinematic momentum operator corresponding to the classical kinematic momentum $m\mathbf{v}$ rather than the canonical momentum $\mathbf{p}$.

(c) Calculate the equal-time commutators $[\hat{x}_i, \hat{\pi}_j]$ and $[\hat{\pi}_i, \hat{\pi}_j]$.

(d) Finally, calculate the commutators of the $\hat{\mathbf{x}}$ and $\hat{\vec{\pi}}$ operators with the Hamiltonian (6) and compare the resulting Heisenberg equations with the classical equations of motion for the charged particle.

2. Next, consider the free electromagnetic fields. For the EM fields coupled to electric currents, the Hamiltonian formalism — and hence the canonical quantization — involves the potetials $A^\mu(x)$, which need to be gauge fixed. Alas, the gauge redundancy does not agree with the Hamiltonian formalism, which complicates the quantization of interacting EM fields; so in this class I shall postpone this issue till November. However, for the *free* EM fields — *i.e.*, not coupled to any electric charges or currents — the quantum theory can be reduced to the quantum tension fields $\hat{\mathbf{E}}(\mathbf{x},t)$ and $\hat{\mathbf{B}}(\mathbf{x},t)$, and that's what this problem is about.

   In the quantum theory, the time-independent Maxwell equations are implemented as operatorial identities

$$\nabla \cdot \hat{\mathbf{E}}(\mathbf{x},t) = \nabla \cdot \hat{\mathbf{B}}(\mathbf{x},t) = 0 \tag{8}$$

   in the Hilbert space, while the time-dependent Maxwell equations follows from the Hamiltonian

$$\hat{H}_{EM} = \int d^3\mathbf{x} \left(\tfrac{1}{2}\hat{\mathbf{E}}^2 + \tfrac{1}{2}\hat{\mathbf{B}}^2\right) \tag{9}$$

   and the equal-time commutation relations

$$\begin{aligned} \left[\hat{E}_i(\mathbf{x},t), \hat{E}_j(\mathbf{x}',t'=t)\right] &= 0, \\ \left[\hat{B}_i(\mathbf{x},t), \hat{B}_j(\mathbf{x}',t'=t)\right] &= 0, \\ \left[\hat{E}_i(\mathbf{x},t), \hat{B}_j(\mathbf{x}',t'=t)\right] &= -i\hbar c \epsilon_{ijk} \frac{\partial}{\partial x_k} \delta^{(3)}(\mathbf{x}-\mathbf{x}'). \end{aligned} \tag{10}$$

   (a) Verify that the commutation relations (10) are consistent with the time-independent Maxwell equations (8).

   (b) Derive the time-dependent Maxwell equations from the Hamiltonian (9) and the commutation relations (10).

3. Next, consider the *massive* relativistic vector field $A^\mu(x)$. with classical Lagrangian density (in $\hbar = c = 1$ units)

$$\mathcal{L} \;=\; -\tfrac{1}{4}\,F_{\mu\nu}F^{\mu\nu} \;+\; \tfrac{1}{2}m^2\,A_\mu A^\mu \;-\; A^\mu J_\mu \tag{11}$$

where the current $J^\mu(x)$ is a fixed source for the $A^\mu(x)$ field. Because of the mass term, the Lagrangian (11) is *not* gauge invariant. However, we *assume* that the current $J^\mu(x)$ is conserved, $\partial_\mu J^\mu(x) = 0$.

Back in homework set#1 (problem 1) we have derived the Euler–Lagrange equations for the massive vector field. In this problem, we develop the Hamiltonian formalism for the $A^\mu(x)$. Our first step is to identify the canonically conjugate "momentum" fields.

(a) Show that $\partial\mathcal{L}/\partial\dot{\mathbf{A}} = -\mathbf{E}$ but $\partial\mathcal{L}/\partial\dot{A}_0 \equiv 0$.

In other words, the canonically conjugate field to $\mathbf{A}(\mathbf{x})$ is $-\mathbf{E}(\mathbf{x})$ but the $A_0(\mathbf{x})$ does not have a canonical conjugate! Consequently,

$$H \;=\; \int\! d^3\mathbf{x}\left(-\dot{\mathbf{A}}(\mathbf{x})\cdot\mathbf{E}(\mathbf{x}) \;-\; \mathcal{L}\right). \tag{12}$$

(b) Show that in terms of the $\mathbf{A}$, $\mathbf{E}$, and A_0 fields, and their *space* derivatives,

$$H \;=\; \int\! d^3\mathbf{x}\left\{\tfrac{1}{2}\mathbf{E}^2 \;+\; A_0\,(J_0 - \nabla\cdot\mathbf{E}) \;-\; \tfrac{1}{2}m^2A_0^2 \;+\; \tfrac{1}{2}\,(\nabla\times\mathbf{A})^2 \;+\; \tfrac{1}{2}m^2\mathbf{A}^2 \;-\; \mathbf{J}\cdot\mathbf{A}\right\}. \tag{13}$$

Because the A_0 field does not have a canonical conjugate, the Hamiltonian formalism does not produce an equation for the time-dependence of this field. Instead, it gives us a time-independent equation relating the $A_0(\mathbf{x},t)$ to the values of other fields *at the same time* t. Specifically, we have

$$\begin{pmatrix}\text{variational}\\ \text{derivative}\end{pmatrix}\quad \frac{\delta H}{\delta A_0(\mathbf{x})} \;\equiv\; \frac{\partial\mathcal{H}}{\partial A_0}\bigg|_{\mathbf{x}} \;-\; \nabla\cdot\frac{\partial\mathcal{H}}{\partial(\nabla A_0)}\bigg|_{\mathbf{x}} \;=\; 0. \tag{14}$$

At the same time, the vector fields $\mathbf{A}$ and $\mathbf{E}$ satisfy the Hamiltonian equations of motion,

$$\begin{aligned}
\frac{\partial}{\partial t}\mathbf{A}(\mathbf{x},t) \;&=\; -\,\frac{\delta H}{\delta\mathbf{E}(\mathbf{x})}\bigg|_t \;\equiv\; -\left[\frac{\partial\mathcal{H}}{\partial\mathbf{E}} \;-\; \nabla_i\frac{\partial\mathcal{H}}{\partial(\nabla_i\mathbf{E})}\right]_{(\mathbf{x},t)},\\
\frac{\partial}{\partial t}\mathbf{E}(\mathbf{x},t) \;&=\; +\,\frac{\delta H}{\delta\mathbf{A}(\mathbf{x})}\bigg|_t \;\equiv\; +\left[\frac{\partial\mathcal{H}}{\partial\mathbf{A}} \;-\; \nabla_i\frac{\partial\mathcal{H}}{\partial(\nabla_i\mathbf{A})}\right]_{(\mathbf{x},t)}.
\end{aligned} \tag{15}$$

(c) Write down the explicit form of all these equations.

(d) Verify that the equations you have just written down are equivalent to the relativistic Euler–Lagrange equations for the $A^\mu(x)$, namely

$$(\partial^\mu\partial_\mu + m^2)A^\nu \;=\; \partial^\nu(\partial_\mu A^\mu) \;+\; J^\nu \tag{16}$$

and hence $\partial_\mu A^\mu(x) = 0$ and $(\partial^\nu\partial_\nu + m^2)A^\mu = 0$ when $\partial_\mu J^\mu \equiv 0$, *cf.* homework #1.

4. Finally, let's quantize the massive vector field from the previous problem. Since classically the $-\mathbf{E}(\mathbf{x})$ fields are canonically conjugate momenta to the $\mathbf{A}(\mathbf{x})$ fields, the corresponding quantum fields $\hat{\mathbf{E}}(\mathbf{x})$ and $\hat{\mathbf{A}}(\mathbf{x})$ satisfy the canonical equal-time commutation relations

$$\begin{aligned} [\hat{A}_i(\mathbf{x},t), \hat{A}_j(\mathbf{y},t)] &= 0, \\ [\hat{E}_i(\mathbf{x},t), \hat{E}_j(\mathbf{y},t)] &= 0, \\ [\hat{A}_i(\mathbf{x},t), \hat{E}_j(\mathbf{y},t)] &= -i\delta_{ij}\delta^{(3)}(\mathbf{x}-\mathbf{y}) \end{aligned} \tag{17}$$

(in the $\hbar = c = 1$ units). The currents also become quantum fields $\hat{J}^\mu(\mathbf{x},t)$, but they are composed of some kind of charged degrees of freedom independent from the vector fields in question. Consequently, *at equal times* the currents $\hat{J}^\mu(\mathbf{x},t)$ commute with both the $\hat{\mathbf{E}}(\mathbf{y},t)$ and the $\hat{\mathbf{A}}(\mathbf{y},t)$ fields.

The classical $A^0(\mathbf{x},t)$ field does not have a canonical conjugate and its equation of motion does not involve time derivatives. In the quantum theory, $\hat{A}^0(\mathbf{x},t)$ satisfies a similar time-independent constraint

$$m^2\hat{A}^0(\mathbf{x},t) \;=\; \hat{J}^0(\mathbf{x},t) \;-\; \nabla\cdot\hat{\mathbf{E}}(\mathbf{x},t), \tag{18}$$

but from the Hilbert space point of view this is an operatorial identity rather than an equation of motion. Consequently, the commutation relations of the scalar potential field follow from eqs. (17); in particular, at equal times the $\hat{A}^0(\mathbf{x},t)$ commutes with the $\hat{\mathbf{E}}(\mathbf{y},t)$ but does not commute with the $\hat{\mathbf{A}}(\mathbf{y},t)$.

Finally, the Hamiltonian operator follows from the classical eq. (13), namely

$$\begin{aligned}\hat{H} &= \int d^3\mathbf{x} \left\{ \tfrac{1}{2}\hat{\mathbf{E}}^2 + \hat{A}_0 \left(\hat{J}_0 - \nabla \cdot \hat{\mathbf{E}} \right) - \tfrac{1}{2}m^2 \hat{A}_0^2 + \tfrac{1}{2}\left(\nabla \times \hat{\mathbf{A}} \right)^2 + \tfrac{1}{2}m^2 \hat{\mathbf{A}}^2 - \hat{\mathbf{J}} \cdot \hat{\mathbf{A}} \right\} \\ &= \int d^3\mathbf{x} \left\{ \tfrac{1}{2}\hat{\mathbf{E}}^2 + \frac{1}{2m^2} \left(\hat{J}_0 - \nabla \cdot \hat{\mathbf{E}} \right)^2 + \tfrac{1}{2}\left(\nabla \times \hat{\mathbf{A}} \right)^2 + \tfrac{1}{2}m^2 \hat{\mathbf{A}}^2 - \hat{\mathbf{J}} \cdot \hat{\mathbf{A}} \right\}\end{aligned} \tag{19}$$

where the second line follows from the first and eq. (18).

Your task is to calculate the commutators $[\hat{A}_i(\mathbf{x}, t), \hat{H}]$ and $[\hat{E}_i(\mathbf{x}, t), \hat{H}]$ and write down the Heisenberg equations for the quantum vector fields. Make sure those equations are similar to the Hamilton equations for the classical fields.