1. Consider once again the massive vector field $\hat{A}^\mu(x)$. In the previous homework (set#3, problem 2), you (should have) expanded the free vector field into the creation and annihilation operators multiplied by the plane-waves according to

$$\hat{A}^\mu(x) = \int \frac{d^3\mathbf{k}}{(2\pi)^3} \frac{1}{2\omega_\mathbf{k}} \sum_\lambda \left(e^{-ikx} \times f^\mu_{\mathbf{k},\lambda} \times \hat{a}_{\mathbf{k},\lambda} + e^{+ikx} \times f^{*\mu}_{\mathbf{k},\lambda} \times \hat{a}^\dagger_{\mathbf{k},\lambda} \right)^{k^0=+\omega_\mathbf{k}}. \quad (1)$$

The λ here labels the independent polarizations of a vector particle (for example, the helicities $\lambda = -1, 0, +1$), while $f^\mu_{\mathbf{k},\lambda}$ are the polarization vectors obeying

$$k_\mu f^\mu_{\mathbf{k},\lambda} = 0, \quad g_{\mu\nu} f^\mu_{\mathbf{k},\lambda} f^{*\nu}_{\mathbf{k},\lambda'} = -\delta_{\lambda,\lambda'}. \quad (2)$$

In this problem, we shall calculate the Feynman propagator for the massive vector field (1).

(a) First, a lemma: Show that any polarization vectors obeying the constraints (2) also obey

$$\sum_\lambda f^\mu_{\mathbf{k},\lambda} f^{*\nu}_{\mathbf{k},\lambda} = -g^{\mu\nu} + \frac{k^\mu k^\nu}{m^2}. \quad (3)$$

(b) Next, calculate the "vacuum sandwich" of two vector fields and show that

$$\langle 0| \hat{A}^\mu(x)\hat{A}^\nu(y) |0\rangle = \int \frac{d^3\mathbf{k}}{(2\pi)^3} \frac{1}{2\omega_\mathbf{k}} \left[\left(-g^{\mu\nu} + \frac{k^\mu k^\nu}{m^2} \right) e^{-ik(x-y)} \right]_{k^0=+\omega_\mathbf{k}}$$

$$= \left(-g^{\mu\nu} - \frac{\partial^\mu \partial^\nu}{m^2} \right) D(x-y). \quad (4)$$

(c) Now consider a free scalar field (of the same mass m as the vector field) and its Feynman propagator $G_F^{\text{scalar}}(x-y)$. Show that

$$\left(-g^{\mu\nu} - \frac{\partial^\mu \partial^\nu}{m^2} \right) G_F^{\text{scalar}}(x-y) = \langle 0| \mathbf{T}\hat{A}^\mu(x)\hat{A}^\nu(y) |0\rangle + \frac{i}{m^2} \delta^{\mu 0}\delta^{\nu 0}\delta^{(4)}(x-y). \quad (5)$$

To avoid the δ–function singularity in formulae like (5), the time-ordered product of the vector fields (or rather, just of their $\hat{A}^0$ components) is *modified*[★] according to

$$\mathbf{T}^* \hat{A}^\mu(x)\hat{A}^\nu(y) \;=\; \mathbf{T}\hat{A}^\mu(x)\hat{A}^\nu(y) \;+\; \frac{i}{m^2}\,\delta^{\mu 0}\delta^{\nu 0}\delta^{(4)}(x-y). \tag{6}$$

Consequently, the Feynman propagator for the massive vector field is defined using the modified time-ordered product of the two fields,

$$G_F^{\mu\nu}(x-y) \;\overset{\text{def}}{=}\; \langle 0|\,\mathbf{T}^* \hat{A}^\mu(x)\hat{A}^\nu(y)\,|0\rangle \tag{7}$$

(d) Show that this propagator obtains as

$$G_F^{\mu\nu}(x-y) \;=\; \int \frac{d^4\mathbf{k}}{(2\pi)^4}\left(-g^{\mu\nu} + \frac{k^\mu k^\nu}{m^2}\right) \times \frac{ie^{-ik(x-y)}}{k^2 - m^2 + i0}\,. \tag{8}$$

(e) Finally, write the classical action for the free vector field as

$$S \;=\; \tfrac{1}{2}\int d^4x\, A_\mu(x)\,\mathcal{D}^{\mu\nu} A_\nu(x) \tag{9}$$

where $\mathcal{D}^{\mu\nu}$ is a differential operator, and show that the Feynman propagator (8) is a Green's function of this operator,

$$\mathcal{D}_x^{\mu\nu} G_{\nu\lambda}^F(x-y) \;=\; +i\delta_\lambda^\mu \delta^{(4)}(x-y). \tag{10}$$

2. Next, a reading assignment. To help you understand the relations between the continuous symmetries, their generators, the multiplets, and the representations of the generators and of the finite symmetries, read about the rotational symmetry and its generators in chapter 3 of the J. J. Sakurai's book *Modern Quantum Mechanics.*[†] Please focus on sections 1, 2, 3, second half of section 5 (representations of the rotation operators), and section 10; the other sections 4, 6, 7, 8, and 9 are not relevant to the present class material.

PS: If you have already read the Sakurai's book before but it has been a while, please read it again.

[★] See *Quantum Field Theory* by Claude Itzykson and Jean–Bernard Zuber.
[†] The UT Math–Physics–Astronomy library has several hard copies but no electronic copies of the book. However, you can find several pirate scans of the book (in PDF format) all over the web; Google them up if you cannot find a legitimate copy.

3. Finally, consider a complex scalar field $\Phi(x) \neq \Phi^*(x)$ with the classical Lagrangian density

$$\mathcal{L} = \partial_\mu \Phi^* \partial^\mu \Phi - m^2 \Phi^* \Phi - \frac{\lambda}{2} (\Phi^* \Phi)^2. \tag{11}$$

This Lagrangian has a global phase symmetry

$$\Phi(x) \rightarrow \Phi'(x) = e^{-i\theta} \Phi(x), \quad \Phi^*(x) \rightarrow \Phi^{*\prime}(x) = e^{+i\theta} \Phi^*(x). \tag{12}$$

According to the Noether theorem — which I should soon explain in class — this symmetry gives rise to a conserved current

$$J^\mu = i\Phi^* \partial^\mu \Phi - i\Phi \partial^\mu \Phi^*. \tag{13}$$

(a) Write down the classical equations for the fields Φ and Φ^* — treat them as independent variables — and verify that these field equations indeed lead to the conservation of the current (13), $\partial_\mu J^\mu = 0$.

Canonical quantization of the complex field yields non-hermitian quantum fields $\hat{\Phi}(x) \neq \hat{\Phi}^\dagger(x)$ and $\hat{\Pi}(x) \neq \hat{\Pi}^\dagger(x)$ and the Hamiltonian

$$\hat{H} = \int d^3\mathbf{x} \left(\hat{\Pi}^\dagger \hat{\Pi} + \nabla\hat{\Phi}^\dagger \cdot \nabla\hat{\Phi} + m^2 \hat{\Phi}^\dagger \hat{\Phi} + \frac{\lambda}{2} \hat{\Phi}^\dagger \hat{\Phi}^\dagger \hat{\Phi} \hat{\Phi} \right). \tag{14}$$

(b) Derive this Hamiltonian and write down the equal-time commutation relations for all the quantum fields.

In the quantum theory, the conserved current (13) becomes operator valued

$$\hat{\mathbf{J}} = -i\hat{\Phi}^\dagger \nabla\hat{\Phi} + i\hat{\Phi} \nabla\hat{\Phi}^\dagger,$$
$$\hat{J}^0 = \tfrac{i}{2}\{\hat{\Pi}^\dagger, \hat{\Phi}^\dagger\} - \tfrac{i}{2}\{\hat{\Pi}, \hat{\Phi}\} \tag{15}$$

$$\text{modulo operator ordering ambiguity,}$$

with the net charge operator being

$$\hat{Q} = \int d^3\mathbf{x}\, \hat{J}^0. \tag{16}$$

(c) Show that $[\hat{Q}, \hat{\Phi}(x)] = -\hat{\Phi}(x)$ while $[\hat{Q}, \hat{\Phi}^\dagger(x)] = +\hat{\Phi}^\dagger(x)$ and therefore the charge

operator $\hat{Q}$ *generates* the phase symmetry (12) according to

$$\exp(+i\theta\hat{Q})\hat{\Phi}(x)\exp(-i\theta\hat{Q}) \;=\; e^{-i\theta}\hat{\Phi}(x), \quad \exp(+i\theta\hat{Q})\hat{\Phi}^\dagger(x)\exp(-i\theta\hat{Q}) \;=\; e^{+i\theta}\hat{\Phi}^\dagger(x).$$
(17)

(d) Verify that the net charge operator commutes with the Hamiltonian (14) — that's what charge conservation means in quantum mechanics.

(e) Finally, for the free complex fields (*i.e.*, for $\lambda = 0$), expand the quantum fields into creation and annihilation operators (for both particles and antiparticles), then show that

$$\hat{Q} \;=\; \int \frac{d^3\mathbf{k}}{(2\pi)^3} \frac{1}{2\omega_\mathbf{k}} \left(\hat{a}_\mathbf{k}^\dagger \hat{a}_\mathbf{k} - \hat{b}_\mathbf{k}^\dagger \hat{b}_\mathbf{k} \right) \;=\; \#\text{particles} \;-\; \#\text{antiparticles}.$$
(18)