1. According to the Noether theorem, a translationally invariant system of classical fields $\phi_a(x)$ has a conserved stress-energy tensor

$$T^{\mu\nu}_{\text{Noether}} = \sum_a \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\,\partial^\nu\phi_a \;-\; g^{\mu\nu}\,\mathcal{L}. \tag{1}$$

Actually, to assure the symmetry of the stress-energy tensor, $T^{\mu\nu} = T^{\nu\mu}$ (which is necessary for the angular momentum conservation), one sometimes has to add a total divergence,

$$T^{\mu\nu} \;=\; T^{\mu\nu}_{\text{Noether}} \;+\; \partial_\lambda\mathcal{K}^{\lambda\mu\,\nu}, \tag{2}$$

where $\mathcal{K}^{\lambda\mu\,\nu} \equiv -\mathcal{K}^{\mu\lambda\,\nu}$ is some 3–index Lorentz tensor antisymmetric in its first two indices.

(a) Show that regardless of the specific form of the $\mathcal{K}^{\lambda\mu\,\nu}(\phi,\partial\phi)$ as a function of the fields and their derivatives, we have

$$\begin{aligned} \partial_\mu T^{\mu\nu} \;&=\; \partial_\mu T^{\mu\nu}_{\text{Noether}} \;= \text{(hopefully)} =\; 0 \\ \text{and}\quad P^\mu_{\text{net}} \;\equiv \int\! d^3\mathbf{x}\, T^{0\mu} \;&=\; \int\! d^3\mathbf{x}\, T^{0\mu}_{\text{Noether}}\,. \end{aligned} \tag{3}$$

Note: Assume that all the fields go to zero for $|\mathbf{x}| \to \infty$ fast enough that all the surface integrals over the boundary of 3D space vanish when we push the boundary to infinity.

For the scalar fields, real or complex, the $T^{\mu\nu}_{\text{Noether}}$ is properly symmetric and one simply has $T^{\mu\nu} = T^{\mu\nu}_{\text{Noether}}$. Unfortunately, the situation is more complicated for the vector, tensor or spinor fields. To illustrate the problem, consider the free electromagnetic fields described by the Lagrangian

$$\mathcal{L}(A_\mu, \partial_\nu A_\mu) \;=\; -\tfrac{1}{4}\,F_{\mu\nu}F^{\mu\nu} \tag{4}$$

where A_μ is a real vector field and $F_{\mu\nu} \stackrel{\text{def}}{=} \partial_\mu A_\nu - \partial_\nu A_\mu$.

(b) Write down $T^{\mu\nu}_{\text{Noether}}$ for the free electromagnetic fields and show that it is neither symmetric nor gauge invariant.

(c) The properly symmetric — and also gauge invariant — stress-energy tensor for the free electromagnetism is

$$T^{\mu\nu}_{\text{EM}} \;=\; -F^{\mu\lambda}F^{\nu}_{\ \lambda} \;+\; \tfrac{1}{4}\,g^{\mu\nu}\,F_{\kappa\lambda}F^{\kappa\lambda}. \tag{5}$$

Show that this expression indeed has form (2) for some $\mathcal{K}^{\lambda\mu\nu}$.

(d) Write down the components of the stress-energy tensor (5) in non-relativistic notations and make sure you have the familiar electromagnetic energy density, momentum density, and stress.

Next, consider the electromagnetic fields coupled to the electric current J^μ of some charged "matter" fields. Because of this coupling, only the *net* energy-momentum of the whole field system should be conserved, but not the separate P^μ_{EM} and P^μ_{mat}. Consequently, we should have

$$\partial_\mu T^{\mu\nu}_{\text{net}} \;=\; 0 \qquad \text{for} \quad T^{\mu\nu}_{\text{net}} \;=\; T^{\mu\nu}_{\text{EM}} \;+\; T^{\mu\nu}_{\text{mat}} \tag{6}$$

but generally $\partial_\mu T^{\mu\nu}_{\text{EM}} \neq 0$ and $\partial_\mu T^{\mu\nu}_{\text{mat}} \neq 0$.

(e) Use Maxwell's equations to show that

$$\partial_\mu T^{\mu\nu}_{\text{EM}} \;=\; -F^{\nu\lambda}J_\lambda \tag{7}$$

(in $c = 1$ units), and therefore any system of charged matter fields should have its stress-energy tensor related to the electric current J_λ according to

$$\partial_\mu T^{\mu\nu}_{\text{mat}} \;=\; +F^{\nu\lambda}J_\lambda. \tag{8}$$

(f) Rewrite eq. (7) in non-relativistic notations and explain its physical meaning in terms of the electromagnetic energy, momentum, work, and forces.

2. Continuing problem 1, consider the EM fields coupled to a specific model of charged matter, namely a complex scalar field $\Phi(x) \neq \Phi^*(x)$ of electric charge $q \neq 0$. Altogether, the net Lagrangian for the A^μ, Φ, and Φ^* fields is

$$\mathcal{L}_{\text{net}} = D^\mu\Phi^* D_\mu\Phi - m^2\Phi^*\Phi - \tfrac{1}{4}F^{\mu\nu}F_{\mu\nu} \tag{9}$$

where

$$D_\mu\Phi = (\partial_\mu + iqA_\mu)\Phi \quad \text{and} \quad D_\mu\Phi^* = (\partial_\mu - iqA_\mu)\Phi^* \tag{10}$$

are the *covariant* derivatives.

(a) Write down the equation of motion for all fields in a covariant from. Also, write down the electric current

$$J^\mu \stackrel{\text{def}}{=} -\frac{\partial\mathcal{L}}{\partial A_\mu} \tag{11}$$

in a manifestly gauge-invariant form and verify its conservation, $\partial_\mu J^\mu = 0$ (as long as the scalar fields satisfy their equations of motion).

(b) Write down the Noether stress-energy tensor for the whole system and show that

$$T^{\mu\nu}_{\text{net}} \equiv T^{\mu\nu}_{\text{EM}} + T^{\mu\nu}_{\text{mat}} = T^{\mu\nu}_{\text{Noether}} + \partial_\lambda\mathcal{K}^{\lambda\mu\nu}, \tag{12}$$

where $T^{\mu\nu}_{\text{EM}}$ is exactly as in eq. (5) for the free EM fields, the improvement tensor $\mathcal{K}^{\lambda\mu\nu} = -\mathcal{K}^{\mu\lambda\nu}$ is also exactly as in problem **1**, and

$$T^{\mu\nu}_{\text{mat}} = D^\mu\Phi^*\, D^\nu\Phi + D^\nu\Phi^*\, D^\mu\Phi - g^{\mu\nu}\big(D_\lambda\Phi^*\, D^\lambda\Phi - m^2\Phi^*\,\Phi\big). \tag{13}$$

Note: although the improvement tensor $\mathcal{K}^{\lambda\mu\nu}$ for the EM + matter system is the same as for the free EM fields, in presence of an electric current J^μ its derivative $\partial_\lambda\mathcal{K}^{\lambda\mu\nu}$ contains an extra $J^\mu A^\nu$ term. Pay attention to this term — it is important for obtaining the gauge-invariant stress-energy tensor (13) for the scalar field.

(c) Use the scalar fields' equations of motion and the non-commutativity of covariant derivatives

$$[D_\mu, D_\nu]\Phi \;=\; iqF_{\mu\nu}\Phi, \qquad [D_\mu, D_\nu]\Phi^* \;=\; -iqF_{\mu\nu}\Phi^* \tag{14}$$

to show that

$$\partial_\mu T^{\mu\nu}_{\text{mat}} \;=\; +F^{\nu\lambda}J_\lambda \tag{15}$$

exactly as in eq. (8), and therefore the *net* stress-energy tensor (12) is conserved, *cf.* problem **1**(e).

3. Next, consider the Noether currents of an internal rather than translational symmetry. Let $\Phi^a(x)$ be N complex scalar fields — of similar masses and electric charges — which interact with each other and with the EM fields A^μ according to the Lagrangian

$$\mathcal{L} \;=\; \sum_a D_\mu\Phi^*_a\, D^\mu\Phi^a \;-\; m^2\sum_a \Phi^*_a\Phi^a \;-\; \frac{\lambda}{4}\left(\sum_a \Phi^*_a\Phi^a\right)^2 \;-\; \tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}. \tag{16}$$

The $D_\mu\Phi^a$ and the $D_\mu\Phi^*_a$ here are as in eq. (10) — they are covariant derivatives WRT the local $U(1)$ symmetry associated with the EM fields. Specifically, all the Φ^a fields have the same electric charge $+q$ while the complex-conjugate fields Φ^*_a have charge $-q$.

Both the electric charges and the scalar potentials are invariant under global unitary "rotations" of the Φ^a fields into each other. (But not into the conjugate Φ^*_a fields with opposite-sign electric charges.) Such symmetries act according to

$$\Phi^a(x) \;\to\; \sum_b U^a_{\ b}\times\Phi^b(x), \qquad \Phi^*_a(x) \;\to\; \sum_b \Phi^*_b(x)\times\left(U^\dagger\right)^b_{\ a}, \qquad A^\mu(x)\ \text{unchanged}, \tag{17}$$

where $U = \|U^a_{\ b}\|$ is any *unitary* $N\times N$ matrix. The group of such matrices — and hence of symmetries (17) — is called the $U(N)$.

(a) Check that the Lagrangian (16) is invariant under any $U(N)$ symmetry (17).

(b) The infinitesimal $U(N)$ symmetries have form $U = 1 - i\epsilon T$ — *i.e.*, $U^a_{\ b} = \delta^a_b - i\epsilon T^a_{\ b}$ — for *hermitian* matrices T.

Derive the Noether current J^μ_T for any given hermitian matrix T and show that it has the form

$$J^\mu_T \;=\; \mathrm{tr}\big(TJ^\mu\big) \;=\; \sum_{a,b} T^b_{\ a} J^{\mu a}_{\ \ \ b} \tag{18}$$

where $J^{\mu a}_{\ \ \ b}$ is the $N \times N$ hermitian matrix of currents

$$J^{\mu a}_{\ \ \ b} \;=\; -i\Phi^a D^\mu \Phi^*_b \;+\; i\Phi^*_b D^\mu \Phi^a \;=\; \big(J^{\mu b}_{\ \ \ a}\big)^*. \tag{19}$$

(c) Verify the conservation of the N^2 currents (19) and hence of J^μ_T for any hermitian matrix T.

The scalar potential in the Lagrangian (16) has a bigger symmetry than the $U(N)$, namely the $SO(2N)$ which rotates the real and imaginary parts of the $\Phi_a(x)$ fields as if they were $2N$ unrelated real fields. But the $SO(2N)$ symmetries outside of the $U(N)$ do not commute with the local $U(1)$ symmetry of the charged fields and hence do not preserve their couplings to the EM fields.

(d) Work this out.

The infinitesimal form of an $SO(2N)$ symmetry outside of the $U(N)$ is

$$\delta\Phi^a(x) \;=\; \epsilon \sum_b C^{ab}\Phi^*_b(x), \qquad \delta\Phi^*_a(x) \;=\; \epsilon \sum_b C^*_{ab}\Phi^b(x) \tag{20}$$

for a complex antisymmetric matrix $C^{ab} = -C^{ba}$.

(e) Write down the Noether current for such a would-be symmetry and show that it is NOT conserved (unless $A^\mu = 0$).

4. Finally, consider the quantum field theory based on the classical theory of the previous problem. The charged quantum fields are

$$\hat{\Phi}^a(x), \quad \hat{\Phi}^\dagger_a(x), \quad \hat{\Pi}_a(x) \;=\; D_0\hat{\Phi}^\dagger_a(x), \quad \hat{\Pi}^{\dagger a}(x) \;=\; D_0\hat{\Phi}^a(x), \tag{21}$$

there are also quantum EM fields in some gauge, and the Hamiltonian is

$$\hat{H} \;=\; \hat{H}_{\rm EM} \;+\int\! d^3\mathbf{x}\left[\sum_a\left(\hat{\Pi}^{\dagger a}\hat{\Pi}_a \;+\; q\hat{A}_0\hat{J}^{0a}{}_a \;+\; \mathbf{D}\hat{\Phi}^\dagger_a\cdot\mathbf{D}\hat{\Phi}^a\right) \;+\; \hat{V}\right] \tag{22}$$

for

$$\hat{V} \;=\; m^2\sum_a \hat{\Phi}^\dagger_a\hat{\Phi}^a \;+\; \frac{\lambda}{2}\left(\sum_a \hat{\Phi}^\dagger_a\hat{\Phi}^a\right)^2. \tag{23}$$

The $U(N)$ symmetry currents (19) become operators

$$\hat{\mathbf{J}}^a{}_b \;=\; i\hat{\Phi}^a\mathbf{D}\hat{\Phi}^\dagger_b \;-\; i\hat{\Phi}^\dagger_b\mathbf{D}\hat{\Phi}^a, \tag{24}$$

$$\begin{aligned}
\hat{J}^{0a}{}_b \;&=\; -\tfrac{i}{2}\{\hat{\Phi}^a,\hat{\Pi}_b\} \;+\; \tfrac{i}{2}\{\hat{\Phi}^\dagger_b,\hat{\Pi}^{\dagger a}\} \\
&=\; -i\hat{\Pi}_b\hat{\Phi}^a \;+\; i\hat{\Pi}^{\dagger a}\hat{\Phi}^\dagger_b \;+\; \delta^a_b\times\begin{pmatrix}\text{a divergent}\\ \text{c-number}\end{pmatrix},
\end{aligned} \tag{25}$$

and the corresponding net charge operators

$$\hat{Q}^a{}_b \;=\int\! d^3\mathbf{x}\,\hat{J}^{0a}{}_b(\mathbf{x}) \tag{26}$$

commute with the Hamiltonian (22).

(a) Verify the commutation relations of the charges (26) with the quantum fields

$$\left[\hat{Q}^a{}_b,\hat{\Phi}^c(x)\right] \;=\; -\delta^c_b\hat{\Phi}^a(x), \tag{27.a}$$

$$\left[\hat{Q}^a{}_b,\hat{\Phi}^\dagger_c(x)\right] \;=\; +\delta^a_c\hat{\Phi}^\dagger_b(x), \tag{27.b}$$

$$\left[\hat{Q}^a{}_b,\hat{\Pi}_c(x)\right] \;=\; +\delta^a_c\hat{\Pi}_b(x), \tag{27.c}$$

$$\left[\hat{Q}^a{}_b,\hat{\Pi}^{\dagger c}(x)\right] \;=\; -\delta^c_b\hat{\Pi}^{\dagger a}(x). \tag{27.d}$$

(b) Show that

$$\left[\hat{Q}^a{}_b, \hat{Q}^c{}_d\right] \;=\; -\delta^c_b \hat{Q}^a{}_d \;+\; \delta^a_d \hat{Q}^c{}_b\,, \tag{28}$$

and consequently the operators $\hat{Q}_T = \mathrm{tr}(T\hat{Q})$ representing the $U(N)$ generators T in the Hilbert space of the QFT obey the same commutation relations as the hermitian $N \times N$ matrices T themselves. That is, for any hermitian $N \times N$ matrices (with c-number matrix elements) T_1, T_2 and their matrix commutator $[T_1, T_2] = iT_3$, the corresponding operators

$$\hat{Q}_{T1} \;=\; \mathrm{tr}(\hat{Q}T_1), \quad \hat{Q}_{T2} \;=\; \mathrm{tr}(\hat{Q}T_2), \quad \hat{Q}_{T3} \;=\; \mathrm{tr}(\hat{Q}T_3) \tag{29}$$

commute with each other just like the matrices themselves:

$$\left[\hat{Q}_{T1}, \hat{Q}_{T2}\right] \;=\; i\hat{Q}_{T3}\,. \tag{30}$$