

1. According to the Noether theorem, a translationally invariant system of classical fields $\phi_a(x)$ has a conserved stress-energy tensor

$$T_{\text{Noether}}^{\mu\nu} = \sum_a \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \partial^\nu \phi_a - g^{\mu\nu} \mathcal{L}. \quad (1)$$

Actually, to assure the symmetry of the stress-energy tensor, $T^{\mu\nu} = T^{\nu\mu}$ (which is necessary for the angular momentum conservation), one sometimes has to add a total divergence,

$$T^{\mu\nu} = T_{\text{Noether}}^{\mu\nu} + \partial_\lambda \mathcal{K}^{\lambda\mu\nu}, \quad (2)$$

where $\mathcal{K}^{\lambda\mu\nu} \equiv -\mathcal{K}^{\mu\lambda\nu}$ is some 3-index Lorentz tensor antisymmetric in its first two indices.

- (a) Show that regardless of the specific form of the $\mathcal{K}^{\lambda\mu\nu}(\phi, \partial\phi)$ as a function of the fields and their derivatives, we have

$$\begin{aligned} \partial_\mu T^{\mu\nu} &= \partial_\mu T_{\text{Noether}}^{\mu\nu} = (\text{hopefully}) = 0 \\ \text{and } P_{\text{net}}^\mu &\equiv \int d^3\mathbf{x} T^{0\mu} = \int d^3\mathbf{x} T_{\text{Noether}}^{0\mu}. \end{aligned} \quad (3)$$

Note: Assume that all the fields go to zero for $|\mathbf{x}| \rightarrow \infty$ fast enough that all the surface integrals over the boundary of 3D space vanish when we push the boundary to infinity.

For the scalar fields, real or complex, the $T_{\text{Noether}}^{\mu\nu}$ is properly symmetric and one simply has $T^{\mu\nu} = T_{\text{Noether}}^{\mu\nu}$. Unfortunately, the situation is more complicated for the vector, tensor or spinor fields. To illustrate the problem, consider the free electromagnetic fields described by the Lagrangian

$$\mathcal{L}(A_\mu, \partial_\nu A_\mu) = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \quad (4)$$

where A_μ is a real vector field and $F_{\mu\nu} \stackrel{\text{def}}{=} \partial_\mu A_\nu - \partial_\nu A_\mu$.

- (b) Write down $T_{\text{Noether}}^{\mu\nu}$ for the free electromagnetic fields and show that it is neither symmetric nor gauge invariant.
- (c) The properly symmetric — and also gauge invariant — stress-energy tensor for the free electromagnetism is

$$T_{\text{EM}}^{\mu\nu} = -F^{\mu\lambda}F^\nu{}_\lambda + \frac{1}{4}g^{\mu\nu}F_{\kappa\lambda}F^{\kappa\lambda}. \quad (5)$$

Show that this expression indeed has form (2) for some $\mathcal{K}^{\lambda\mu\nu}$.

- (d) Write down the components of the stress-energy tensor (5) in non-relativistic notations and make sure you have the familiar electromagnetic energy density, momentum density, and stress.

Next, consider the electromagnetic fields coupled to the electric current J^μ of some charged “matter” fields. Because of this coupling, only the *net* energy-momentum of the whole field system should be conserved, but not the separate P_{EM}^μ and P_{mat}^μ . Consequently, we should have

$$\partial_\mu T_{\text{net}}^{\mu\nu} = 0 \quad \text{for} \quad T_{\text{net}}^{\mu\nu} = T_{\text{EM}}^{\mu\nu} + T_{\text{mat}}^{\mu\nu} \quad (6)$$

but generally $\partial_\mu T_{\text{EM}}^{\mu\nu} \neq 0$ and $\partial_\mu T_{\text{mat}}^{\mu\nu} \neq 0$.

- (e) Use Maxwell’s equations to show that

$$\partial_\mu T_{\text{EM}}^{\mu\nu} = -F^{\nu\lambda}J_\lambda \quad (7)$$

(in $c = 1$ units), and therefore any system of charged matter fields should have its stress-energy tensor related to the electric current J_λ according to

$$\partial_\mu T_{\text{mat}}^{\mu\nu} = +F^{\nu\lambda}J_\lambda. \quad (8)$$

- (f) Rewrite eq. (7) in non-relativistic notations and explain its physical meaning in terms of the electromagnetic energy, momentum, work, and forces.

2. Continuing problem 1, consider the EM fields coupled to a specific model of charged matter, namely a complex scalar field $\Phi(x) \neq \Phi^*(x)$ of electric charge $q \neq 0$. Altogether, the net Lagrangian for the A^μ , Φ , and Φ^* fields is

$$\mathcal{L}_{\text{net}} = D^\mu \Phi^* D_\mu \Phi - m^2 \Phi^* \Phi - \frac{1}{4} F^{\mu\nu} F_{\mu\nu} \quad (9)$$

where

$$D_\mu \Phi = (\partial_\mu + iqA_\mu)\Phi \quad \text{and} \quad D_\mu \Phi^* = (\partial_\mu - iqA_\mu)\Phi^* \quad (10)$$

are the *covariant* derivatives.

- (a) Write down the equation of motion for all fields in a covariant form. Also, write down the electric current

$$J^\mu \stackrel{\text{def}}{=} -\frac{\partial \mathcal{L}}{\partial A_\mu} \quad (11)$$

in a manifestly gauge-invariant form and verify its conservation, $\partial_\mu J^\mu = 0$ (as long as the scalar fields satisfy their equations of motion).

- (b) Write down the Noether stress-energy tensor for the whole system and show that

$$T_{\text{net}}^{\mu\nu} \equiv T_{\text{EM}}^{\mu\nu} + T_{\text{mat}}^{\mu\nu} = T_{\text{Noether}}^{\mu\nu} + \partial_\lambda \mathcal{K}^{\lambda\mu\nu}, \quad (12)$$

where $T_{\text{EM}}^{\mu\nu}$ is exactly as in eq. (5) for the free EM fields, the improvement tensor $\mathcal{K}^{\lambda\mu\nu} = -\mathcal{K}^{\mu\lambda\nu}$ is also exactly as in problem 1, and

$$T_{\text{mat}}^{\mu\nu} = D^\mu \Phi^* D^\nu \Phi + D^\nu \Phi^* D^\mu \Phi - g^{\mu\nu} (D_\lambda \Phi^* D^\lambda \Phi - m^2 \Phi^* \Phi). \quad (13)$$

Note: although the improvement tensor $\mathcal{K}^{\lambda\mu\nu}$ for the EM + matter system is the same as for the free EM fields, in presence of an electric current J^μ its derivative $\partial_\lambda \mathcal{K}^{\lambda\mu\nu}$ contains an extra $J^\mu A^\nu$ term. Pay attention to this term — it is important for obtaining the gauge-invariant stress-energy tensor (13) for the scalar field.

- (c) Use the scalar fields' equations of motion and the non-commutativity of covariant derivatives

$$[D_\mu, D_\nu]\Phi = iqF_{\mu\nu}\Phi, \quad [D_\mu, D_\nu]\Phi^* = -iqF_{\mu\nu}\Phi^* \quad (14)$$

to show that

$$\partial_\mu T_{\text{mat}}^{\mu\nu} = +F^{\nu\lambda}J_\lambda \quad (15)$$

exactly as in eq. (8), and therefore the *net* stress-energy tensor (12) is conserved, *cf.* problem 1(e).

3. Next, consider the Noether currents of an internal rather than translational symmetry. Let $\Phi^a(x)$ be N complex scalar fields — of similar masses and electric charges — which interact with each other and with the EM fields A^μ according to the Lagrangian

$$\mathcal{L} = \sum_a D_\mu \Phi_a^* D^\mu \Phi^a - m^2 \sum_a \Phi_a^* \Phi^a - \frac{\lambda}{4} \left(\sum_a \Phi_a^* \Phi^a \right)^2 - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}. \quad (16)$$

The $D_\mu \Phi^a$ and the $D_\mu \Phi_a^*$ here are as in eq. (10) — they are covariant derivatives WRT the local $U(1)$ symmetry associated with the EM fields. Specifically, all the Φ^a fields have the same electric charge $+q$ while the complex-conjugate fields Φ_a^* have charge $-q$.

Both the electric charges and the scalar potentials are invariant under global unitary “rotations” of the Φ^a fields into each other. (But not into the conjugate Φ_a^* fields with opposite-sign electric charges.) Such symmetries act according to

$$\Phi^a(x) \rightarrow \sum_b U_b^a \times \Phi^b(x), \quad \Phi_a^*(x) \rightarrow \sum_b \Phi_b^*(x) \times (U^\dagger)_a^b, \quad A^\mu(x) \text{ unchanged}, \quad (17)$$

where $U = \|U_b^a\|$ is any *unitary* $N \times N$ matrix. The group of such matrices — and hence of symmetries (17) — is called the $U(N)$.

- (a) Check that the Lagrangian (16) is invariant under any $U(N)$ symmetry (17).

- (b) The infinitesimal $U(N)$ symmetries have form $U = 1 - i\epsilon T$ — *i.e.*, $U_b^a = \delta_b^a - i\epsilon T_b^a$ — for *hermitian* matrices T .

Derive the Noether current J_T^μ for any given hermitian matrix T and show that it has the form

$$J_T^\mu = \text{tr}(T J^\mu) = \sum_{a,b} T_b^a J^{\mu a}_b \quad (18)$$

where $J^{\mu a}_b$ is the $N \times N$ hermitian matrix of currents

$$J^{\mu a}_b = -i\Phi^a D^\mu \Phi_b^* + i\Phi_b^* D^\mu \Phi^a = (J^{\mu b}_a)^*. \quad (19)$$

- (c) Verify the conservation of the N^2 currents (19) and hence of J_T^μ for any hermitian matrix T .

The scalar potential in the Lagrangian (16) has a bigger symmetry than the $U(N)$, namely the $SO(2N)$ which rotates the real and imaginary parts of the $\Phi_a(x)$ fields as if they were $2N$ unrelated real fields. But the $SO(2N)$ symmetries outside of the $U(N)$ do not commute with the local $U(1)$ symmetry of the charged fields and hence do not preserve their couplings to the EM fields.

- (d) Work this out.

The infinitesimal form of an $SO(2N)$ symmetry outside of the $U(N)$ is

$$\delta\Phi^a(x) = \epsilon \sum_b C^{ab} \Phi_b^*(x), \quad \delta\Phi_a^*(x) = \epsilon \sum_b C_{ab}^* \Phi^b(x) \quad (20)$$

for a complex antisymmetric matrix $C^{ab} = -C^{ba}$.

- (e) Write down the Noether current for such a would-be symmetry and show that it is NOT conserved (unless $A^\mu = 0$).

4. Finally, consider the quantum field theory based on the classical theory of the previous problem. The charged quantum fields are

$$\hat{\Phi}^a(x), \quad \hat{\Phi}_a^\dagger(x), \quad \hat{\Pi}_a(x) = D_0 \hat{\Phi}_a^\dagger(x), \quad \hat{\Pi}^{\dagger a}(x) = D_0 \hat{\Phi}^a(x), \quad (21)$$

there are also quantum EM fields in some gauge, and the Hamiltonian is

$$\hat{H} = \hat{H}_{\text{EM}} + \int d^3\mathbf{x} \left[\sum_a \left(\hat{\Pi}^{\dagger a} \hat{\Pi}_a + q \hat{A}_0 \hat{J}^{0a}_a + \mathbf{D} \hat{\Phi}_a^\dagger \cdot \mathbf{D} \hat{\Phi}^a \right) + \hat{V} \right] \quad (22)$$

for

$$\hat{V} = m^2 \sum_a \hat{\Phi}_a^\dagger \hat{\Phi}^a + \frac{\lambda}{2} \left(\sum_a \hat{\Phi}_a^\dagger \hat{\Phi}^a \right)^2. \quad (23)$$

The $U(N)$ symmetry currents (19) become operators

$$\hat{\mathbf{J}}_b^a = i \hat{\Phi}^a \mathbf{D} \hat{\Phi}_b^\dagger - i \hat{\Phi}_b^\dagger \mathbf{D} \hat{\Phi}^a, \quad (24)$$

$$\begin{aligned} \hat{J}^{0a}_b &= -\frac{i}{2} \{ \hat{\Phi}^a, \hat{\Pi}_b \} + \frac{i}{2} \{ \hat{\Phi}_b^\dagger, \hat{\Pi}^{\dagger a} \} \\ &= -i \hat{\Pi}_b \hat{\Phi}^a + i \hat{\Pi}^{\dagger a} \hat{\Phi}_b^\dagger + \delta_b^a \times \begin{pmatrix} \text{a divergent} \\ \text{c-number} \end{pmatrix}, \end{aligned} \quad (25)$$

and the corresponding net charge operators

$$\hat{Q}_b^a = \int d^3\mathbf{x} \hat{J}^{0a}_b(\mathbf{x}) \quad (26)$$

commute with the Hamiltonian (22).

(a) Verify the commutation relations of the charges (26) with the quantum fields

$$[\hat{Q}_b^a, \hat{\Phi}^c(x)] = -\delta_b^c \hat{\Phi}^a(x), \quad (27.a)$$

$$[\hat{Q}_b^a, \hat{\Phi}_c^\dagger(x)] = +\delta_c^a \hat{\Phi}_b^\dagger(x), \quad (27.b)$$

$$[\hat{Q}_b^a, \hat{\Pi}_c(x)] = +\delta_c^a \hat{\Pi}_b(x), \quad (27.c)$$

$$[\hat{Q}_b^a, \hat{\Pi}^{\dagger c}(x)] = -\delta_b^c \hat{\Pi}^{\dagger a}(x). \quad (27.d)$$

(b) Show that

$$[\hat{Q}_b^a, \hat{Q}_d^c] = -\delta_b^c \hat{Q}_d^a + \delta_d^a \hat{Q}_b^c, \quad (28)$$

and consequently the operators $\hat{Q}_T = \text{tr}(T\hat{Q})$ representing the $U(N)$ generators T in the Hilbert space of the QFT obey the same commutation relations as the hermitian $N \times N$ matrices T themselves. That is, for any hermitian $N \times N$ matrices (with c-number matrix elements) T_1, T_2 and their matrix commutator $[T_1, T_2] = iT_3$, the corresponding operators

$$\hat{Q}_{T_1} = \text{tr}(\hat{Q}T_1), \quad \hat{Q}_{T_2} = \text{tr}(\hat{Q}T_2), \quad \hat{Q}_{T_3} = \text{tr}(\hat{Q}T_3) \quad (29)$$

commute with each other just like the matrices themselves:

$$[\hat{Q}_{T_1}, \hat{Q}_{T_2}] = i\hat{Q}_{T_3}. \quad (30)$$