1. In class, I have focused on the *fundamental multiplet* of the local $SU(N)$ symmetry, *i.e.*, a set of N fields (complex scalars or Dirac fermions) which transform as a complex N–vector,

$$\Psi'(x) \;=\; U(x)\Psi(x) \quad i.e. \quad \Psi'_i(x) \;=\sum_j U_i^{\ j}(x)\Psi_j(x), \quad i,j=1,2,\ldots,N \tag{1}$$

where $U(x)$ is an x–dependent unitary $N\times N$ matrix, $\det U(x)\equiv 1$. Now consider N^2-1 real fields $\Phi^a(x)$ forming an *adjoint multiplet*: In matrix form

$$\Phi(x) \;=\; \sum_a \Phi^a(x)\times\frac{\lambda^a}{2} \tag{2}$$

is a traceless hermitian $N\times N$ matrix which transforms under the local $SU(N)$ symmetry as

$$\Phi'(x) \;=\; U(x)\Phi(x)U^\dagger(x). \tag{3}$$

Note that this transformation law preserves the $\Phi^\dagger=\Phi$ and the $\mathrm{tr}(\Phi)=0$ conditions.

The covariant derivatives D_μ act on an adjoint multiplet according to

$$D_\mu\Phi(x) \;=\; \partial_\mu\Phi(x) \;+\; i[\mathcal{A}_\mu(x),\Phi(x)] \;\equiv\; \partial_\mu\Phi(x) \;+\; i\mathcal{A}_\mu(x)\Phi(x) \;-\; i\Phi(x)\mathcal{A}_\mu(x), \tag{4}$$

or in components

$$D_\mu\Phi^a(x) = \;\partial_\mu\Phi_a(x) \;-\; f^{abc}\mathcal{A}^b_\mu(x)\Phi^c(x). \tag{5}$$

(a) Verify that these derivatives are indeed covariant under finite gauge transforms.

(b) Verify the Leibniz rule for covariant derivatives of matrix products. Let $\Phi(x)$ and $\Xi(x)$ be two adjoint multiplets while $\Psi(x)$ is a fundamental multiplet and $\Psi^\dagger(x)$ is its hermitian conjugate (row vector of Ψ_i^*). Show that

$$\begin{aligned} D_\mu(\Phi\Xi) &= (D_\mu\Phi)\Xi + \Phi(D_\mu\Xi), \\ D_\mu(\Phi\Psi) &= (D_\mu\Phi)\Psi + \Phi(D_\mu\Psi), \\ D_\mu(\Psi^\dagger\Xi) &= (D_\mu\Psi^\dagger)\Xi + \Psi^\dagger(D_\mu\Xi). \end{aligned} \tag{6}$$

(c) Show that for an adjoint multiplet $\Phi(x)$,

$$[D_\mu, D_\nu]\Phi(x) = i[\mathcal{F}_{\mu\nu}(x), \Phi(x)] = ig[F_{\mu\nu}(x), \Phi(x)] \tag{7}$$

or in components $[D_\mu, D_\nu]\Phi^a(x) = -gf^{abc}F^b_{\mu\nu}(x)\Phi^c(x)$.

- In my notations A_μ and $F_{\mu\nu}$ are canonically normalized fields while $\mathcal{A}_\mu = gA_\mu$ and $\mathcal{F}_{\mu\nu} = gF_{\mu\nu}$ are normalized by the symmetry action.

In class, I have argued (using covariant derivatives) that the tension fields $\mathcal{F}_{\mu\nu}(x)$ themselves transform according to eq. (3). In other words, the $\mathcal{F}^a_{\mu\nu}(x)$ form an adjoint multiplet of the $SU(N)$ symmetry group.

(d) Verify the $\mathcal{F}'_{\mu\nu}(x) = U(x)\mathcal{F}_{\mu\nu}(x)U^\dagger(x)$ transformation law directly from the definition $\mathcal{F}_{\mu\nu} \stackrel{\text{def}}{=} \partial_\mu\mathcal{A}_\nu - \partial_\nu\mathcal{A}_\mu + i[\mathcal{A}_\mu, \mathcal{A}_\nu]$ and the non-abelian gauge transform of the $\mathcal{A}_\mu$ fields.

(e) Verify the Bianchi identity for the non-abelian tension fields $\mathcal{F}_{\mu\nu}(x)$:

$$D_\lambda\mathcal{F}_{\mu\nu} + D_\mu\mathcal{F}_{\nu\lambda} + D_\nu\mathcal{F}_{\lambda\mu} = 0. \tag{8}$$

Note the covariant derivatives in this equation.

Finally, consider the $SU(N)$ Yang–Mills theory — the non-abelian gauge theory that does not have any fields except $\mathcal{A}^a(x)$ and $\mathcal{F}^a(x)$; its Lagrangian is

$$\mathcal{L}_{\text{YM}} = -\frac{1}{2g^2}\,\text{tr}\big(\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu}\big) = \sum_a \tfrac{-1}{4}F^a_{\mu\nu}F^{a\mu\nu}. \tag{9}$$

(f) Show that the Euler–Lagrange field equations for the Yang–Mills theory can be written in covariant form as $D_\mu\mathcal{F}^{\mu\nu} = 0$.
Hint: first show that for an infinitesimal variation $\delta\mathcal{A}_\mu(x)$ of the non-abelian gauge fields, the tension fields vary according to $\delta\mathcal{F}_{\mu\nu}(x) = D_\mu\delta\mathcal{A}_\nu(x) - D_\nu\delta\mathcal{A}_\mu(x)$.

2. Continuing the previous problem, consider an $SU(N)$ gauge theory in which N^2-1 vector fields $A^a_\mu(x)$ interact with some "matter" fields $\phi_\alpha(x)$,

$$\mathcal{L} \;=\; -\frac{1}{2g^2}\,\mathrm{tr}\big(\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu}\big) \;+\; \mathcal{L}_{\mathrm{mat}}(\phi, D_\mu\phi). \tag{10}$$

For the moment, let me keep the matter fields completely generic — they can be scalars, or vectors, or spinors, or whatever, and form any kind of a multiplet of the local $SU(N)$ symmetry as long as such multiplet is complete and non-trivial. All we need to know right now is that there are well-defined covariant derivatives $D_\mu\phi$ that depend on the gauge fields A^a_μ, which give rise to the currents

$$J^{a\mu} \;=\; -\frac{\partial\mathcal{L}_{\mathrm{mat}}}{\partial A^a_\mu} \;=\; -\sum_\phi \frac{\partial\mathcal{L}_{\mathrm{mat}}}{\partial(D_\mu\phi)}\times ig\hat{T}^a\phi. \tag{11}$$

Collectively, these N^2-1 currents should form an adjoint multiplet $J^\mu = \sum_a(\frac{1}{2}\lambda^a)J^{a\mu}$ of the $SU(N)$ symmetry.

(a) Show that in this theory the equation of motion for the A^a_μ fields are $D_\mu F^{a\mu\nu} = J^{a\nu}$ and that consistency of these equations requires require the currents to be *covariantly conserved,*

$$D_\mu J^\mu \;=\; \partial_\mu J^\mu \;+\; i[\mathcal{A}_\mu, J^\mu] \;=\; 0, \tag{12}$$

or in components, $\partial_\mu J^{a\mu} - f^{abc}\mathcal{A}^b_\mu J^{c\mu} = 0$.

Note: a covariantly conserved current does *not* lead to a conserved charge, $(d/dt)\int d^3\mathbf{x}\, J^{a0}(\mathbf{x},t) \neq 0$!

Now consider a simple example of matter fields — a fundamental multiplet $\Psi(x)$ of N scalar fields $\Psi^i(x)$, with a Lagrangian

$$\mathcal{L}_{\mathrm{mat}} \;=\; D_\mu\Psi^\dagger D^\mu\Psi - m^2\Psi^\dagger\Psi - \frac{\lambda}{4}\big(\Psi^\dagger\Psi\big)^2, \qquad \mathcal{L}_{\mathrm{net}} \;=\; \mathcal{L}_{\mathrm{mat}} - \frac{1}{2g^2}\,\mathrm{tr}\big(\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu}\big). \tag{13}$$

(b) Derive the $SU(N)$ currents $J^{a\mu}$ for this set of fields and verify that under $SU(N)$ symmetries the currents transform covariantly into each other as members of the adjoint multiplet. That is, the $N\times N$ matrix $J^\mu = \sum_a (\frac{1}{2}\lambda^a) J^{a\mu}$ transforms according to eq. (3).

Hint: for any complex vectors Ψ and Ψ', $\sum_a (\Psi^\dagger\lambda^a\Psi')\lambda^a = 2\Psi'\otimes\Psi^\dagger - \frac{2}{N}(\Psi^\dagger\Psi')\times\mathbf{1}$.

(c) Finally, verify the covariant conservation $D_\mu J^{a\mu}$ of these currents when the scalar fields $\Psi^i(x)$ and $\Psi^\dagger_i(x)$ obey their equations of motion.

3. This problem is an extra challenge for the students familiar with the basic theory of Lie groups. The students who struggle with this subject should skip this problem but instead read Howard Georgi's 1999 book *"Lie Algebras in Particle Physics: from Isospin to Unified Theories"*; here is link to UT library ebook. There is no deadline for this reading assignment, but the sooner the better.

Consider a general gauge theory with some simple compact gauge symmetry group G. The generators $\hat{T}^a$ of G form a Lie algebra with commutation relations $[\hat{T}^a,\hat{T}^b] = if^{abc}\hat{T}^c$, and the gauge fields $A^a_\mu(x)$ form the Lie-algebra-valued connection $\mathcal{A}_\mu(x) = gA^a_\mu(x)\hat{T}^a$. Besides the gauge fields, the theory also has matter fields $\Psi^\alpha(x)$ in some multiplet (m) of G. These matter fields could be scalar, or spinor, or whatever, — it does not matter for the purpose of this problem.

Under finite gauge symmetries $u(x)\in G$, the matter fields transform as

$$\Psi'^\alpha(x) \;=\; \left[R_{(m)}(u(x))\right]^\alpha_{\ \beta}\Psi^\beta(x) \tag{14}$$

where $R_{(m)}(u)$ is the matrix representing u in the multiplet (m), while the gauge fields themselves transform according to

$$\mathcal{A}'_\mu(x) \;=\; i(\partial_\mu u(x))u^{-1}(x) \;+\; u(x)\mathcal{A}_\mu(x)u^{-1}(x). \tag{15}$$

In class (*cf.* my notes) we saw that the covariant derivatives of matter fields

$$D_\mu\Psi^\alpha(x) \;=\; \partial_\mu\Psi^\alpha(x) \;+\; igA^a_\mu(x)\left[T^a_{(m)}\right]^\alpha_{\ \beta}\psi^\beta(x) \tag{16}$$

transform covariantly — *i.e.*, just like the $\Psi^\alpha(x)$ themselves — under the *infinitesimal* local symmetries.

- Your task is to show that under the finite gauge transforms (14) and (15), the covariant derivatives (16) also transform like the matter fields themselves.

Hint: use the following **Lemma:**
In any representation (m) of the group G, the matrices $T^a_{(m)}$ representing the Lie algebra's generators $\hat{T}^a$ form an adjoint multiplet. That is, for any group element $u \in G$ and the matrix $R_{(m)}(u)$ representing u in the multiplet (m),

$$\text{matrix product } R_{(m)}(u) \times T^a_{(m)} \times R^{-1}_{(m)}(u) \;=\; \sum_b T^b_{(m)} \times R^{ba}_{\text{adj}}(u). \qquad (17)$$

Note that the same $R^{ba}_{\text{adj}}(u)$ appears on right hand sides of eqs. (17) for all multiplets (m) of G — and that's what allows us to use the same gauge fields $\mathcal{A}^a_\mu(x)$ to make covariant derivatives (16) for all multiplets of the gauge group G.