This homework set has 7 problems, but none of them are long or hard. Also, 2 out of 7 problems are optional, for extra challenge.

The the first 3 problems of this set are about the generators and the representations of the Lorentz symmetry. Problems 4, 6 and the optional problem 5 are about the Dirac matrices and Dirac spinor fields. Finally, for the students interested in BEC and superfluidity, there is an extra reading assignment (problem 7) based on my extra lecture on 10/11.

1. Consider the continuous Lorentz group $SO^+(3,1)$ and its generators $\hat{J}^{\mu\nu} = -\hat{J}^{\nu\mu}$. In 3D terms, the six independent $\hat{J}^{\mu\nu}$ generators comprise the 3 components of the angular momentum $\hat{J}^i = \frac{1}{2}\epsilon^{ijk}\hat{J}^{jk}$ — which generate the rotations of space — plus 3 generators $\hat{K}^i = \hat{J}^{0i} = -\hat{J}^{i0}$ of the Lorentz boosts.

   (a) In 4D terms, the commutation relations of the Lorentz generators are

$$\left[\hat{J}^{\alpha\beta}, \hat{J}^{\mu\nu}\right] = ig^{\beta\mu}\hat{J}^{\alpha\nu} - ig^{\alpha\mu}\hat{J}^{\beta\nu} - ig^{\beta\nu}\hat{J}^{\alpha\mu} + ig^{\alpha\nu}\hat{J}^{\beta\mu}. \tag{1}$$

   Show that in 3D terms, these relations become

$$\left[\hat{J}^i, \hat{J}^j\right] = i\epsilon^{ijk}\hat{J}^k, \quad \left[\hat{J}^i, \hat{K}^j\right] = i\epsilon^{ijk}\hat{K}^k, \quad \left[\hat{K}^i, \hat{K}^j\right] = -i\epsilon^{ijk}\hat{J}^k. \tag{2}$$

The Lorentz symmetry dictates the commutation relations of the $\hat{J}^{\mu\nu}$ with any operators comprising a Lorentz multiplet. In particular, for any Lorentz vector $\hat{V}^\mu$

$$\left[\hat{V}^\lambda, \hat{J}^{\mu\nu}\right] = ig^{\lambda\mu}\hat{V}^\nu - ig^{\lambda\nu}\hat{V}^\mu. \tag{3}$$

   (b) Spell out these commutation relations in 3D terms, then use them to show that the Lorentz boost generators $\hat{\mathbf{K}}$ do not commute with the Hamiltonian $\hat{H}$.

(c) Show that even in the non-relativistic limit, the Galilean boosts $t' = t$, $\mathbf{x}' = \mathbf{x} + \mathbf{v}t$ and their generators $\hat{\mathbf{K}}_G$ do not commute with the Hamiltonian.

Note: Only the *time-independent* symmetries commute with the Hamiltonian. But when the action of a symmetry is manifestly time dependent — like a Galilean boost $\mathbf{x}' = \mathbf{x} + \mathbf{v}t$ or a Lorentz boost — the symmetry operators do not commute with the time evolution and hence with the Hamiltonian.

2. Next, consider the little group $G(p)$ of Lorentz symmetries preserving some momentum 4–vector p^μ. For the moment, allow the p^μ to be time-like, light-like, or even space-like — anything goes as long as $p \neq 0$.

(d) Show that the little group $G(p)$ is generated by the 3 components of the vector

$$\hat{\mathbf{R}} \;=\; p^0 \hat{\mathbf{J}} \;+\; \mathbf{p} \times \hat{\mathbf{K}} \tag{4}$$

after a suitable component-by-component rescaling.

Suppose the momentum p^μ belongs to a massive particle, thus $p^\mu p_\mu = m^2 > 0$. For simplicity, assume the particle moves in z direction with velocity β, thus $p^\mu = (E, 0, 0, p)$ for $E = \gamma m$ and $p = \beta\gamma m$. In this case, the properly normalized generators of the little group $G(p)$ are the

$$\begin{aligned} \tilde{J}^x &= \frac{1}{m}\,\hat{R}^x \;=\; \gamma \hat{J}^x \;-\; \beta\gamma \hat{K}^y, \\ \tilde{J}^y &= \frac{1}{m}\,\hat{R}^y \;=\; \gamma \hat{J}^y \;+\; \beta\gamma \hat{K}^x, \\ \tilde{J}^z &= \frac{1}{\gamma m}\,\hat{R}^z \;=\; \hat{J}^z, \quad \text{the helicity.} \end{aligned} \tag{5}$$

(e) Show that these generators have angular-momentum-like commutators with each other, $[\tilde{J}^i, \tilde{J}^j] = i\epsilon^{ijk}\tilde{J}^k$. Consequently, the little group $G(p)$ is isomorphic to the rotation group $SO(3)$.

Now suppose the momentum p^μ belongs to a massless particle, $p^\mu p_\mu = 0$. Again, assume for simplicity that the particle moves in the z direction, thus $p^\mu = (E, 0, 0, E)$. In this case, we cannot normalize the generators of the little group as in eq. (5); instead, let's normalize them according to

$$\hat{\mathbf{I}} \;=\; \frac{1}{E}\hat{\mathbf{R}} \;=\; \hat{\mathbf{J}} \;+\; \vec{\beta}\times\hat{\mathbf{K}}, \tag{6}$$

or in components,

$$\hat{I}^x \;=\; \hat{J}^x \,-\, \hat{K}^y, \quad \hat{I}^y \;=\; \hat{J}^y + \hat{K}^x, \quad \hat{I}^z \;=\; \hat{J}^z. \tag{7}$$

(f) Show that these generators obey similar commutation relations to the $\hat{p}^x$, $\hat{p}^y$, and $\hat{J}^z$ operators, namely

$$[\hat{J}^z, \hat{I}^x] \;=\; +i\hat{I}^y, \quad [\hat{J}^z, \hat{I}^y] \;=\; -i\hat{I}^x, \quad [\hat{I}^x, \hat{I}^y] \;=\; 0. \tag{8}$$

Consequently, the little group $G(p)$ is isomorphic to the ISO(2) group of *rotations and translations* in the xy plane.

(g) Finally, show that for a tachyonic momentum with $p^\mu p_\mu < 0$, the properly normalized generators of the little group have similar commutation relations to the $\hat{K}^x$, $\hat{K}^y$, and $\hat{J}^z$ operators. Consequently, the little group $G(p)$ is isomorphic to the $SO^+(2,1)$, the continuous Lorentz group in $2+1$ spacetime dimensions.

3. Now let's focus on the massless particles. As explained in class, the finite unitary multiplets of the $G(p) \cong \mathrm{ISO}(2)$ group generated by the (7) operators are singlets $|\lambda\rangle$, although they are non-trivial singlets for $\lambda \neq 0$. Specifically, the state $|\lambda\rangle$ is an eigenstate of the helicity operator $\hat{J}^z$ (for the momentum in the z direction) and are annihilated by the $\hat{I}^{x,y}$ operators,

$$\hat{J}^z\,|\lambda\rangle \;=\; \lambda\,|\lambda\rangle\,, \quad \hat{I}^x\,|\lambda\rangle \;=\; 0, \quad \hat{I}^y\,|\lambda\rangle \;=\; 0. \tag{9}$$

(a) Show that in 4D terms the state $|p,\lambda\rangle$ of a *massless* particle satisfies

$$\epsilon_{\alpha\beta\gamma\delta}\hat{J}^{\beta\gamma}\hat{P}^\delta\,|p,\lambda\rangle \;=\; 2\lambda\hat{P}_\alpha\,|p,\lambda\rangle\,. \tag{10}$$

(b) Use eq. (10) to show that *continuous* Lorentz transforms do not change helicities of *massless* particles,

$$\forall L \in \mathrm{SO}^+(3,1), \quad \widehat{\mathcal{D}}(L)\,|p,\lambda\rangle \;=\; |Lp,\,\text{same}\,\lambda\rangle \times e^{i\,\text{phase}}. \tag{11}$$

4. Next, an exercise in Dirac matrices γ^μ. In this problem, you should not assume any explicit matrices for the γ^μ but simply use the anticommutation relations

$$\gamma^\mu\gamma^\nu \;+\; \gamma^\nu\gamma^\mu \;=\; 2g^{\mu\nu}. \tag{12}$$

When necessary, you may also assume that the Dirac matrices are 4×4, and the γ^0 matrix is hermitian while the $\gamma^1, \gamma^2, \gamma^3$ matrices are antihermitian, $(\gamma^0)^\dagger = +\gamma^0$ while $(\gamma^i)^\dagger = -\gamma^i$ for $i = 1, 2, 3$.

(a) Show that $\gamma^\alpha\gamma_\alpha = 4$, $\gamma^\alpha\gamma^\nu\gamma_\alpha = -2\gamma^\nu$, $\gamma^\alpha\gamma^\mu\gamma^\nu\gamma_\alpha = 4g^{\mu\nu}$, and $\gamma^\alpha\gamma^\lambda\gamma^\mu\gamma^\nu\gamma_\alpha = -2\gamma^\nu\gamma^\mu\gamma^\lambda$.
Hint: use $\gamma^\alpha\gamma^\nu = 2g^{\nu\alpha} - \gamma^\nu\gamma^\alpha$ repeatedly.

(b) The electron field in the EM background obeys the *covariant Dirac equation* $\bigl(i\gamma^\mu D_\mu - m\bigr)\Psi(x) = 0$ where $D_\mu\Psi = \partial_\mu\Psi - ieA_\mu\Psi$. Show that this equation implies

$$\bigl(D_\mu D^\mu \,+\, m^2 \,-\, eF_{\mu\nu}S^{\mu\nu}\bigr)\,\Psi(x) \;=\; 0. \tag{13}$$

Besides the 4 Dirac matrices γ^μ, there is another useful matrix $\gamma^5 \stackrel{\text{def}}{=} i\gamma^0\gamma^1\gamma^2\gamma^3$.

(c) Show that the γ^5 anticommutes with each of the γ^μ matrices — $\gamma^5\gamma^\mu = -\gamma^\mu\gamma^5$ — and commutes with all the spin matrices, $\gamma^5 S^{\mu\nu} = +S^{\mu\nu}\gamma^5$.

(d) Show that the γ^5 is hermitian and that $(\gamma^5)^2 = 1$.

(e) Show that $\gamma^5 = (i/24)\epsilon_{\kappa\lambda\mu\nu}\gamma^\kappa\gamma^\lambda\gamma^\mu\gamma^\nu$ and that $\gamma^{[\kappa}\gamma^\lambda\gamma^\mu\gamma^{\nu]} = +24i\epsilon^{\kappa\lambda\mu\nu}\,\gamma^5$.

(f) Show that $\gamma^{[\lambda}\gamma^\mu\gamma^{\nu]} = +6i\epsilon^{\kappa\lambda\mu\nu}\,\gamma_\kappa\gamma^5$.

(g) Show that any 4×4 matrix Γ is a unique linear combination of the following 16 matrices: 1, γ^μ, $\frac{1}{2}\gamma^{[\mu}\gamma^{\nu]} = -2iS^{\mu\nu}$, $\gamma^5\gamma^\mu$, and γ^5.

$*$ My conventions here are: $\epsilon^{0123} = -1$, $\epsilon_{0123} = +1$, $\gamma^{[\mu}\gamma^{\nu]} = \gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu$,
$\gamma^{[\lambda}\gamma^\mu\gamma^{\nu]} = \gamma^\lambda\gamma^\mu\gamma^\nu - \gamma^\lambda\gamma^\nu\gamma^\mu + \gamma^\mu\gamma^\nu\gamma^\lambda - \gamma^\mu\gamma^\lambda\gamma^\nu + \gamma^\nu\gamma^\lambda\gamma^\mu - \gamma^\nu\gamma^\mu\gamma^\lambda$, *etc.*

5. This is an optional exercise, for extra challenge. Let's generalize the Dirac matrices to spacetime dimensions $d \neq 4$. Such matrices always satisfy the Clifford algebra (12), but their sizes depend on d.

   Generalization of the γ^5 to d dimensions is $\Gamma = i^n \gamma^0 \gamma^1 \cdots \gamma^{d-1}$, where the pre-factor $i^n = \pm i$ or ± 1 is chosen such that $\Gamma = \Gamma^\dagger$ and $\Gamma^2 = +1$.

   (a) *For even* d, Γ anticommutes with all the γ^μ. Prove this, then use this fact to show that there are 2^d independent products of the γ^μ matrices, and consequently the matrices should be $2^{d/2} \times 2^{d/2}$.

   (b) *For odd* d, Γ commutes with all the γ^μ — prove this. Consequently, one can set $\Gamma = +1$ or $\Gamma = -1$; the two choices lead to in-equivalent sets of the γ^μ.

   Classify the independent products of the γ^μ for odd d and show that their net number is 2^{d-1}; consequently, the matrices should be $2^{(d-1)/2} \times 2^{(d-1)/2}$.

6. Now let's go back to $d = 3 + 1$ and learn about the Weyl spinors and Weyl spinor fields. Since all the spin matrices $S^{\mu\nu}$ commute with the γ^5, for all *continuous* Lorentz symmetries $L^\mu_{\ \nu}$ their Dirac-spinor representations $M_D(L) = \exp\left(-\frac{i}{2}\Theta_{\alpha\beta}S^{\alpha\beta}\right)$ are block-diagonal in the eigenbasis of the γ^5. This makes the Dirac spinor Ψ a *reducible* multiplet of the continuous Lorentz group $SO^+(3,1)$ — it comprises two different irreducible 2-component spinor multiplets, called the left-handed Weyl spinor ψ_L and the right-handed Weyl spinor ψ_R.

   This decomposition becomes clear in the Weyl convention for the Dirac matrices where

$$\gamma^\mu = \begin{pmatrix} 0 & \bar{\sigma}^\mu \\ \sigma^\mu & 0 \end{pmatrix} \quad \text{where} \quad \begin{aligned} \sigma^\mu &\stackrel{\text{def}}{=} \left(\mathbf{1}_{2\times 2}, -\boldsymbol{\sigma}\right), \\ \bar{\sigma}^\mu &\stackrel{\text{def}}{=} \left(\mathbf{1}_{2\times 2}, +\boldsymbol{\sigma}\right), \end{aligned} \tag{14}$$

   and consequently

$$\gamma^5 = \begin{pmatrix} -1 & 0 \\ 0 & +1 \end{pmatrix} \quad \Longrightarrow \quad M_D(L) = \begin{pmatrix} M_L(L) & 0 \\ 0 & M_R(L) \end{pmatrix}. \tag{15}$$

   (a) Check that the γ^5 matrix indeed has this form and write down explicit matrices for the $S^{\mu\nu}$ in the Weyl convention.

(b) Show that for a space rotation R through angle θ around axis $\mathbf{n}$,

$$M_L(R) \;=\; M_R(R) \;=\; \exp\bigl(-\tfrac{i}{2}\,\theta\mathbf{n}\cdot\boldsymbol{\sigma}\bigr). \tag{16}$$

Likewise, show that for a Lorentz boost B of speed v in the direction $\mathbf{n}$,

$$M_L(B) \;=\; \exp(-\tfrac{1}{2}\,r\,\mathbf{n}\cdot\boldsymbol{\sigma}\bigr) \quad \text{while} \quad M_R(B) \;=\; \exp(+\tfrac{1}{2}\,r\,\mathbf{n}\cdot\boldsymbol{\sigma}\bigr) \tag{17}$$

where $r = \operatorname{artanh}(v)$ is the *rapidity* of the boost. For successive boosts in the same direction, the rapidities add up, $r_{1+2} = r_1 + r_2$. Consequently, a finite Lorentz boost of rapidity r in the direction $\mathbf{n}$ is $B = \exp(r\mathbf{n}\cdot\hat{\mathbf{K}})$.

(c) The more familiar β and γ parameters of a Lorentz boost are related to the rapidity as

$$\beta \;=\; \tanh(r), \quad \gamma \;=\; \cosh(r), \quad \beta\gamma \;=\; \sinh(r). \tag{18}$$

Show that in terms of these parameters, eqs. (17) translate to

$$M_L(B) \;=\; \sqrt{\gamma}\times\sqrt{1\,-\,\beta\,\mathbf{n}\cdot\boldsymbol{\sigma}}\,, \qquad M_R(B) \;=\; \sqrt{\gamma}\times\sqrt{1\,+\,\beta\,\mathbf{n}\cdot\boldsymbol{\sigma}}\,. \tag{19}$$

(d) Show that for any continuous Lorentz symmetry L, the $M_L(L)$ and the $M_R(L)$ matrices are related to each other according to

$$M_R(L) \;=\; \sigma_2\times M_L^*(L)\times\sigma_2\,, \quad M_L(L) \;=\; \sigma_2\times M_R^*(L)\times\sigma_2\,. \tag{20}$$

Hint: all 3 Pauli matrices σ_i, are related to their complex conjugates σ_i^* according to $\sigma_2\sigma_i^*\sigma_2 = -\sigma_i$,

In the Weyl convention for the Dirac matrices, the Dirac spinor field $\Psi(x)$ splits into the left-handed Weyl spinor field $\psi_L(x)$ and the right-handed Weyl spinor field $\psi_R(x)$ according to

$$\Psi_{\text{Dirac}}(x) \;=\; \begin{pmatrix}\psi_L(x),\\ \psi_R(x)\end{pmatrix} \quad \text{where} \quad \begin{aligned}\psi_L'(x') \;&=\; M_L(L)\psi_L(x),\\ \psi_R'(x') \;&=\; M_R(L)\psi_R(x).\end{aligned} \tag{21}$$

(e) Show that the hermitian conjugate of each Weyl spinor transforms equivalently to the other spinor. Specifically, the $\sigma_2 \times \psi_L^*(x)$ transforms under continuous Lorentz symmetries like the $\psi_R(x)$, while the $\sigma_2 \times \psi_R^*(x)$ transforms like the $\psi_L(x)$.

Note: the * superscript on a multi-component quantum field means hermitian conjugation of each component field but without transposing the components, thus

$$\psi_L = \begin{pmatrix} \psi_{L1} \\ \psi_{L2} \end{pmatrix}, \quad \psi_L^* = \begin{pmatrix} \psi_{L1}^\dagger \\ \psi_{L2}^\dagger \end{pmatrix}, \quad \text{while} \quad \psi_L^\dagger = (\, \psi_{L1}^\dagger \quad \psi_{L2}^\dagger \,), \tag{22}$$

and likewise for the ψ_R and its conjugates.

Finally, consider the Dirac Lagrangian $\mathcal{L} = \overline{\Psi}(i\gamma^\mu\partial_\mu - m)\Psi$.

(f) Express this Lagrangian in terms of the Weyl spinor fields $\psi_L(x)$ and $\psi_R(x)$ (and their conjugates $\psi_L^\dagger(x)$ and $\psi_R^\dagger(x)$).

(g) Show that for $m = 0$ — and only for $m = 0$ — the two Weyl spinor fields become independent from each other.

7. Finally, for the students who came to my extra lecture on 10/11 about BEC and superfluidity, there is an extra reading assignment, namely my notes on the subject. In particular, please read the solutions to all the lemmas.