1. Consider the plane-wave solutions of the Dirac equation, $\Psi_\alpha(x) = u_\alpha \times e^{-ipx}$ and $\Psi_\alpha(x) = v_\alpha \times e^{+ipx}$ for some x-independent Dirac spinors $u_\alpha(p,s)$ and $v_\alpha(p,s)$.

   (a) Check that these waves indeed solve the Dirac equation provided $p^2 = m^2$ while

$$(\not{p} - m)u(p,s) \;=\; 0, \quad (\not{p} + m)v(p,s) \;=\; 0. \tag{1}$$

By convention, we always take $E = p^0 = +\sqrt{\mathbf{p}^2 + m^2}$ — that's why we have both $e^{-ipx}u_\alpha$ and $e^{+ipx}v_\alpha$ types of wave — while the spinor coefficients $u(p,s)$ and $v(p,s)$ are normalized to

$$u^\dagger(p,s)u(p,s') \;=\; v^\dagger(p,s)v(p,s') \;=\; 2E\delta_{s,s'}\,. \tag{2}$$

In this problem we shall write down explicit formulae for these spinors in the Weyl basis for the γ^μ matrices.

   (b) Show that for $\mathbf{p} = 0$,

$$u(\mathbf{p} = \mathbf{0}, s) \;=\; \begin{pmatrix} \sqrt{m}\,\xi_s \\ \sqrt{m}\,\xi_s \end{pmatrix} \tag{3}$$

   where ξ_s is a two-component $SO(3)$ spinor encoding the electron's spin state. The ξ_s are normalized to $\xi_s^\dagger \xi_{s'} = \delta_{s,s'}$.

   (c) For other momenta, $u(p,s) = M_D(\text{boost}) \times u(\mathbf{p} = 0, s)$ for the boost that turns $(m, \vec{0})$ into p^μ. Use eqs. (7.19) from the previous homework set#7 to show that

$$u(p,s) \;=\; \begin{pmatrix} \sqrt{E - \mathbf{p}\cdot\boldsymbol{\sigma}}\,\xi_s \\ \sqrt{E + \mathbf{p}\cdot\boldsymbol{\sigma}}\,\xi_s \end{pmatrix} \;=\; \begin{pmatrix} \sqrt{p_\mu\bar{\sigma}^\mu}\,\xi_s \\ \sqrt{p_\mu\sigma^\mu}\,\xi_s \end{pmatrix}. \tag{4}$$

   (d) Use similar arguments to show that

$$v(p,s) \;=\; \begin{pmatrix} +\sqrt{E - \mathbf{p}\cdot\boldsymbol{\sigma}}\,\eta_s \\ -\sqrt{E + \mathbf{p}\cdot\boldsymbol{\sigma}}\,\eta_s \end{pmatrix} \;=\; \begin{pmatrix} +\sqrt{p_\mu\bar{\sigma}^\mu}\,\eta_s \\ -\sqrt{p_\mu\sigma^\mu}\,\eta_s \end{pmatrix} \tag{5}$$

   where η_s are two-component $SO(3)$ spinors normalized to $\eta_s^\dagger \eta_{s'} = \delta_{s,s'}$.

Physically, the η_s should have opposite spins from ξ_s — the holes in the Dirac sea have opposite spins (as well as p^μ) from the missing negative-energy particles. Mathematically, this requires $\eta_s^\dagger \mathbf{S} \eta_s = -\xi_s^\dagger \mathbf{S} \xi_s$; we may solve this condition by letting $\eta_s = \sigma_2 \xi_s^* = \pm i\xi_{-s}^*$.

(e) Check that $\eta_s = \sigma_2 \xi_s^* = \pm i\xi_{-s}^*$ indeed provides for the $\eta_s^\dagger \mathbf{S} \eta_s = -\xi_s^\dagger \mathbf{S} \xi_s$, then show that this leads to

$$v(p,s) \;= \gamma^2 u^*(p,s) \quad \text{and} \quad u(p,s) \;=\; \gamma^2 v^*(p,s). \tag{6}$$

(f) Show that for the ultra-relativistic electrons or positrons of definite helicity $\lambda = \pm\frac{1}{2}$, the Dirac plane waves become *chiral* — *i.e.*, dominated by one of the two irreducible Weyl spinor components $\psi_L(x)$ or $\psi_R(x)$ of the Dirac spinor $\Psi(x)$, while the other component becomes negligible. Specifically,

$$\begin{aligned} u(p,-\tfrac{1}{2}) \;&\approx\; \sqrt{2E}\begin{pmatrix}\xi_L\\ 0\end{pmatrix}, \qquad & u(p,+\tfrac{1}{2}) \;&\approx\; \sqrt{2E}\begin{pmatrix}0\\ \xi_R\end{pmatrix}, \\ v(p,-\tfrac{1}{2}) \;&\approx\; -\sqrt{2E}\begin{pmatrix}0\\ \eta_L\end{pmatrix}, \qquad & v(p,+\tfrac{1}{2}) \;&\approx\; \sqrt{2E}\begin{pmatrix}\eta_R\\ 0\end{pmatrix}. \end{aligned} \tag{7}$$

Note that for the electron waves the helicity agrees with the chirality — they are both left or both right, — but for the positron waves the chirality is opposite from the helicity.

In the previous homework#7 (problem 6.g), we saw that for $m = 0$ the LH and the RH Weyl spinor fields decouple from each other. Now this exercise show us which particle modes comprise each Weyl spinor: *The $\psi_L(x)$ and its hermitian conjugate $\psi_L^\dagger(x)$ contain the left-handed fermions and the right-handed antifermions, while the $\psi_R(x)$ and the $\psi_R^\dagger(x)$ contain the right-handed fermions and the left-handed antifermions.*

2. Next, let's establish some basis-independent properties of the Dirac spinors $u(p,s)$ and $v(p,s)$ — although you may use the Weyl basis to verify them. We shall use these properties in class when we get to Quantum Electrodynamics (QED).

(a) Show that

$$\bar{u}(p,s)u(p,s') \;=\; +2m\delta_{s,s'}\,, \quad \bar{v}(p,s)v(p,s') \;=\; -2m\delta_{s,s'}\,; \tag{8}$$

note the $\pm 2m$ normalization factors here, unlike the $+2E$ factors in eq. (2) for the $u^\dagger u$ and the $v^\dagger v$.

(b) There are only two independent $SO(3)$ spinors, hence $\sum_s \xi_s \xi_s^\dagger = \sum_s \eta_s \eta_s^\dagger = \mathbf{1}_{2\times 2}$. Use this fact to show that

$$\sum_{s=1,2} u_\alpha(p,s)\bar{u}_\beta(p,s) \;=\; (\not{p}+m)_{\alpha\beta} \quad \text{and} \quad \sum_{s=1,2} v_\alpha(p,s)\bar{v}_\beta(p,s) \;=\; (\not{p}-m)_{\alpha\beta}\,. \tag{9}$$

3. Now consider the charge conjugation symmetry $\mathbf{C}$ which exchanges particles with antiparticles, for example the electrons e^- with the positrons e^+,

$$\widehat{\mathbf{C}}\,\big|e^-(\mathbf{p},s)\big\rangle \;=\; \big|e^+(\mathbf{p},s)\big\rangle\,, \quad \widehat{\mathbf{C}}\,\big|e^+(\mathbf{p},s)\big\rangle \;=\; \big|e^-(\mathbf{p},s)\big\rangle\,. \tag{10}$$

Note that the operator $\widehat{\mathbf{C}}$ is unitary and squares to one (repeating the exchange brings us back to the original particles), hence $\widehat{\mathbf{C}}^\dagger = \widehat{\mathbf{C}}^{-1} = \widehat{\mathbf{C}}$.

(a) In the fermionic Fock space, the $\widehat{\mathbf{C}}$ operator act on multi-particle states by turning each particle into an antiparticle and vice verse according to eqs. (10). Show that this action implies

$$\widehat{\mathbf{C}}\,\hat{a}^\dagger_{\mathbf{p},s}\widehat{\mathbf{C}} \;=\; \hat{b}^\dagger_{\mathbf{p},s}\,, \quad \widehat{\mathbf{C}}\,\hat{b}^\dagger_{\mathbf{p},s}\widehat{\mathbf{C}} \;=\; \hat{a}^\dagger_{\mathbf{p},s}\,, \quad \widehat{\mathbf{C}}\,\hat{a}_{\mathbf{p},s}\widehat{\mathbf{C}} \;=\; \hat{b}_{\mathbf{p},s}\,, \quad \widehat{\mathbf{C}}\,\hat{b}_{\mathbf{p},s}\widehat{\mathbf{C}} \;=\; \hat{a}_{\mathbf{p},s}\,. \tag{11}$$

(b) The quantum Dirac fields $\widehat{\Psi}(x)$ and $\widehat{\overline{\Psi}}(x)$ are linear combinations of creation and annihilation operators. Use eqs. (11) and the plane-wave relations (6) to show that

$$\widehat{\mathbf{C}}\widehat{\Psi}(x)\widehat{\mathbf{C}} \;=\; \gamma^2\widehat{\Psi}^*(x) \quad \text{and} \quad \widehat{\mathbf{C}}\widehat{\overline{\Psi}}(x)\widehat{\mathbf{C}} \;=\; \widehat{\overline{\Psi}}^*(x)\gamma^2 \tag{12}$$

where $*$ stands for the hermitian conjugation of the component fields but without transposing a column vector (of 4 Dirac components) into a row vector or vice verse, thus

$$\widehat{\Psi} \;=\; \begin{pmatrix} \hat{\psi}_1 \\ \hat{\psi}_2 \\ \hat{\psi}_3 \\ \hat{\psi}_4 \end{pmatrix}, \qquad \widehat{\Psi}^* \;=\; \begin{pmatrix} \hat{\psi}_1^\dagger \\ \hat{\psi}_2^\dagger \\ \hat{\psi}_3^\dagger \\ \hat{\psi}_4^\dagger \end{pmatrix}, \qquad \begin{aligned} \widehat{\overline{\Psi}} \;&=\; \left(\hat{\psi}_1^\dagger, \hat{\psi}_2^\dagger, \hat{\psi}_3^\dagger, \hat{\psi}_4^\dagger\right) \times \gamma^0, \\ \widehat{\overline{\Psi}}^* \;&=\; \left(\hat{\psi}_1, \hat{\psi}_2, \hat{\psi}_3, \hat{\psi}_4\right) \times \gamma^0. \end{aligned} \tag{13}$$

(c) Show that the Dirac equation transforms covariantly under the charge conjugation (12). Hint: prove and use $\gamma^\mu\gamma^2 = -\gamma^2(\gamma^\mu)^*$ for all γ^μ in the Weyl basis.

(d) Show that that the *classical* Dirac Lagrangian is invariant under the charge conjugation (up to a total spacetime derivative). Note that in the classical limit the Dirac fields *anticommute* with each other, $\Psi^*_\alpha \Psi_\beta = -\Psi_\beta \Psi^*_\alpha$. Also, similar to the hermitian conjugation of quantum fields, the complex conjugation of fermionic fields reverses their order: $(F_1 F_2)^* = F_2^* F_1^* = -F_1^* F_2^*$.

4. Another important discrete symmetry is the *parity* $\mathbf{P}$, the im-proper Lorentz symmetry that reflects the space but not the time, $(\mathbf{x}, t) \to (-\mathbf{x}, +t)$. This symmetry acts on the Dirac spinor fields according to

$$\widehat{\Psi}'(-\mathbf{x}, +t) \;=\; \pm\gamma^0 \widehat{\Psi}(+\mathbf{x}, +t) \tag{14}$$

where the overall $\pm$ sign is the *intrinsic parity* of the fermion species described by the $\widehat{\Psi}$ field.

(a) Verify that the Dirac equation transforms covariantly under (14) and that the Dirac Lagrangian is invariant (apart from $\mathcal{L}(\mathbf{x}, t) \to \mathcal{L}(-\mathbf{x}, t)$).

In the Fock space, eq. (14) becomes

$$\widehat{\mathbf{P}} \widehat{\Psi}(\mathbf{x}, t) \widehat{\mathbf{P}} \;=\; \pm\gamma^0 \widehat{\Psi}(-\mathbf{x}, t) \tag{15}$$

for some unitary operator $\widehat{\mathbf{P}}$ that squares to one. Let's find how this operator acts on the particles and their states.

(b) First, check the plane-wave solutions $u(\mathbf{p}, s)$ and $v(\mathbf{p}, s)$ from problem (1) and show that $u(-\mathbf{p}, s) = +\gamma^0 u(\mathbf{p}, s)$ while $v(-\mathbf{p}, s) = -\gamma^0 v(\mathbf{p}, s)$.

(c) Now show that eq. (15) implies

$$\begin{aligned} \widehat{\mathbf{P}}\, \hat{a}_{\mathbf{p},s}\, \widehat{\mathbf{P}} \;&=\; \pm\hat{a}_{-\mathbf{p},+s}\,, \quad & \widehat{\mathbf{P}}\, \hat{a}^\dagger_{\mathbf{p},s}\, \widehat{\mathbf{P}} \;&=\; \pm\hat{a}^\dagger_{-\mathbf{p},+s}\,, \\ \widehat{\mathbf{P}}\, \hat{b}_{\mathbf{p},s}\, \widehat{\mathbf{P}} \;&=\; \mp\hat{b}_{-\mathbf{p},+s}\,, \quad & \widehat{\mathbf{P}}\, \hat{b}^\dagger_{\mathbf{p},s}\, \widehat{\mathbf{P}} \;&=\; \mp\hat{b}^\dagger_{-\mathbf{p},+s}\,, \end{aligned} \tag{16}$$

and hence

$$\widehat{\mathbf{P}}\, |F(\mathbf{p}, s)\rangle \;=\; \pm\, |F(-\mathbf{p}, +s)\rangle \quad \text{and} \quad \widehat{\mathbf{P}}\, \big|\overline{F}(\mathbf{p}, s)\big\rangle \;=\; \mp\, \big|\overline{F}(-\mathbf{p}, +s)\big\rangle\,. \tag{17}$$

Note that the fermion F and the antifermion $\overline{F}$ have opposite intrinsic parities!

5. A Dirac spinor field $\Psi(x)$ comprises two 2-component Weyl spinor fields,

$$\widehat{\Psi}(x) = \begin{pmatrix} \hat{\psi}_L(x) \\ \hat{\psi}_R(x) \end{pmatrix}. \tag{18}$$

Spell out the actions of **C**, **P**, and **CP** (combined —bf $C and$**P** symmetries on the Weyl spinors. In particular, show that **C** and **P** interchange the two spinors, while the combined **CP** symmetry acts on the ψ_L and the ψ_R independently from each other.

6. Consider the bilinear products of a Dirac field $\Psi(x)$ and its conjugate $\overline{\Psi}(x)$. Generally, such products have form $\overline{\Psi}\Gamma\Psi$ where Γ is one of 16 matrices discussed in the previous homework#7 (problem 4.g); altogether, we have

$$S = \overline{\Psi}\Psi, \quad V^\mu = \overline{\Psi}\gamma^\mu\Psi, \quad T^{\mu\nu} = \overline{\Psi}\tfrac{i}{2}\gamma^{[\mu}\gamma^{\nu]}\Psi, \quad A^\mu = \overline{\Psi}\gamma^5\gamma^\mu\Psi, \quad P = \overline{\Psi}i\gamma^5\Psi. \tag{19}$$

(a) Show that all the bilinears (19) are Hermitian.
Hint: First, show that $\left(\overline{\Psi}\Gamma\Psi\right)^\dagger = \overline{\Psi}\overline{\Gamma}\Psi$.
Note: despite the Fermi statistics, $\left(\Psi^\dagger_\alpha\Psi_\beta\right)^\dagger = +\Psi^\dagger_\beta\Psi_\alpha$.

(b) Show that under *continuous* Lorentz symmetries, the S and the P transform as scalars, the V^μ and the A^μ as vectors, and the $T^{\mu\nu}$ as an antisymmetric tensor.

(c) Find the transformation rules of the bilinears (19) under parity and show that while S is a true scalar and V is a true (polar) vector, P is a pseudoscalar and A is an axial vector.

Finally, consider the charge-conjugation properties of the Dirac bilinears. To avoid the operator-ordering problems, take the classical limit where $\Psi(x)$ and $\Psi^\dagger(x)$ *anticommute* with each other, $\Psi_\alpha\Psi^\dagger_\beta = -\Psi^\dagger_\beta\Psi_\alpha$.

(d) Show that **C** turns $\overline{\Psi}\Gamma\Psi$ into $\overline{\Psi}\Gamma^c\Psi$ where $\Gamma^c = \gamma^0\gamma^2\Gamma^\top\gamma^0\gamma^2$.

(e) Calculate Γ^c for all 16 independent matrices Γ and find out which Dirac bilinears are C–even and which are C–odd.