1. As a warm-up exercise, consider two species of scalar fields, Φ and ϕ, with a cubic coupling to each other,

$$\mathcal{L} \;=\; \frac{1}{2}(\partial_\mu\Phi)^2 \;-\; \frac{M^2}{2}\,\Phi^2 \;+\; \frac{1}{2}(\partial_\mu\phi)^2 \;-\; \frac{m^2}{2}\,\phi^2 \;-\; \frac{\mu}{2}\,\Phi\phi^2. \tag{1}$$

   (a) Write down the vertices and the propagators for the Feynman rules for this theory.

   (b) Suppose $M > 2m$, so a single Φ particle may decay to two ϕ particles. Calculate the rate Γ of this decay (in the rest frame of the original Φ) to lowest order in perturbation theory.

2. Now consider N scalar fields ϕ_i of the same mass m and with $O(N)$ symmetric quartic couplings to each other,

$$\mathcal{L} \;=\; \frac{1}{2}\sum_i(\partial_\mu\phi_i)^2 \;-\; \frac{m^2}{2}\sum_i\phi_i^2 \;-\; \frac{\lambda}{8}\left(\sum_i\phi_i^2\right)^2. \tag{2}$$

   (a) Write down the Feynman propagators and vertices for this theory.

   (b) Calculate the tree-level scattering amplitudes $\mathcal{M}$, the partial cross-sections $d\sigma/d\Omega_{\rm cm}$ (in the center-of-mass frame), and the total cross-sections for the following 3 processes:

      (i) $\phi_1 + \phi_2 \to \phi_1 + \phi_2$.

      (ii) $\phi_1 + \phi_1 \to \phi_2 + \phi_2$.

      (iii) $\phi_1 + \phi_1 \to \phi_1 + \phi_1$.

3. Next, consider the so-called *linear sigma model* comprising N massless scalar or pseudoscalar fields π_i and a massive scalar field σ with both quartic and cubic couplings to the pions, specifically

$$\begin{aligned}\mathcal{L} &= \frac{1}{2}\sum\nolimits_i(\partial_\mu\pi_i)^2 + \frac{1}{2}(\partial_\mu\sigma)^2 - \frac{\lambda}{8}\Big(\sum\nolimits_i \pi_i^2 + \sigma^2 + 2f\sigma\Big)^2 \\ &= \frac{1}{2}\sum_i(\partial_\mu\pi_i)^2 + \frac{1}{2}(\partial_\mu\sigma)^2 - \frac{\lambda f^2}{2}\times\sigma^2 \\ &\qquad - \frac{\lambda f}{2}\times\Big(\sigma^3 + \sigma\sum\nolimits_i \pi_i^2\Big) - \frac{\lambda}{8}\Big(\sum\nolimits_i \pi_i^2 + \sigma^2\Big)^2\end{aligned} \tag{3}$$

Both the masslessness of the π_i fields and the specific relations between the quartic couplings, the cubic couplings, and the sigma's mass $M_\sigma^2 = \lambda f^2$ in this model stem from the *spontaneous breaking down* of the $O(N+1)$ symmetry, which I shall explain in class later this semester. I shall also explain the relation of this model to the approximate chiral symmetry of QCD and hence to the real-life pi-mesons and their low-energy scattering amplitudes.

But in this homework, you should simply take the Lagrangian (3) as it is, and explore its implications for the scattering of π particles.

(a) Write down all the vertices and all the propagators for the Feynman rules for this theory.

(b) Draw *all* the tree diagrams and calculate the tree-level scattering amplitudes of two pions to two pions, $\mathcal{M}_{\text{tree}}(\pi^j + \pi^k \to \pi^\ell + \pi^m)$.

(c) Show that due to specific relations between the quartic and the cubic couplings in the Lagrangian (3), in the low-energy limit $E_{\text{tot}} \ll M_\sigma$, all the amplitudes $\mathcal{M}_{\text{tree}}(\pi^j + \pi^k \to \pi^\ell + \pi^m)$ become small as $O(E_{\text{tot}}^2/M_\sigma^2)$ or smaller.

Then use Mandelstam's variables s, t, u to show that when any of the incoming or outgoing pions' energy becomes small (while the other pions' energies are $O(M_\sigma)$), the scattering amplitudes become small as $O(E_{\text{small}}/M_\sigma)$ or smaller.

Later in class, we shall learn that this behavior stems from the *Goldstone–Nambu theorem.*

(d) Write down specific tree-level amplitudes, partial cross-sections (in the CM frame), and total cross-sections for the processes

(i) $\pi^1 + \pi^2 \to \pi^1 + \pi^2$

(ii) $\pi^1 + \pi^1 \to \pi^2 + \pi^2$

(iii) $\pi^1 + \pi^1 \to \pi^1 + \pi^1$

in the low-energy limit $E_{\text{cm}} \ll M_\sigma$.

4. Finally, an exercise in phase-space calculation: the muon decay. Most of the time, a muon decays into an electron, an electron-flavored antineutrino, and a muon-flavored neutrino, $\mu^- \to e^- \bar{\nu}_e \nu_\mu$. In a later homework, we shall calculate the tree-level amplitude for this process in the Standard Model, and then we shall learn how to sum this amplitude — or rather the $|\mathcal{M}|^2$ — over the spin states of the outgoing electron, neutrino, and antineutrino, and average over the spin states of the initial muon. But for the present purposes, let me simply give you the result:

$$\overline{|\mathcal{M}|^2} \;=\; \frac{1}{2} \sum_{\substack{\text{all} \\ \text{spins}}} \left| \left\langle e^-, \bar{\nu}_e, \nu_\mu \right| \mathcal{M} \left| \mu^- \right\rangle \right|^2 \;=\; 64 G_F^2 (p_\mu \cdot p_{\bar{\nu}})\, (p_e \cdot p_\nu), \tag{4}$$

where $G_F \approx 1.17 \cdot 10^{-5}\,\text{GeV}^{-2}$ is the Fermi constant.

The following lemma is very useful for three-body decays like $\mu^- \to e^- + \nu_\mu + \bar{\nu}_e$:
For a decay of initial particle of mass M_0 into three final particles of respective masses m_1, m_2, and m_3, the partial decay rate in the rest frame of the original particle is

$$d\Gamma \;=\; \frac{1}{2M_0} \times \overline{|\mathcal{M}|^2} \times \frac{d^3\Omega}{256\pi^5} \times dE_1\, dE_2\, dE_3\, \delta(E_1 + E_2 + E_3 - M_0), \tag{5}$$

where $d^3\Omega$ comprises three angular variables parametrizing the directions of the three final-state particles relative to some external frame, but not affecting the angles between the three momenta. For example, one may use two angles to describe the orientation of the decay plane (the three momenta are coplanar, $\mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3 = 0$) and one more angle to fix the direction of *e.g.*, $\mathbf{p}_1$ in that plane. Altogether, $\int d^3\Omega = 4\pi \times 2\pi = 8\pi^2$.

(a) Prove this lemma.

The electron and the neutrinos are so much lighter then the muon that in most decay events all three final-state particles are ultra-relativistic. This allows us to approximate $m_e \approx m_\nu \approx m_{\bar{\nu}} \approx 0$, which gives us rather simple limits for the final particles' energies.

(b) Show that when $m_1 = m_2 = m_3 = 0$, the kinematically allowed range of the final particles' energies is given by

$$0 \;\le\; E_1, E_2, E_3 \;\le \tfrac{1}{2} M_0 \qquad \text{while} \quad E_1 + E_2 + E_3 \;=\; M_0. \tag{6}$$

Note however that for non-zero masses $m_{1,2,3}$, the allowed energy range becomes much more complicated.

Experimentally, the neutrinos and the antineutrinos are hard to detect. But it is easy to measure the muon's net decay rate $\Gamma = 1/\tau_\mu$ and the energy distribution $d\Gamma/dE_e$ of the electrons produced by decaying muons.

(c) Integrate the muon's partial decay rate over the final particle energies and derive first the $d\Gamma/dE_e$ and then the total decay rate.