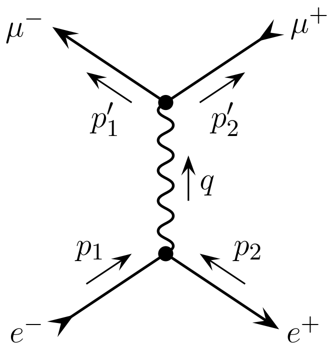
1. First, a reading assignment: §4.7 of the *Peskin&Schroeder* textbook about the Feynman rules of the Yukawa theory. Find out where the sign rules for the fermionic lines come from. Also find out the origin of the Yukawa potential $V(r) \propto e^{-mr}/r$. (There is also a much shorter explanation of the Yukawa theory and the Yukawa potential on the last three pages of my notes on QED Feynman rules.

2. Second, a simple QED problem: pair production of muons in electron-positron collisions, $e^-e^+ \to \mu^-\mu^+$. As I explained in class, there is only one tree diagram for this process,



which yields the amplitude

$$\left\langle \mu^-, \mu^+ \right| \mathcal{M} \left| e^-, e^+ \right\rangle = \frac{e^2}{s} \times \bar{u}(\mu^-)\gamma^\nu v(\mu^+) \times \bar{v}(e^+)\gamma_\nu u(e^-). \tag{1}$$

In class I have focused on the un-polarized pair-production cross-section — see my notes on the subject, — but in this exercise you should focus on the polarized amplitudes for definite helicities of all 4 particles involved.

For simplicity, let us assume that all the particles are ultra-relativistic so that their Dirac spinors $u(e^-)$, $v(e^+)$, $u(\mu^-)$, $v(\mu^+)$ all have definite chiralities,

$$\begin{aligned} u_L &\approx \sqrt{2E}\begin{pmatrix}\xi_L\\ 0\end{pmatrix}, & u_R &\approx \sqrt{2E}\begin{pmatrix}0\\ \xi_R\end{pmatrix}, \\ v_L &\approx -\sqrt{2E}\begin{pmatrix}0\\ \eta_L\end{pmatrix}, & v_R &\approx \sqrt{2E}\begin{pmatrix}\eta_R\\ 0\end{pmatrix}. \end{aligned} \tag{2}$$

cf. homework set#8, eq. (7).

(a) Show that in the approximation (2),

$$\bar{v}(e_L^+)\gamma_\nu u(e_L^-) \;=\; \bar{v}(e_R^+)\gamma_\nu u(e_R^-) \;=\; 0, \tag{3}$$

which means there is no muon pairs production unless the initial electron and positron have *opposite helicities*.

(b) Show that the μ^- and the μ^+ must also have *opposite helicities* because

$$\bar{u}(\mu_L^-)\gamma^\nu v(\mu_L^+) \;=\; \bar{u}(\mu_R^-)\gamma^\nu v(\mu_R^+) \;=\; 0. \tag{4}$$

(c) Let's work in the center-of-mass frame where the initial e^- and e^+ collide along the z axis, $p_1^\nu = (E, 0, 0, +E)$, $p_2^\nu = (E, 0, 0, -E)$. Calculate the 4–vector $\bar{v}(e^+)\gamma^\nu u(e^-)$ in this frame and show that

$$\bar{v}(e_L^+)\gamma_\nu u(e_R^-) \;=\; 2E \times (0, -i, +1, 0)_\nu\,, \qquad \bar{v}(e_R^+)\gamma_\nu u(e_L^-) \;=\; 2E \times (0, +i, +1, 0)_\nu\,. \tag{5}$$

(d) In the CM frame the muons fly away in opposite directions at some angle θ to the electron / positron directions. Without loss of generality we may assume the muons' momenta being in the xz plane, thus

$$p_1'^\nu \;=\; (E, +E\sin\theta, 0, +E\cos\theta), \qquad p_1'^\nu \;=\; (E, -E\sin\theta, 0, -E\cos\theta) \tag{6}$$

Calculate the 4–vector $\bar{u}(\mu^-)\gamma_\nu v(\mu^+)$ for the muons and show that

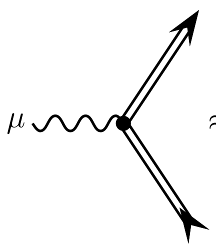
$$\begin{aligned} \bar{u}(\mu_R^-)\gamma^\nu v(\mu_L^+) &= 2E \times (0, -i\cos\theta, -1, +i\sin\theta), \\ \bar{u}(\mu_L^-)\gamma^\nu v(\mu_R^+) &= 2E \times (0, +i\cos\theta, -1, -i\sin\theta). \end{aligned} \tag{7}$$

(e) Now calculate the amplitudes (1) for all possible combinations of particles' helicities, calculate the partial cross-sections, and show that

$$\begin{aligned}
\frac{d\sigma(e_L^- + e_R^+ \to \mu_L^- + \mu_R^+)}{d\Omega_{\text{c.m.}}} &= \frac{d\sigma(e_R^- + e_L^+ \to \mu_R^- + \mu_L^+)}{d\Omega_{\text{c.m.}}} = \frac{\alpha^2}{4s} \times (1+\cos\theta)^2, \\
\frac{d\sigma(e_L^- + e_R^+ \to \mu_R^- + \mu_L^+)}{d\Omega_{\text{c.m.}}} &= \frac{d\sigma(e_R^- + e_L^+ \to \mu_L^- + \mu_R^+)}{d\Omega_{\text{c.m.}}} = \frac{\alpha^2}{4s} \times (1-\cos\theta)^2, \\
\frac{d\sigma(e_L^- + e_L^+ \to \mu_{\text{any}}^- + \mu_{\text{any}}^+)}{d\Omega_{\text{c.m.}}} &= \frac{d\sigma(e_R^- + e_R^+ \to \mu_{\text{any}}^- + \mu_{\text{any}}^+)}{d\Omega_{\text{c.m.}}} = 0, \\
\frac{d\sigma(e_{\text{any}}^- + e_{\text{any}}^+ \to \mu_L^- + \mu_L^+)}{d\Omega_{\text{c.m.}}} &= \frac{d\sigma(e_{\text{any}}^- + e_{\text{any}}^+ \to \mu_R^- + \mu_R^+)}{d\Omega_{\text{c.m.}}} = 0.
\end{aligned} \tag{8}$$

(f) Finally, sum / average over the helicities and calculate the un-polarized cross-section for the muon pair production.

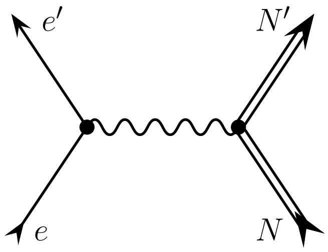
3. Third, another simple QED problem: *Mott scattering* of a relativistic electron off a heavy nucleus of charge $+Ze$ and mass $M_N \gg m_e$. As long as the electron's energy E_e is no larger than a few tens of MeV, we may treat the nucleus as a point source of the electric field, and we may also neglect its recoil. Hence, in the CM frame — which is essentially the nucleus's frame — we may approximate the nucleus-nucleus-photon vertex as



$$\approx\ -iZe \times \begin{cases} 2M_N & \text{for } \mu = 0, \\ 0 & \text{for } \mu = 1,2,3. \end{cases} \tag{9}$$

To be precise, this formula includes the vertex and the external leg factors for the incoming and outgoing nucleus, but it does not include the photon's propagator.

In QED, there is only one tree diagram for the Mott scattering, namely



(10)

(a) Evaluate this diagram and write down the amplitude $\mathcal{M}$ in terms of $q = p' - p$ and

$\bar{u}(p', s')\gamma^0 u(p, s)$.

(b) Assume the initial electron beam is un-polarized (*i.e.*, both values of spin s are equally likely) and the detector for the scattered electron does not measure its spin s' but only momentum s'. Show that for such an experiment,

$$\frac{d\sigma}{d\Omega} = \frac{(Z\alpha)^2}{(\mathbf{q}^2)^2} \times \frac{1}{2}\sum_{s,s'} \left|\bar{u}(p', s')\gamma^0 u(p, s)\right|^2 \tag{11}$$

where $\alpha = e^2/4\pi$ (or in conventional units, $\alpha = e^2/\hbar c$; anyhow, $\alpha \approx 1/137$.)

(c) Sum over the electron spins and show that

$$\frac{1}{2}\sum_{s,s'} \left|\bar{u}(p', s')\gamma^0 u(p, s)\right|^2 = 2\left(m_e^2 + EE' + \mathbf{p}\cdot\mathbf{p}'\right). \tag{12}$$

(d) Finally, assemble all the factors together and derive Mott formula

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{Mott}} = \left(\frac{d\sigma}{d\Omega}\right)_{\text{Rutherford}} \times \frac{1 - \beta^2 \sin^2(\theta/2)}{\gamma^2} \tag{13}$$

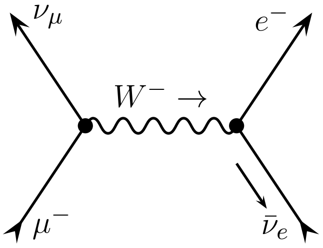
where β is the electron's speed (in $c = 1$ units), $\gamma = E/m_e$, and

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{Rutherford}} = \frac{(Z\alpha)^2}{4m_e^2\beta^4 \sin^4(\theta/2)} \tag{14}$$

is the classical Rutherford scattering cross-section (translated into $\hbar = c = 1$ units).

4. Finally, a non-QED problem on Dirac trace technology. Consider the muon decay into an electron, an electron-flavored antineutrino, and a muon-flavored neutrino, $\mu^- \to e^- \bar{\nu}_e \nu_\mu$. In the previous homework#9 (problem 4) you have calculated the phase space factor for this decay. In the current problem, we shall derive eq. (9.4) for the spin-summed amplitude you have used last week to calculate the muon decay rate.

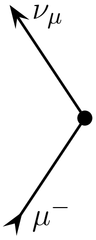
At the tree level of the Standard model, this decay proceeds through a single Feynman diagram



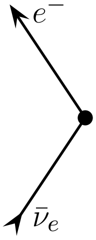
$$(15)$$

Since I have not yet explained the Standard Model in class — although I plan to do it in December, — let me simply spell out the Feynman rules relevant to this diagram.

- The vertices and the external fermionic legs attached to them are



$$= \bar{u}(\nu_\mu)\left(-ig_2\gamma^\kappa\,\frac{1-\gamma^5}{2}\right)u(\mu^-),$$



$$= \bar{u}(e^-)\left(-ig_2\gamma^\lambda\,\frac{1-\gamma^5}{2}\right)v(\bar{\nu}_e), \tag{16}$$

  where g_2 is the $SU(2)_W$ gauge coupling.

- W^- is a massive vector particle, so its propagator is

$$= \frac{-i}{q^2 - M_W^2}\left(g_{\kappa\lambda} - \frac{k_\kappa k_\lambda}{M_W^2}\right) \xrightarrow[|q|\ll M_W]{} \frac{ig_{\kappa\lambda}}{M_W^2}\,. \tag{17}$$

  The approximation here corresponds to the effective Fermi theory of weak interactions. It is valid for all nuclear beta decays as well as weak decays of all particles much lighter than $M_W \approx 80$ GeV. In particular, it is valid for the muon decay in question.

(a) Assemble the muon decay amplitude (in the Fermi theory approximation) to

$$\mathcal{M} \;=\; -\frac{G_F}{\sqrt{2}} \times \left[\bar{u}(\nu_\mu)\gamma^\lambda(1-\gamma^5)u(\mu^-)\right] \times \left[\bar{u}(e^-)\gamma_\lambda(1-\gamma^5)v(\bar{\nu}_e)\right] \tag{18}$$

where

$$G_F \;=\; \frac{g_2^2}{2\sqrt{2}M_W^2} \;\approx\; 1.17\cdot 10^{-5}\,\mathrm{GeV}^{-2} \tag{19}$$

is the Fermi constant of low-energy weak interactions.

(b) Sum the absolute square of the amplitude (18) over the final particle spins, average over the initial muon's spin, and write the result as a product of two Dirac traces,

$$\overline{|\mathcal{M}|^2} \stackrel{\text{def}}{=} \frac{1}{2}\sum_{\substack{\text{all}\\\text{spins}}} |\mathcal{M}|^2 \;=\; \frac{G_F^2}{4} \times \operatorname{tr}(\text{matrix product } \#1) \times \operatorname{tr}(\text{matrix product } \#2). \tag{20}$$

(c) Evaluate the traces in eq. (20).

(d) Sum over the Lorentz indices and show that altogether

$$\overline{|\mathcal{M}|^2} \;=\; 64G_F^2(p_\mu \cdot p_{\bar{\nu}})\,(p_e \cdot p_\nu), \tag{21}$$

exactly as in eq. (9.4) from the previous homework#9.