1. Let's start with an exercise in the Higgs mechanism. In the previous homework (set#11, problem#5), we had the continuous global symmetry group $G = SU(N)_L \times SU(N)_R \times U(1)$ spontaneously broken down to its $H = SU(N)_V$ subgroup. Now let's gauge the entire $G = SU(N)_L \times SU(N)_R \times U(1)$ symmetry and work out the Higgs mechanism.

   The present theory comprises N^2 complex scalar fields $\Phi_i^{\ j}(x)$ organized into an $N \times N$ matrix, and $2N^2 - 1$ real vector fields $B_\mu(x)$, $L^a_\mu(x)$, and $R^a_\mu(x)$, the latter organized into traceless hermitian matrices $L_\mu(x) = \sum_a L^a_\mu(x) \times \frac{1}{2}\lambda^a$ and $R_\mu(x) = \sum_a R^a_\mu(x) \times \frac{1}{2}\lambda^a$, where $a = 1, \ldots, (N^2 - 1)$ and λ^a are the Gell-Mann matrices of $SU(N)$. The Lagrangian is

$$\mathcal{L} \;=\; -\tfrac{1}{4} B_{\mu\nu} B^{\mu\nu} \;-\; \tfrac{1}{2} \operatorname{tr}\left(L_{\mu\nu} L^{\mu\nu}\right) \;-\; \tfrac{1}{2} \operatorname{tr}\left(R_{\mu\nu} R^{\mu\nu}\right) \;+\; \operatorname{tr}\left(D^\mu \Phi^\dagger \, D_\mu \Phi\right) - \; V(\Phi^\dagger \Phi), \quad (1)$$

   where

$$\begin{aligned}
B_{\mu\nu} \;&=\; \partial_\mu B_\nu \;-\; \partial_\nu B_\mu \,, \\
L_{\mu\nu} \;&=\; \partial_\mu L_\nu \;-\; \partial_\nu L_\mu \;+\; ig[L_\mu, L_\nu], \\
R_{\mu\nu} \;&=\; \partial_\mu R_\nu \;-\; \partial_\nu R_\mu \;+\; ig[R_\mu, R_\nu], \\
D_\mu \Phi \;&=\; \partial_\mu \Phi \;+\; ig' B_\mu \Phi \;+\; ig L_\mu \Phi \;-\; ig \Phi R_\mu \,, \\
D_\mu \Phi^\dagger \;&=\; (D_\mu \Phi)^\dagger \;=\; \partial_\mu \Phi^\dagger \;-\; ig' B_\mu \Phi^\dagger \;+\; ig R_\mu \Phi^\dagger \;-\; ig \Phi^\dagger L_\mu \,.
\end{aligned} \qquad (2)$$

   For simplicity, I assume equal gauge couplings $g_L = g_R = g$ for the two $SU(N)$ factors of the gauge group, but the abelian coupling g' is different.

   The scalar potential V is precisely as in the previous homework,

$$V \;=\; \frac{\alpha}{2} \operatorname{tr}\bigl(\Phi^\dagger \Phi \Phi^\dagger \Phi\bigr) \;+\; \frac{\beta}{2} \operatorname{tr}^2\bigl(\Phi^\dagger \Phi\bigr) \;+\; m^2 \operatorname{tr}\bigl(\Phi^\dagger \Phi\bigr), \qquad \alpha, \beta > 0, \quad m^2 < 0, \qquad (3)$$

   hence similar VEVs of the scalar fields: up to a gauge symmetry,

$$\langle \Phi \rangle \;=\; C \times \mathbf{1}_{N \times N} \quad \text{where} \quad C \;=\; \sqrt{\frac{-m^2}{\alpha + N\beta}}\,, \qquad (4)$$

   which breaks the $G = SU(N)_L \times SU(N)_R \times U(1)$ symmetry down to the $SU(N)_V$ subgroup.

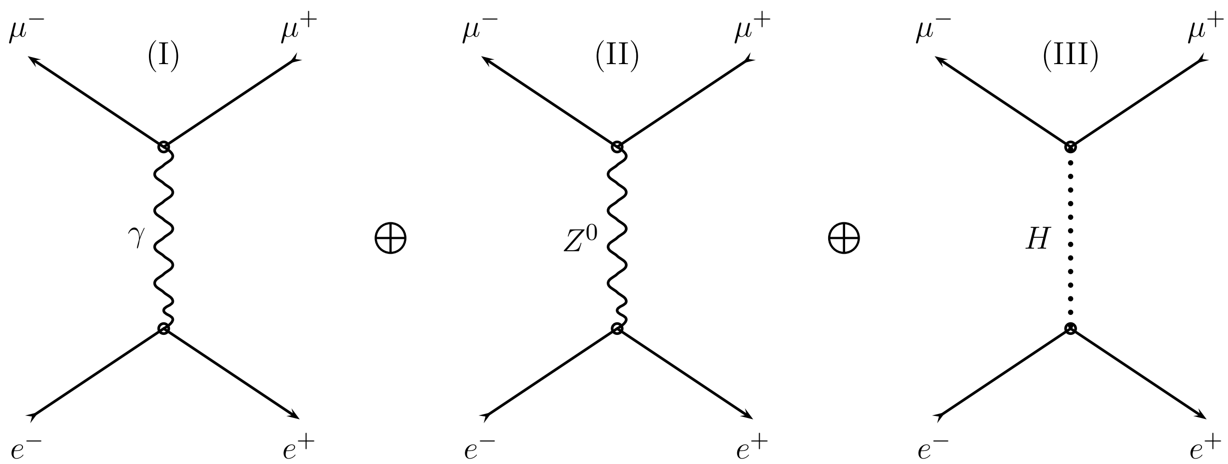
(a) The Higgs mechanism makes N^2 out of $2N^2-1$ vector fields massive. Calculate their masses by plugging $\langle\Phi\rangle$ for the $\Phi(x)$ into the $\mathrm{tr}(D_\mu\Phi^\dagger\, D^\mu\Phi)$ term of the Lagrangian. In particular, show that the abelian gauge field B_μ and the $X^a_\mu = \frac{1}{\sqrt{2}}(L^a_\mu - R^a_\mu)$ combinations of the $SU(N)$ gauge fields become massive, while the $V^a_\mu = \frac{1}{\sqrt{2}}(L^a_\mu + R^a_\mu)$ combinations remain massless.

(b) Find the effective Lagrangian for the massless vector fields $V^a_\mu(x)$ by freezing all the other fields, *i.e.* setting $B_\mu(x)\equiv 0$, $X^a_\mu(x)\equiv 0$, and $\Phi(x)\equiv\langle\Phi\rangle$. Show that this Lagrangian describes a Yang–Mills theory with gauge group $SU(N)_V$ and gauge coupling $g_V = g/\sqrt{2}$.

$\star$ For extra challenge, allow for un-equal gauge couplings $g_L \neq g_R$. Find which combinations of the $L^a_\mu(x)$ and $R^a_\mu(x)$ fields remain massless in this case, then derive the effective Lagrangian for these massless fields by freezing everything else. As in part (b), you should get an $SU(N)$ Yang–Mills theory, but this time the gauge coupling is

$$g_v = \frac{g_L g_R}{\sqrt{g_L^2 + g_R^2}}\,. \tag{5}$$

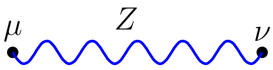
2. Now consider the $e^+e^- \to \mu^+\mu^-$ pair production in the Standard Model of electroweak interactions rather than in just QED. Unlike QED, the Standard Model has 3 three diagrams contributing to this process: one with the virtual photon in the s channel, one with the virtual Z^0 gauge boson, and one with the virtual Higgs scalar,



$$\mathcal{M}(e^-,e^+\to\mu^-,\mu^+)\;=\;\mathcal{M}_\gamma\;+\;\mathcal{M}_Z\;+\;\mathcal{M}_H\,. \tag{6}$$

The first diagram was studied in class and also in the homework set#10. In this problem, we shall focus on the other two diagrams — especially the diagram (II) with a virtual Z^0 — and on their interference with the first diagram.

For simplicity, let's work in the unitary gauge where the 'eaten-up' scalars do not have any vertices or propagators, while the massive gauge bosons like Z_0 have propagators



$$=\;\frac{i}{q^2-M_Z^2+iM_Z\Gamma_Z}\times\left(-g^{\mu\nu}\;+\;\frac{q^\mu q^\nu}{M_Z^2}\right). \tag{7}$$

(a) Derive the electron-Z and muon-Z vertices in diagram (II) from the neutral week current,

$$\mathcal{L}\;\supset\;-g'Z_\lambda\times\sum_{\substack{\text{quarks and}\\ \text{leptons}}}\overline{\Psi}\gamma^\lambda\left(T^3\frac{1-\gamma^5}{2}\;-\;Q\sin^2\theta\right)\Psi, \tag{8}$$

cf. my notes on the electroweak interactions of quarks and leptons, and write down the amplitude $\mathcal{M}_Z$. For simplicity, approximate $\sin^2\theta\approx\frac{1}{4}$ (experimentally, $\sin^2\theta\approx 0.233$) so that for the charged leptons like the electron or the muon

$$T^3\frac{1-\gamma^5}{2}\;-\;Q\sin^2\theta\;=\;\frac{-1+\gamma^5}{4}\;+\;\sin^2\theta\;\approx\;\frac{\gamma^5}{4}\,, \tag{9}$$

(b) Assume both the electrons and the muons to be ultra-relativistic ($E_{\text{c.m.}}=O(M_Z)\gg m_\mu,m_e$) and evaluate the amplitude $\mathcal{M}_Z$ for all possible particle helicities. (Use the center-of-mass frame.)
Hint: proceed exactly as in homework set#10 (problem 1) for the $\mathcal{M}_\gamma$ amplitude, but mind the γ^5 factors in the vertices and the massive vector propagator for the Z_0.

(c) Write down the amplitude $\mathcal{M}_H$ due to virtual Higgs scalar (diagram III). Also, relate the Yukawa couplings of the Higgs to the electrons and the muons to the fermion masses, then argue that these couplings are so much smaller than the gauge couplings e or g' that the $\mathcal{M}_H$ is negligibly small compared to the $\mathcal{M}_Z$ or $\mathcal{M}_\gamma$ amplitudes.

(d) Combine the amplitudes due to virtual Z and virtual photon and calculate the polarized partial cross-sections $d\sigma(e^-e^+ \to \mu^-\mu^+)/d\Omega$ as functions of EM energy$^2 = s$, scattering angle θ, and helicities of all 4 fermions involved. Specifically, show that

$$\begin{aligned}
\frac{d\sigma(e^-_L + e^+_L \to \mu^-_{\text{any}} + \mu^+_{\text{any}})}{d\Omega_{\text{c.m.}}} &= \frac{d\sigma(e^-_R + e^+_R \to \mu^-_{\text{any}} + \mu^+_{\text{any}})}{d\Omega_{\text{c.m.}}} = 0, \\
\frac{d\sigma(e^-_{\text{any}} + e^+_{\text{any}} \to \mu^-_L + \mu^+_L)}{d\Omega_{\text{c.m.}}} &= \frac{d\sigma(e^-_{\text{any}} + e^+_{\text{any}} \to \mu^-_R + \mu^+_R)}{d\Omega_{\text{c.m.}}} = 0.
\end{aligned} \tag{10}$$

while

$$\begin{aligned}
\frac{d\sigma(e^-_L + e^+_R \to \mu^-_L + \mu^+_R)}{d\Omega_{\text{c.m.}}} &= \frac{d\sigma(e^-_R + e^+_L \to \mu^-_R + \mu^+_L)}{d\Omega_{\text{c.m.}}} \\
&= \frac{\alpha^2}{4s} \times |1 + F(s)|^2 \times (1 + \cos\theta)^2, \\
\frac{d\sigma(e^-_L + e^+_R \to \mu^-_R + \mu^+_L)}{d\Omega_{\text{c.m.}}} &= \frac{d\sigma(e^-_R + e^+_L \to \mu^-_L + \mu^+_R)}{d\Omega_{\text{c.m.}}} \\
&= \frac{\alpha^2}{4s} \times |1 - F(s)|^2 \times (1 - \cos\theta)^2,
\end{aligned} \tag{11}$$

where the $|1 \pm F(s)|^2$ factor stem from the interference between the virtual-photon and virtual-Z diagrams.

(e) Finally, assume un-polarized electron and positron beams and a spin-blind muon detector. Calculate the total cross section $\sigma(e^+e^- \to \mu^+\mu^-)$ and the forward-backward asymmetry

$$A = \frac{\sigma(\theta < \pi/2) - \sigma(\theta > \pi/2)}{\sigma(\theta < \pi/2) + \sigma(\theta > \pi/2)} \tag{12}$$

as functions of the total energy $E_{\text{c.m.}}$.

Note: In QED, the tree-level pair production is symmetric with respect to $\theta \to \pi - \theta$; the asymmetry in the Standard Model arises from the interference between the virtual-photon and virtual-Z diagrams.

3. Finally, another Standard Model problem. While most weak decays or quarks or leptons involve a virtual W^+ or W^- gauge boson, the top quark is so heavy that it decays into a real (*i.e.*, on-shell) W^+ and the bottom quark. The interactions relevant to this process are contained in

$$\mathcal{L} \supset -\frac{g}{2\sqrt{2}} W_\mu^- \overline{\Psi}_b \gamma^\mu (1-\gamma^5) \Psi_t \;-\; \frac{g}{2\sqrt{2}} W_\mu^+ \overline{\Psi}_t \gamma^\mu (1-\gamma^5) \Psi_b \,. \tag{13}$$

For your information, $\alpha_w \equiv (g^2/4\pi) \approx 1/30$, $M_t \approx 173$ GeV, $M_W \approx 80.5$ GeV, and $M_b \approx 4.5$ GeV.

(a) Write down the Feynman vertices for the interactions (13), draw tree diagram(s) for the $t \to W^+ + b$ decay, and write down the tree-level decay amplitude.

(b) This amplitude does *not* satisfy the Ward identity. Write down a simple formula for $k_W^\mu \times \mathcal{M}_\mu$.

(c) The W^+ gauge boson is a massive particle, so it has 3 distinct spin/polarization states. Show that its polarization vectors $\mathcal{E}^\mu(k,\lambda)$ satisfy

$$\sum_\lambda \mathcal{E}^\mu(k,\lambda) \times \mathcal{E}^{*\nu}(k,\lambda) \;=\; -g^{\mu\nu} \;+\; \frac{k_W^\mu k_W^\nu}{M_W^2}\,, \tag{14}$$

and consequently, for the W emission amplitude of the form $\mathcal{M} = \mathcal{M}_\mu \times \mathcal{E}^\mu(k,\lambda)$, summing $|\mathcal{M}|^2$ over the W polarizations yields

$$\sum_\lambda |\mathcal{M}|^2 \;=\; -\mathcal{M}^\mu \mathcal{M}_\mu^* \;+\; \frac{|\mathcal{M}_\mu k_W^\mu|^2}{M_W^2}\,. \tag{15}$$

(d) Going back to the top quark decay $t \to b + W^+$, sum the $|\mathcal{M}|^2$ over both final particles' spins, average over the initial top quark's spin, and calculate the decay rate.

For simplicity, you may neglect the bottom quark's mass compared to masses of the top quark and of the W boson.