

1. Let's start with an exercise in the Higgs mechanism. In the [previous homework](#) (set#11, problem#5), we had the continuous global symmetry group $G = SU(N)_L \times SU(N)_R \times U(1)$ spontaneously broken down to its $H = SU(N)_V$ subgroup. Now let's gauge the entire $G = SU(N)_L \times SU(N)_R \times U(1)$ symmetry and work out the Higgs mechanism.

The present theory comprises N^2 complex scalar fields $\Phi_i^j(x)$ organized into an $N \times N$ matrix, and $2N^2 - 1$ real vector fields $B_\mu(x)$, $L_\mu^a(x)$, and $R_\mu^a(x)$, the latter organized into traceless hermitian matrices $L_\mu(x) = \sum_a L_\mu^a(x) \times \frac{1}{2} \lambda^a$ and $R_\mu(x) = \sum_a R_\mu^a(x) \times \frac{1}{2} \lambda^a$, where $a = 1, \dots, (N^2 - 1)$ and λ^a are the Gell-Mann matrices of $SU(N)$. The Lagrangian is

$$\mathcal{L} = -\frac{1}{4} B_{\mu\nu} B^{\mu\nu} - \frac{1}{2} \text{tr}(L_{\mu\nu} L^{\mu\nu}) - \frac{1}{2} \text{tr}(R_{\mu\nu} R^{\mu\nu}) + \text{tr}(D^\mu \Phi^\dagger D_\mu \Phi) - V(\Phi^\dagger \Phi), \quad (1)$$

where

$$\begin{aligned} B_{\mu\nu} &= \partial_\mu B_\nu - \partial_\nu B_\mu, \\ L_{\mu\nu} &= \partial_\mu L_\nu - \partial_\nu L_\mu + ig[L_\mu, L_\nu], \\ R_{\mu\nu} &= \partial_\mu R_\nu - \partial_\nu R_\mu + ig[R_\mu, R_\nu], \\ D_\mu \Phi &= \partial_\mu \Phi + ig' B_\mu \Phi + ig L_\mu \Phi - ig \Phi R_\mu, \\ D_\mu \Phi^\dagger &= (D_\mu \Phi)^\dagger = \partial_\mu \Phi^\dagger - ig' B_\mu \Phi^\dagger + ig R_\mu \Phi^\dagger - ig \Phi^\dagger L_\mu. \end{aligned} \quad (2)$$

For simplicity, I assume equal gauge couplings $g_L = g_R = g$ for the two $SU(N)$ factors of the gauge group, but the abelian coupling g' is different.

The scalar potential V is precisely as in the previous homework,

$$V = \frac{\alpha}{2} \text{tr}(\Phi^\dagger \Phi \Phi^\dagger \Phi) + \frac{\beta}{2} \text{tr}^2(\Phi^\dagger \Phi) + m^2 \text{tr}(\Phi^\dagger \Phi), \quad \alpha, \beta > 0, \quad m^2 < 0, \quad (3)$$

hence similar VEVs of the scalar fields: up to a gauge symmetry,

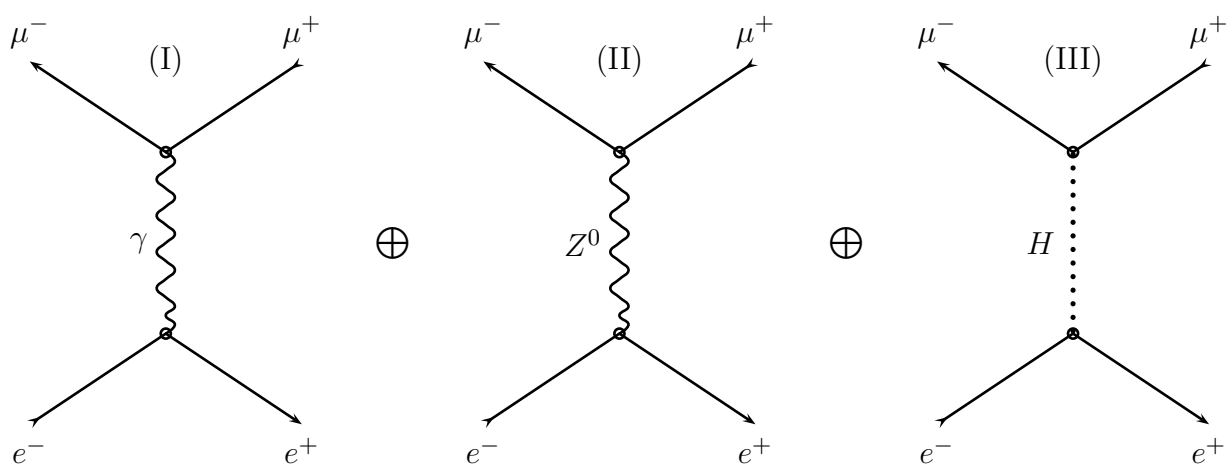
$$\langle \Phi \rangle = C \times \mathbf{1}_{N \times N} \quad \text{where} \quad C = \sqrt{\frac{-m^2}{\alpha + N\beta}}, \quad (4)$$

which breaks the $G = SU(N)_L \times SU(N)_R \times U(1)$ symmetry down to the $SU(N)_V$ subgroup.

- (a) The Higgs mechanism makes N^2 out of $2N^2 - 1$ vector fields massive. Calculate their masses by plugging $\langle \Phi \rangle$ for the $\Phi(x)$ into the $\text{tr}(D_\mu \Phi^\dagger D^\mu \Phi)$ term of the Lagrangian. In particular, show that the abelian gauge field B_μ and the $X_\mu^a = \frac{1}{\sqrt{2}}(L_\mu^a - R_\mu^a)$ combinations of the $SU(N)$ gauge fields become massive, while the $V_\mu^a = \frac{1}{\sqrt{2}}(L_\mu^a + R_\mu^a)$ combinations remain massless.
- (b) Find the effective Lagrangian for the massless vector fields $V_\mu^a(x)$ by freezing all the other fields, *i.e.* setting $B_\mu(x) \equiv 0$, $X_\mu^a(x) \equiv 0$, and $\Phi(x) \equiv \langle \Phi \rangle$. Show that this Lagrangian describes a Yang–Mills theory with gauge group $SU(N)_V$ and gauge coupling $g_V = g/\sqrt{2}$.
- ★ For extra challenge, allow for unequal gauge couplings $g_L \neq g_R$. Find which combinations of the $L_\mu^a(x)$ and $R_\mu^a(x)$ fields remain massless in this case, then derive the effective Lagrangian for these massless fields by freezing everything else. As in part (b), you should get an $SU(N)$ Yang–Mills theory, but this time the gauge coupling is

$$g_v = \frac{g_L g_R}{\sqrt{g_L^2 + g_R^2}}. \quad (5)$$

2. Now consider the $e^+e^- \rightarrow \mu^+\mu^-$ pair production in the Standard Model of electroweak interactions rather than in just QED. Unlike QED, the Standard Model has 3 three diagrams contributing to this process: one with the virtual photon in the s channel, one with the virtual Z^0 gauge boson, and one with the virtual Higgs scalar,



- (d) Combine the amplitudes due to virtual Z and virtual photon and calculate the polarized partial cross-sections $d\sigma(e^-e^+ \rightarrow \mu^-\mu^+)/d\Omega$ as functions of EM energy² = s , scattering angle θ , and helicities of all 4 fermions involved. Specifically, show that

$$\begin{aligned} \frac{d\sigma(e_L^- + e_L^+ \rightarrow \mu_{\text{any}}^- + \mu_{\text{any}}^+)}{d\Omega_{\text{c.m.}}} &= \frac{d\sigma(e_R^- + e_R^+ \rightarrow \mu_{\text{any}}^- + \mu_{\text{any}}^+)}{d\Omega_{\text{c.m.}}} = 0, \\ \frac{d\sigma(e_{\text{any}}^- + e_{\text{any}}^+ \rightarrow \mu_L^- + \mu_L^+)}{d\Omega_{\text{c.m.}}} &= \frac{d\sigma(e_{\text{any}}^- + e_{\text{any}}^+ \rightarrow \mu_R^- + \mu_R^+)}{d\Omega_{\text{c.m.}}} = 0. \end{aligned} \quad (10)$$

while

$$\begin{aligned} \frac{d\sigma(e_L^- + e_R^+ \rightarrow \mu_L^- + \mu_R^+)}{d\Omega_{\text{c.m.}}} &= \frac{d\sigma(e_R^- + e_L^+ \rightarrow \mu_R^- + \mu_L^+)}{d\Omega_{\text{c.m.}}} \\ &= \frac{\alpha^2}{4s} \times |1 + F(s)|^2 \times (1 + \cos\theta)^2, \\ \frac{d\sigma(e_L^- + e_R^+ \rightarrow \mu_R^- + \mu_L^+)}{d\Omega_{\text{c.m.}}} &= \frac{d\sigma(e_R^- + e_L^+ \rightarrow \mu_L^- + \mu_R^+)}{d\Omega_{\text{c.m.}}} \\ &= \frac{\alpha^2}{4s} \times |1 - F(s)|^2 \times (1 - \cos\theta)^2, \end{aligned} \quad (11)$$

where the $|1 \pm F(s)|^2$ factor stem from the interference between the virtual-photon and virtual- Z diagrams.

- (e) Finally, assume un-polarized electron and positron beams and a spin-blind muon detector. Calculate the total cross section $\sigma(e^+e^- \rightarrow \mu^+\mu^-)$ and the forward-backward asymmetry

$$A = \frac{\sigma(\theta < \pi/2) - \sigma(\theta > \pi/2)}{\sigma(\theta < \pi/2) + \sigma(\theta > \pi/2)} \quad (12)$$

as functions of the total energy $E_{\text{c.m.}}$.

Note: In QED, the tree-level pair production is symmetric with respect to $\theta \rightarrow \pi - \theta$; the asymmetry in the Standard Model arises from the interference between the virtual-photon and virtual- Z diagrams.

3. Finally, another Standard Model problem. While most weak decays of quarks or leptons involve a virtual W^+ or W^- gauge boson, the top quark is so heavy that it decays into a real (*i.e.*, on-shell) W^+ and the bottom quark. The interactions relevant to this process are contained in

$$\mathcal{L} \supset -\frac{g}{2\sqrt{2}} W_\mu^- \bar{\Psi}_b \gamma^\mu (1 - \gamma^5) \Psi_t - \frac{g}{2\sqrt{2}} W_\mu^+ \bar{\Psi}_t \gamma^\mu (1 - \gamma^5) \Psi_b. \quad (13)$$

For your information, $\alpha_w \equiv (g^2/4\pi) \approx 1/30$, $M_t \approx 173$ GeV, $M_W \approx 80.5$ GeV, and $M_b \approx 4.5$ GeV.

- (a) Write down the Feynman vertices for the interactions (13), draw tree diagram(s) for the $t \rightarrow W^+ + b$ decay, and write down the tree-level decay amplitude.
- (b) This amplitude does *not* satisfy the Ward identity. Write down a simple formula for $k_W^\mu \times \mathcal{M}_\mu$.
- (c) The W^+ gauge boson is a massive particle, so it has 3 distinct spin/polarization states. Show that its polarization vectors $\mathcal{E}^\mu(k, \lambda)$ satisfy

$$\sum_\lambda \mathcal{E}^\mu(k, \lambda) \times \mathcal{E}^{*\nu}(k, \lambda) = -g^{\mu\nu} + \frac{k_W^\mu k_W^\nu}{M_W^2}, \quad (14)$$

and consequently, for the W emission amplitude of the form $\mathcal{M} = \mathcal{M}_\mu \times \mathcal{E}^\mu(k, \lambda)$, summing $|\mathcal{M}|^2$ over the W polarizations yields

$$\sum_\lambda |\mathcal{M}|^2 = -\mathcal{M}^\mu \mathcal{M}_\mu^* + \frac{|\mathcal{M}_\mu k_W^\mu|^2}{M_W^2}. \quad (15)$$

- (d) Going back to the top quark decay $t \rightarrow b + W^+$, sum the $|\mathcal{M}|^2$ over both final particles' spins, average over the initial top quark's spin, and calculate the decay rate.

For simplicity, you may neglect the bottom quark's mass compared to masses of the top quark and of the W boson.