Phase Space Factors

For quantum transitions to un-bound states — for example, an atom emitting a photon, or a radioactive decay, or scattering — which is a kind of unbound $\to$ unbound transition, — the transition rate is given by the Fermi's golden rule:

$$\Gamma \stackrel{\text{def}}{=} \frac{d\,\text{probability}}{d\,\text{time}} = \frac{2\pi\rho}{\hbar} \times \left|\langle\text{final}|\,\hat{T}\,|\text{initial}\rangle\right|^2 \tag{1}$$

where $\hat{T} = \hat{H}_{\text{perturbation}} +$ higher order corrections, and ρ is the *density of final states,*

$$\rho = \frac{dN_{\text{final states}}}{dE_{\text{final}}}\,. \tag{2}$$

For an example, consider an atom emitting a photon of specific polarization λ. Using the large-box normalization for the photon's states, we have

$$dN_{\text{final}} = \left(\frac{L}{2\pi}\right)^3 d^3\mathbf{k}_\gamma = \frac{L^3}{(2\pi)^3} \times k_\gamma^2\,dk_\gamma\,d^2\Omega_\gamma \tag{3}$$

while $dE_{\text{final}} = dE_\gamma = \hbar c \times dk_\gamma$, hence

$$\rho = L^3 \times \frac{k_\gamma^2}{(2\pi)^3\hbar c} \times d^2\Omega_\gamma\,. \tag{4}$$

The L^3 factor here cancels against the $L^{-3/2}$ factor in the matrix element $\langle\text{atom}' + \gamma|\,\hat{T}\,|\text{atom}\rangle$ due to the photon's wave function in the large-box normalization. As to the remaining $d^2\Omega_\gamma$ factor, we should integrate over it to get the total decay rate, or divide by it to get the partial emission rate $d\Gamma/d\Omega$ for the photons emitted in a particular direction, thus

$$\begin{aligned} \frac{d\Gamma(\lambda)}{d\Omega} &= \frac{k^2}{(2\pi)^3\hbar c} \times L^3 \left|\big\langle\text{atom}' + \gamma(\mathbf{k},\lambda)\big|\,\hat{T}\,|\text{atom}\rangle\right|^2, \\ \Gamma_{\text{total}} &= \int\!d\Omega \sum_\lambda \frac{k^2}{(2\pi)^3\hbar c} \times L^3 \left|\big\langle\text{atom}' + \gamma(\mathbf{k},\lambda)\big|\,\hat{T}\,|\text{atom}\rangle\right|^2. \end{aligned} \tag{5}$$

In relativistic normalization of quantum states and matrix elements, there are no $L^{-3/2}$ factors but instead there are $\sqrt{2E}$ factors for each final-state or initial state particle, and they must be compensated by dividing the density of states ρ by the $\prod_i(2E_i)$. Also, we must allow for motion of all the final-state particles (*i.e.*, both the photon and the recoiled atom) but impose the momentum conservation as a constraint. Thus, for a decay of 1 initial particle into n final particles,

$$\Gamma = \frac{1}{2E_{\rm in}} \int\frac{d^3\mathbf{p}'_1}{(2\pi)^3\, 2E'_1} \cdots \int\frac{d^3\mathbf{p}'_n}{(2\pi)^3\, 2E'_n}\, \left|\langle p'_1,\ldots,p'_n|\,\mathcal{M}\,|p_{\rm in}\rangle\right|^2 \times (2\pi^4)\delta^{(4)}(p'_1+\cdots+p'_n-p_{\rm in}), \tag{6}$$

where the δ function takes care of both momentum conservation and of the denominator dE_f in the density-of-states factor (2). Likewise, the transition rate for a generic $2\to n$ scattering process is given by

$$\begin{aligned}\Gamma = \frac{1}{2E_1\times 2E_2} \int\frac{d^3\mathbf{p}'_1}{(2\pi)^3\, 2E'_1} \cdots \int\frac{d^3\mathbf{p}'_n}{(2\pi)^3\, 2E'_n}\, \left|\langle p'_1,\ldots,p'_n|\,\mathcal{M}\,|p_1,p_2\rangle\right|^2 \times \\ \times\,(2\pi^4)\delta^{(4)}(p'_1+\cdots+p'_n-p_1-p_2).\end{aligned} \tag{7}$$

In terms of the scattering cross-section σ, the rate (7) is $\Gamma=\sigma\times$ flux of initial particles. In the large-box normalization the flux is $L^{-3}|\mathbf{v}_1-\mathbf{v}_2|$, so in the continuum normalization it's simply the relative speed $|\mathbf{v}_1-\mathbf{v}_2|$. Consequently, the total scattering cross-section is given by

$$\begin{aligned}\sigma_{\rm tot} = \frac{1}{4E_1E_2|\mathbf{v}_1-\mathbf{v}_2|} \int\frac{d^3\mathbf{p}'_1}{(2\pi)^3\, 2E'_1} \cdots \int\frac{d^3\mathbf{p}'_n}{(2\pi)^3\, 2E'_n}\, \left|\langle p'_1,\ldots,p'_n|\,\mathcal{M}\,|p_1,p_2\rangle\right|^2 \times \\ \times\,(2\pi^4)\delta^{(4)}(p'_1+\cdots+p'_n-p_1-p_2).\end{aligned} \tag{8}$$

In particle physics, all the factors in eqs (6) or (8) besides the matrix elements — as well as the integrals over such factors — are collectively called the *phase space* factors.

A note on Lorentz invariance of decay rates or cross-sections. The matrix elements $\langle\text{final}|\,\mathcal{M}\,|\text{initial}\rangle$ are Lorentz invariant, and so are all the integrals over the final-particles' momenta and the δ-functions. The only non-invariant factor in the decay-rate formula (6) is the pre-integral $1/E_{\rm init}$, hence the decay rate of a moving particle is

$$\Gamma(\text{moving}) = \Gamma(\text{rest frame}) \times \frac{M}{E} \tag{9}$$

where M/E is precisely the time dilation factor in the moving frame.

As to the scattering cross-section, it should be invariant under Lorentz boosts along the initial axis of scattering, thus the same cross-section in any frame where $\mathbf{p}_1 \parallel \mathbf{p}_2$. This includes the *lab frame* where one of the two particles is initially at rest, the *center-of-mass frame* where $\mathbf{p}_1 + \mathbf{p}_2 = 0$, and any other frame where the two particles collide head-on. And indeed, the pre-integral factor in eq. (8) for the cross-section

$$\frac{1}{4E_1E_2|\mathbf{v}_1 - \mathbf{v}_2|} = \frac{1}{4|E_1\mathbf{p}_2 - E_2\mathbf{p}_1|} \tag{10}$$

is invariant under Lorentz boosts along the scattering axis.

Let's simplify eq. (8) for a 2 particle $\to$ 2 particle scattering process in the center-of-mass frame where $\mathbf{p}_1 + \mathbf{p}_2 = 0$. In this frame, the pre-integral factor (10) becomes

$$\frac{1}{4|\mathbf{p}| \times (E_1 + E_2)} \tag{11}$$

while the remaining phase space factors amount to

$$\begin{aligned}
\mathcal{P}_{\text{int}} &= \int\frac{d^3\mathbf{p}_1'}{(2\pi)^3\,2E_1'}\int\frac{d^3\mathbf{p}_2'}{(2\pi)^3\,2E_2'}\,(2\pi)^4\delta^{(3)}(\mathbf{p}_1' + \mathbf{p}_2')\delta(E_1' + E_2' - E_{\text{net}}) \\
&= \int\frac{d^3\mathbf{p}_1'}{(2\pi)^3 \times 2E_1' \times 2E_2'}\,(2\pi)\delta(E_1'(\mathbf{p}_1') + E_2'(-\mathbf{p}_1') - E_{\text{net}}) \\
&= \int d^2\Omega_{\mathbf{p}'} \times \int_0^\infty dp'\,\frac{p'^2}{16\pi^2 E_1'E_2'} \times \delta(E_1' + E_2' - E_{\text{tot}}) \\
&= \int d^2\Omega_{\mathbf{p}'}\left[\frac{p'^2}{16\pi^2 E_1'E_2'}\Bigg/\frac{d(E_1' + E_2')}{dp'}\right]^{\text{when}}_{E_1'+E_2'=E_{\text{tot}}}.
\end{aligned} \tag{12}$$

On the last 3 lines here $E_1' = E_1'(\mathbf{p}_1') = \sqrt{p'^2 + m_1'^2}$ while $E_2' = E_2'(\mathbf{p}_2' = -\mathbf{p}_1') = \sqrt{p'^2 + m_2'^2}$. Consequently,

$$\frac{dE_1'}{dp'} = \frac{p'}{E_1'}\,, \quad \frac{dE_2'}{dp'} = \frac{p'}{E_2'}\,, \tag{13}$$

hence

$$\frac{d(E_1' + E_2')}{dp'} = \frac{p'}{E_1'} + \frac{p'}{E_2'} = \frac{p'}{E_1'E_2'} \times (E_2' + E_1' = E_{\text{tot}}), \tag{14}$$

and therefore

$$\mathcal{P}_{\rm int} \;=\; \frac{1}{16\pi^2}\times\frac{p'}{E_{\rm tot}}\times\int\!d^2\Omega_{\mathbf{p}'}\,. \tag{15}$$

Including the pre-integral factor (11), we arrive at the net phase space factor

$$\mathcal{P} \;=\; \frac{p'}{p}\times\frac{1}{64\pi^2E_{\rm tot}^2}\times\int\!d^2\Omega_{\mathbf{p}'}\,. \tag{16}$$

The matrix element $\mathcal{M}$ for the scattering should be put inside the direction-angle integral in this phase-space formula. Thus, the total scattering cross-section is

$$\sigma_{\rm tot}(1+2\to1'+2') \;=\; \frac{p'}{p}\times\frac{1}{64\pi^2E_{\rm cm}^2}\times\int\!d^2\Omega\,\left|\langle p_1'+p_2'|\,\mathcal{M}\,|p_1+p_2\rangle\right|^2, \tag{17}$$

while the partial cross-section for scattering in a particular direction is

$$\frac{d\sigma(1+2\to1'+2')}{d\Omega_{\rm cm}} \;=\; \frac{p'}{p}\times\frac{1}{64\pi^2E_{\rm cm}^2}\times\left|\langle p_1'+p_2'|\,\mathcal{M}\,|p_1+p_2\rangle\right|^2. \tag{18}$$

Note: the total cross-section is the same in frames where the initial momenta are collinear, but in the partial cross-section, $d\Omega$ depends on the frame of reference, so eq. (18) applies only in the center-of mass frame. Also, the $E_{\rm cm}$ factor in denominators of both formulae stands for the net energy in the center-of-mass frame. In frame-independent terms,

$$E_{\rm cm}^2 \;=\; (p_1+p_2)^2 \;=\; (p_1'+p_2')^2 \;=\; s. \tag{19}$$

Finally, let me write down the phase-space factor for a 2-body decay (1 particle $\to$ 2 particles) in the rest frame of the initial particle. The under-the-integral factors for such a decay are the same as in eq. (15) for a $2\to2$ scattering, but the pre-integral factor is $1/2M_{\rm init}$ instead of the (11), thus

$$\mathcal{P} \;=\; \frac{p'}{32\pi^2M^2}\,, \tag{20}$$

meaning

$$\frac{d\Gamma(0\to1'+2')}{d\Omega} \;=\; \frac{p'}{32\pi^2M^2}\times\left|\langle p_1'+p_2'|\,\mathcal{M}\,|p_0\rangle\right|^2, \tag{21}$$

$$\Gamma(0\to1'+2') \;=\; \frac{p'}{32\pi^2M^2}\times\int\!d^2\Omega\,\left|\langle p_1'+p_2'|\,\mathcal{M}\,|p_0\rangle\right|^2. \tag{22}$$