QCD Beta Function

In these notes I shall calculate the one-loop $\beta(g)$ functions for QCD and other non-abelian gauge theories. To keep my formulae generic, I allow for any simple gauge group G — although I call the gauge bosons 'the gluons' — with Dirac fermions — which I call 'the quarks' in some multiplet (Q) of G.

As I explained in class, the non-abelian gauge theories do not obey the QED-like Ward identity $Z_1 = Z_2$. Instead, they have weaker identities

$$\frac{Z_1^{(q)}}{Z_2^{(q)}} = \frac{Z_1^{(gh)}}{Z_2^{(gh)}} = \frac{Z_1^{(3g)}}{Z_3} = \left(\frac{Z_1^{4g}}{Z_3}\right)^{1/2} = \frac{g\sqrt{Z_3}}{g_{\text{bare}}}, \tag{1}$$

which nevertheless assure the universality of renormalized gauge coupling g. That is, we have the same g in all couplings of the non-abelian gauge theory: the quark-gluon coupling, the ghost-gluon coupling, the 3-gluon coupling, and the square of the same g in the 4-gluon coupling. In the MS renormalization schemes, the identities (1) translate to linear relations between the counterterms, or rather the simple $1/\epsilon$ poles of the counterterm. Let $\operatorname{Res}[\delta]$ stand for the residue of the pole at $\epsilon \to 0$ of the counterterm δ, that is, the coefficient of the simple $1/\epsilon$ pole regardless of the higher-order poles $1/\epsilon^2$, $1/\epsilon^3$, *etc.* Then eqs. (1) translate to

$$\operatorname{Res}\big[\delta_1^{(q)} - \delta_2^{(q)}\big] = \operatorname{Res}\big[\delta_1^{(gh)} - \delta_2^{(gh)}\big] = \operatorname{Res}\big[\delta_1^{(3g)} - \delta_3\big] = \tfrac{1}{2}\operatorname{Res}\big[\delta_1^{(4g)} - \delta_3\big]. \tag{2}$$

Moreover, each one of these differences in combinations with the δ_3 counterterm may be used to calculate the β function of the gauge theory:

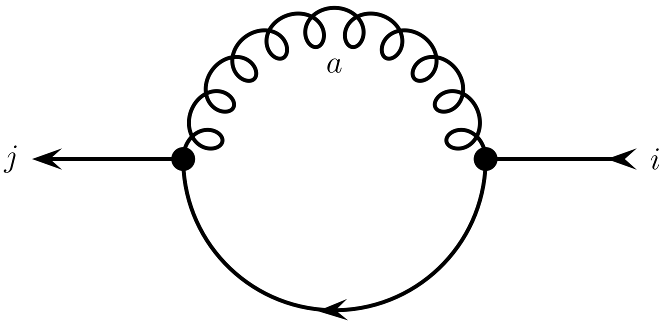
$$\begin{aligned}
\frac{dg(\mu)}{d\log\mu} = \beta(g) &= g\hat{L}\operatorname{Res}\big[2\delta_1^{(q)} - 2\delta_2^{(q)} - \delta_3\big] && (3.1)\\
&= g\hat{L}\operatorname{Res}\big[2\delta_1^{(gh)} - 2\delta_2^{(gh)} - \delta_3\big] && (3.2)\\
&= g\hat{L}\operatorname{Res}\big[2\delta_1^{(3g)} - 3\delta_3\big] && (3.4)\\
&= \frac{g}{2}\hat{L}\operatorname{Res}\big[2\delta_1^{(4g)} - 4\delta_3\big], && (3.4)
\end{aligned}$$

where $\hat{L} = g^2(\partial/\partial g^2)$ is the number-of-loops operator.

In these notes, I shall calculate the $\delta_2^{(q)}$, the $\delta_1^{(q)}$, and the δ_3 counterterms to the one-loop order, and then use eq. (3.1) to calculate the one-loop beta-function.

The $\delta_2^{(q)}$ Counterterm

At the one-loop level, the quark propagator renormalization comes from a single diagram



$$= \left[-i\Sigma(\not{p})\right]^j{}_i = \text{hopefully} = -i\Sigma(\not{p}) \times \delta^j_i \tag{4}$$

which evaluates to

$$\int\!\frac{d^4k}{(2\pi)^4}\,(-ig\gamma_\mu)\,\frac{i}{\not{p}+\not{k}-m+i0}\,(-ig\gamma_\nu)\times\frac{-ig^{\mu\nu}}{k^2+i0}\times\left(t^at^a\right)^j{}_i\,. \tag{5}$$

(For simplicity, I use the Feynman gauge with $\xi = 1$ for the gluon's propagator.) Apart from the group-theoretical factor $\left(t^at^a\right)^j{}_i$, the Dirac indexology and the momentum integral here are exactly as in the electron's propagator correction in QED, so instead of recalculating it from scratch, let me simply copy it from the solutions to homework#17 (problem 3). Focusing on the UV divergence of the integral and disregarding the gory details of the finite part, we have

$$\begin{aligned}
-i\Sigma_0(\not{p}) &= \int\!\frac{d^4k}{(2\pi)^4}\,(-ig\gamma_\mu)\,\frac{i}{\not{p}+\not{k}-m+i0}\,(-ig\gamma_\nu)\times\frac{-ig^{\mu\nu}}{k^2+i0}\\
&= -i\frac{g^2}{16\pi^2}\left((-\not{p}+4m)\times\frac{1}{\epsilon}\;+\;\text{finite}\right).
\end{aligned} \tag{6}$$

Now consider the group theoretical factor. The t^a matrices in the quark-gluon vertices represent the gauge group generators $\hat{T}^a$ in the quark multiplet (Q), so the matrix combination $\sum_a t_at_a$ represent the Casimir operator $\hat{C}_2 = \sum_s \hat{T}^a\hat{T}^a$. In any irreducible multiplet (Q), this Casimir operators becomes a multiplet-dependent number $C(Q)$ times a unit matrix, thus

$$\left(t^a t^a\right)^j{}_i \;=\; C(Q)\times\delta^j_i\,. \tag{7}$$

In QCD, the gauge group is $SU(3)^{\text{color}}$, the quarks belong to the fundamental $\mathbf{3}$ multiplets (one such multiple for each quark flavor), the Casimir of this multiple is $C(\mathbf{3}) = \frac{4}{3}$, and we may just as well plug in this number into eq. (7) and subsequent formulae. However, in order to generalize from QCD to other gauge theories — with other gauge groups, and with fermions belonging to other kinds of multiplets — I am going to keep a generic $C(Q)$ in these notes.

Altogether, the quark's propagator correction is

$$\left[\Sigma(\not{p})\right]^j{}_i \;=\; \delta^j_i\times C(Q)\times\Sigma_0(\not{p}) \;=\; \delta^j_i\times\frac{g^2C(Q)}{16\pi^2}\times\left((-\not{p}+4m)\times\frac{1}{\epsilon}\;+\;\text{finite}\right), \tag{8}$$

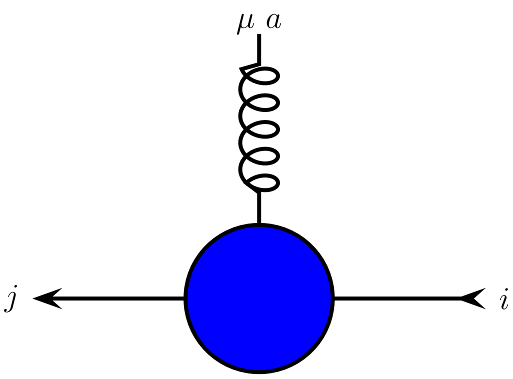
and to cancel the UV divergence here, we need the counterterms

$$\delta_2^{(q)} \;=\; -\frac{g^2C(Q)}{16\pi^2}\times\frac{1}{\epsilon} \quad\text{and}\quad \delta_m^{(q)} \;=\; -m\times\frac{g^2C(Q)}{4\pi^2}\times\frac{1}{\epsilon}\,. \tag{9}$$

The $\delta_m^{(q)}$ counterterm depends on the quark's mass m, so it's different for different quark flavors, but the $\delta_2^{(q)}$ counterterm is the same for all flavors, at least in the MS renormalization scheme.

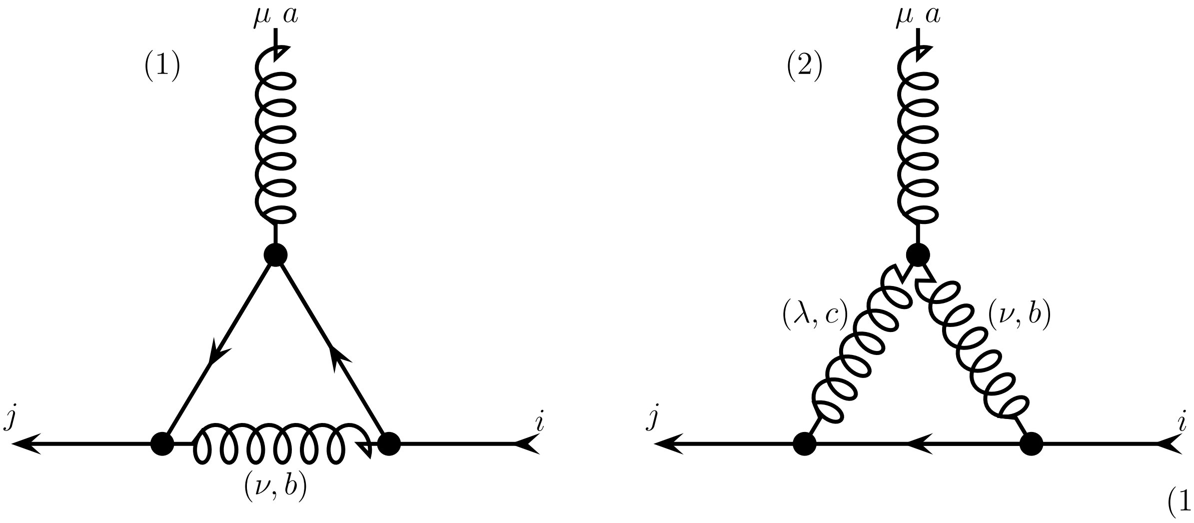
The $\delta_1^{(q)}$ counterterm

The $\delta_1^{(q)}$ counterterm cancels the momentum-independent UV divergence of the quark-antiquark-gluon vertex



$$= \;\left[-ig\Gamma^{\mu,a}\right]^j{}_i \;=\text{hopefully}=\; -ig\Gamma^\mu\times\left(t^a\right)^j{}_i\,. \tag{10}$$

At the one-loop level, there are two diagrams correcting this vertex, namely



(11)

The left diagram here looks just like the electron-photon vertex correction in QED, but the right diagram is new — it appears only in the non-abelian gauge theories like QCD.

Since we are interested only in the UV divergences of the diagrams, and by power-counting such divergences should be independent, let's simplify the calculations by setting all the external momenta to zero. This also avoids the IR divergences — since the zero quark momenta are off-shell — and lets us use the same loop momentum k^μ for all the propagators in the loop. Thus, for the left diagram (11.a) we have

$$\begin{aligned}\left[-ig\Gamma_1^{\mu,a}(0,0)\right]^j_{\ i} \; = \; & \int\frac{d^4k}{(2\pi)^4}\,(-ig\gamma^\nu)\,\frac{i}{\not{k}-m+i0}\,(-ig\gamma^\mu)\,\frac{i}{\not{k}-m+i0}\,(-ig\gamma_\nu)\times\frac{-i}{k^2+i0} \\ & \times\left(t^bt^at^b\right)^j_{\ i}.\end{aligned} \tag{12}$$

The Dirac indexology and the momentum integral on the top line looks exactly like its QED analogue, which was discussed in painful detail in my notes on the QED vertex corrections. For $p'=p=0$, the algebra becomes much simpler:

$$\Gamma^\mu_{\mathrm{QED}}(p'=p=0) \; = \; -ig^2\int\frac{d^4k}{(2\pi)^4}\,\frac{\mathcal{N}^\mu}{(k^2-m^2+i0)^2(k^2+i0)} \tag{13}$$

where

$$
\begin{aligned}
\mathcal{N}^\mu &= \gamma^\nu(\not{k}+m)\gamma^\mu(\not{k}+m)\gamma_\nu \\
&= -2\not{k}\gamma^\mu\not{k} \;+\; 8mk^\mu \;-2m^2\gamma^\mu \;+\; O(\epsilon) \\
&\quad \langle\!\langle\, \text{after averaging over the directions of } k^\nu \,\rangle\!\rangle \\
&\cong -\frac{2k^2}{D}\times\gamma^\nu\gamma^\mu\gamma_\nu \;+\; 0 \;-2m^2\gamma^\mu \;+\; O(\epsilon) \\
&= (k^2-2m^2)\times\gamma^\mu \;+\; O(\epsilon).
\end{aligned}
\tag{14}
$$

Consequently,

$$
\begin{aligned}
\Gamma^\mu_{\text{QED}}(p'=p=0) &= g^2\gamma^\mu\times\int\!\frac{d^4k}{(2\pi)^4}\,\frac{-i(k^2-2m^2)\;+\;O(\epsilon)}{(k^2-m^2+i0)^2(k^2+i0)} \\
&= g^2\gamma^\mu\times\int\!\frac{d^4k_E}{(2\pi)^4}\,\frac{(k_E^2+2m^2)\;+\;O(\epsilon)}{(k_E^2+m^2)^2\times k_E^2} \\
&= \frac{g^2\gamma^\mu}{16\pi^2}\left(\frac{1}{\epsilon}\;+\;\text{a finite constant}\right).
\end{aligned}
\tag{15}
$$

For the finite p and p', we would get the same UV divergence but a much more complicated finite part (*cf.* my notes for the gory details),

$$
\Gamma^\mu_{\text{QED}}(p',p) \;=\; \frac{g^2}{16\pi^2}\left(\frac{\gamma^\mu}{\epsilon}\;+\;\text{finite\_function\_of}(p',p)\right).
\tag{16}
$$

Now consider the group-theoretical factor $\left(t^bt^at^b\right)^j{}_i$ (implicit sum over b) on the second line of eq. (12). First, using $t^at^b=t^bt^a+[t^a,t^b]=t^bt^a+if^{abc}t^c$, we get

$$
t^bt^at^b \;=\; t^b\times t^at^b \;=\; t^bt^b\times t^a \;+\; t^b\times if^{abc}t^c = \;C(Q)\times t^a \;+\; if^{abc}t^bt^c.
\tag{17}
$$

Next, in the second term on the RHS we use the antisymmetry $f^{abc}=-f^{acb}$ to rewrite

$$
if^{abc}t^bt^c \;=\; if^{abc}\times\tfrac{1}{2}[t^b,t^c] \;=\; if^{abc}\times\tfrac{1}{2}\times if^{bcd}t^d \;=\; -\tfrac{1}{2}\big(if^{abc}\times -if^{bcd}\big)\times t^d.
\tag{18}
$$

Finally, we relate the structure constants f^{abc} of the Lie algebra to the matrices representing the generators in the adjoint multiplet, $\big(T^a_{\text{adj}}\big)^{bc}=if^{abc}$ and $\big(T^d_{\text{adj}}\big)^{cb}=if^{dcb}=-if^{bcd}$.

Consequently,

$$\sum_{b,c} if^{abc}\times -if^{bcd} \;=\; +\sum_{b,c}\bigl(T^a_{\rm adj}\bigr)^{bc}\bigl(T^d_{\rm adj}\bigr)^{cb} \;=\; {\rm tr}\bigl(T^a_{\rm adj}T^d_{\rm adj}\bigr) \;\equiv\; {\rm tr}_{\rm adj}\bigl(T^aT^b\bigr) \;=\; R({\rm adj})\times\delta^{ad} \tag{19}$$

where $R(\mathrm{adj})$ is the index of the adjoint multiplet. As you should see in the homework set#22 (problem 1), for the adjoint multiplet the index is equal to the Casimir, $R(\mathrm{adj}) = C(\mathrm{adj})$; it is commonly denoted $C(G)$ where G stands for the gauge group itself. Thus,

$$if^{abc}t^bt^c \;=\; -\tfrac{1}{2}C(G)\times t^a \tag{20}$$

and therefore

$$\sum_b t^bt^at^b \;=\; \bigl(C(Q)-\tfrac{1}{2}C(G)\bigr)\times t^a. \tag{21}$$

Altogether, the first diagram (11.1) yields

$$\begin{aligned} \bigl[-ig\Gamma_1^{\mu,a}(p',p)\bigr]^j_{\ i} \;&=\; -ig\Gamma_1^\mu(p',p)\times(t^a)^j_{\ i} \\ \text{for}\quad \Gamma_1^\mu(p',p) \;&=\; +\bigl(C(Q)-\tfrac{1}{2}C(G)\bigr)\times\frac{g^2}{16\pi^2}\times\frac{\gamma^\mu}{\epsilon}\;+\;\text{finite}. \end{aligned} \tag{22}$$

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Now consider the second diagram (11.2) with two gluon propagators and one three-gluon vertex. At zero external momenta, this diagram evaluates to

$$\begin{aligned} \bigl[-ig\Gamma_2^{\mu,a}(0,0)\bigr]^j_{\ i} \;=\; \int\!\frac{d^4k}{(2\pi)^4}\,(-ig\gamma_\lambda)\frac{i}{\not k-m+i0}(-ig\gamma_\nu)\times\left(\frac{-i}{k^2+i0}\right)^2\times \\ \times\bigl[g^{\lambda\mu}k^\nu\,+\,g^{\mu\nu}k^\lambda\,-\,2g^{\nu\lambda}k^\mu\bigr] \\ \times\bigl(t^bt^c\bigr)^j_{\ i}\times\bigl(-gf^{cab}\bigr). \end{aligned} \tag{23}$$

The second line here (and also the $-gf^{cab}$ factor on the third line) stems from the three-gluon vertex. Note that if we treat all gluons' momenta as flowing in to the vertex, then the left gluon has momentum $k_1=+k$, the right gluon has momentum $k_3=-k$, and the top gluon has $k_2=0$, hence

$$\bigl[g^{\lambda\mu}(k_1-k_2)^\nu\,+\,g^{\mu\nu}(k_2-k_3)^\lambda\,+\,g^{\nu\lambda}(k_3-k_1)^\mu\bigr] \;=\; \bigl[g^{\lambda\mu}k^\nu\,+\,g^{\mu\nu}k^\lambda\,-\,2g^{\nu\lambda}k^\mu\bigr] \tag{24}$$

on the second line of eq. (23).

The group factor on the third line of eq. (23) is an expression we have already evaluated. Indeed, multiplying both sides of eq. (20) by ig, we have

$$-g\big(f^{cab}=f^{abc}\big)\times\big(t^bt^c\big)^j_{\ i}=\ -ig\times\frac{C(G)}{2}\times\big(t^a\big)^j_{\ i}\,. \tag{25}$$

Consequently,

$$\big[-ig\Gamma_2^{\mu,a}\big]^j_{\ i}\ =\ -ig\Gamma^\mu\times(t^a)^j_{\ i} \tag{26}$$

for

$$\Gamma_2^\mu(0,0)\ =\ \frac{ig^2C(G)}{2}\times\int\!\frac{d^4k}{(2\pi)^4}\,\frac{\gamma_\lambda(\not{k}+m)\gamma_\nu}{(k^2-m^2+i0)(k^2+i0)^2}\times\big[k^\lambda g^{\mu\nu}+k^\nu g^{\mu\lambda}-2k^\mu g^{\nu\lambda}\big]. \tag{27}$$

(The color-index dependence (26) is valid for all external momenta, but the integral (27) is specialized for $p'=p=0$ only.)

Combining all the factors in the numerator inside the integral (27), we obtain

$$\begin{aligned}
\mathcal{N}^\mu\ &=\ \gamma_\lambda(\not{k}+m)\gamma_\nu\times\big[k^\lambda g^{\mu\nu}\ +\ k^\nu g^{\mu\lambda}\ -\ 2k^\mu g^{\nu\lambda}\big]\\
&=\not{k}(\not{k}+m)\gamma^\mu\ +\ \gamma^\mu(\not{k}+m)\not{k}\ -\ 2k^\mu\times\gamma^\nu(\not{k}+m)\gamma_\nu\\
&=\ 2k^2\times\gamma^\mu\ +\ 2mk^\mu\ +\ 2k^\mu\times\big((D-2)\not{k}\ -\ Dm\big)\\
&=\ (2k^2g^{\mu\nu}\ +\ 2(D-2)k^\mu k^\nu)\times\gamma_\nu\ -\ 2(D-1)mk^\mu\\
&\qquad\langle\!\langle\text{ after averaging over the directions of }k^\nu\,\rangle\!\rangle\\
&\cong\ \left(2k^2\ +\ \frac{2(D-2)}{D}\,k^2\right)g^{\mu\nu}\times\gamma_\nu\\
&=\ (3+O(\epsilon))\times k^2\times\gamma^\mu,
\end{aligned} \tag{28}$$

and therefore

$$\begin{aligned}
\Gamma_2^\mu(0,0)\ &=\ \frac{3+O(\epsilon)}{2}\times g^2C(G)\times\gamma^\mu\times -i\int\!\frac{d^4k}{(2\pi)^4}\,\frac{k^2}{(k^2-m^2+i0)(k^2+i0)^2}\\
&=\ \frac{3+O(\epsilon)}{2}\times g^2C(G)\times\gamma^\mu\times\int\!\frac{d^4k_E}{(2\pi)^4}\,\frac{k_E^2}{(k_E^2+m^2)\times(k_E^2)^2}\\
&=\ \frac{3+O(\epsilon)}{2}\times g^2C(G)\times\gamma^\mu\times\left(\frac{1}{16\pi^2}\times\frac{1}{\epsilon}\ +\ \text{finite}\right)\\
&=\ \frac{g^2}{16\pi^2}\times\frac{3C(G)}{2}\times\frac{\gamma^\mu}{\epsilon}\ +\ \text{finite}.
\end{aligned} \tag{29}$$

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Combining the two diagrams' (11) contribution, we obtain

$$\left[-ig\Gamma^{\mu,a}_{1+2}(p',p)\right]^j_{\ i} \;=\; -ig\Gamma^{\mu}_{1+2}(p',p)\times(t^a)^j_{\ i}, \tag{30}$$

where at zero momenta $p'=p=0$ we have

$$\Gamma^{\mu}_{1+2}(0,0) \;=\; \gamma^\mu\times\frac{g^2}{16\pi^2}\times\left(\left(C(Q)-\tfrac{1}{2}C(G)+\tfrac{3}{2}C(G)\right)\times\frac{1}{\epsilon}\,+\,\text{finite constant}\right), \tag{31}$$

while at finite momenta p and p' we expect the same UV divergence but a very different finite piece

$$\Gamma^{\mu}_{1+2}(p',p) \;=\; +\frac{g^2}{16\pi^2}\left(C(Q)+C(G)\right)\left(\frac{\gamma^\mu}{\epsilon}\,+\,\mathit{finite\_f}^\mu(p',p)\right). \tag{32}$$

To cancel the UV divergence here, we need the one-loop counterterm

$$\delta^{(q)}_1 \;=\; -\frac{g^2}{16\pi^2}\left(C(Q)+C(G)\right)\times\frac{1}{\epsilon}\,. \tag{33}$$

Note that in QCD — or in any other non-abelian gauge theory — the $\delta^{(q)}_1$ and the $\delta^{(q)}_2$ counterterms are **not** equal to each other — they have different group-theoretical factors: $C(Q)+C(G)$ for the $\delta^{(q)}_1$ versus $C(Q)$ for the $\delta^{(q)}_2$. However, in gauge theories where fermions belong to several *inequivalent* irreducible multiplets (r) of the gauge group G, the difference $\delta^{(r)}_1-\delta^{(r)}_2$ is universal:

$$\text{same }\delta^{(r)}_1\;-\;\delta^{(r)}_2\;=\;-\frac{g^2C(G)}{16\pi^2}\times\frac{1}{\epsilon}\quad\forall(r). \tag{34}$$

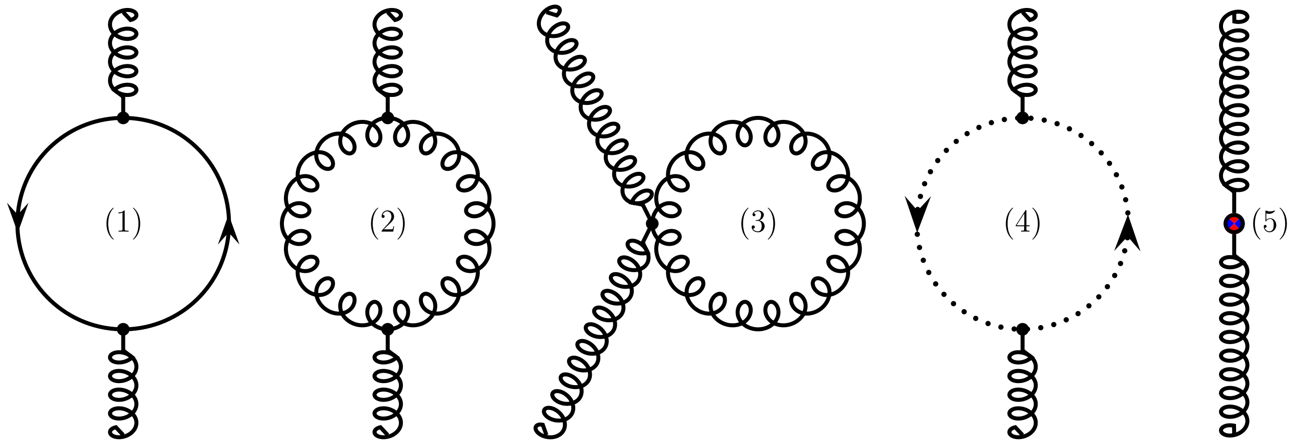
Even at the higher loop orders, the differences $\delta^{(r)}_1-\delta^{(r)}_2$ — or at least the $1/\epsilon$ parts of these differences — are the same for fermions in any multiplet (r) of the gauge group,

$$\text{same }\ \mathrm{Res}\left[\delta^{(r)}_1\;-\;\delta^{(r)}_2\right]\quad\forall(r). \tag{35}$$

This universality is the special case of the relations (2), which assure that all the coupling of the gauge theory have the same renormalized coupling $g(\mu)$.

The δ_3 Counterterm

At the one-loop order, the self-energy corrections to the gluons come from 5 diagrams:



where the fifth diagram's contribution

$$\left[\Sigma_5^{\mu\nu}(k)\right]^{ab} \;=\; -\delta_3 \times \left(k^2 g^{\mu\nu} \;-\; k^\mu k^\nu\right) \times \delta^{ab} \tag{37}$$

cancels the UV divergences of the first 4 diagrams. So let's calculate those divergences.

The first diagram — the quark loop — gives us

$$\begin{aligned}\left[i\Sigma_1^{\mu\nu}(k)\right]^{ab} \;&=\; \text{hopefully} \;=\; \delta^{ab} \times i\Sigma^{\mu\nu}(k)\\ &=\; -\int\!\frac{d^4p}{(2\pi)^4}\,\mathrm{tr}\left((-ig\gamma^\mu)\frac{i}{\not p - m + i0}\,(-ig\gamma^\nu)\frac{i}{\not p + \not k - m_i 0}\right) \times \mathrm{tr}\left(t^a_{(q)}\,t^b_{(q)}\right),\end{aligned} \tag{38}$$

where the first trace is over the Dirac indices while the second trace is over the quarks' colors and flavors. For a single quark multiplet (m) of the gauge group G

$$\mathrm{tr}\left(t^a_{(m)}\,t^b_{(m)}\right) \;=\; \delta^{ab} \times R(m) \tag{39}$$

where $R(m)$ is the *index* of the multiplet (m). For several quark multiplets, their contributions add up, thus

$$\mathrm{tr}\left(t^a_{(q)}\,t^b_{(q)}\right) \;=\; \delta^{ab} \times R_{\mathrm{net}} \;=\; \delta^{ab} \times \sum_{\substack{\text{quark}\\ \text{multiplets}}} R(\text{multiplet}). \tag{40}$$

In particular, in QCD the quarks comprise N_f copies of a fundamental $\mathbf{N}$ multiplet of the

$SU(N)$ gauge group — one fundamental multiplet for each flavor — hence

$$R_{\rm net} \;=\; N_f \times R(\text{fundamental}) \;=\; N_f \times \frac{1}{2}\,. \tag{41}$$

Apart from this group factor, the rest of the quark loop (38) looks exactly like the electron loop in QED. We have calculated that loop back in February — *cf.* my notes — so let me simply recycle the result in the present context:

$$\left[\Sigma_1^{\mu\nu}(k)\right]^{ab} \;=\; \delta^{ab} \times \left(k^2 g^{\mu\nu} \,-\, k^\mu k^\nu\right) \times \frac{-g^2}{12\pi^2} \times R_{\rm net} \;\times \left(\frac{1}{\epsilon} + \text{finite}\right). \tag{42}$$

Consequently, the counterterm needed to cancel this divergence is

$$\delta_3(1^{\rm st}) \;=\; -\frac{g^2}{16\pi^2} \times \frac{1}{\epsilon} \times \frac{4}{3}\, R_{\rm net} \tag{43}$$

for a general gauge theory; for QCD

$$\delta_3(1^{\rm st}) \;=\; -\frac{g^2}{16\pi^2} \times \frac{1}{\epsilon} \times \frac{2}{3}\, N_f\,. \tag{44}$$

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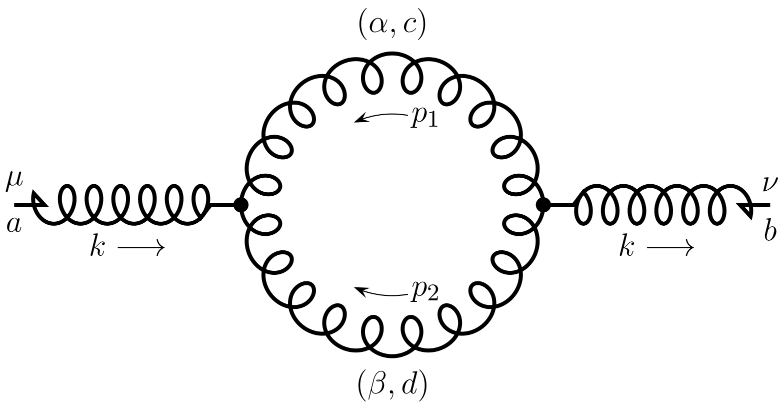
Now consider the second diagram — the gluon loop



(45)

Evaluating this diagram in the Feynman gauge, we get

$$\begin{aligned}\left[i\Sigma_2^{\mu\nu}(k)\right]^{ab} \;=\; \frac{1}{2}\int\frac{d^4p_1}{(2\pi)^4}\,\frac{-i}{p_1^2+i0} \times \frac{-i}{p_2^2+i0} &\times -gf^{acd}V^{\mu\alpha\beta}(k,p_1,p_2) \times \\ &\times -gf^{bcd}V^{\nu}{}_{\alpha\beta}(-k,-p_1,-p_2)\end{aligned} \tag{46}$$

where $\frac{1}{2}$ is the symmetry factor due to 2 similar gluon propagators, their momenta add to $p_1 + p_2 = -k$, and the V's are the momentum- and Lorentz-index-dependent parts of the

3–gluon vertices,

$$\begin{aligned} V^{\mu\alpha\beta}(k,p_1,p_2) &= g^{\alpha\beta}(p_1-p_2)^\mu + g^{\beta\mu}(p_2-k)^\alpha + g^{\mu\alpha}(k-p_1)^\beta, \\ V^{\nu\alpha\beta}(-k,-p_1,-p_2) &= -V^{\nu\alpha\beta}(k,p_1,p_2). \end{aligned} \tag{47}$$

Let's start with the group factor in eq. (46). As we saw a few pages above in eq. (19),

$$\sum_{bc} f^{acd}\times f^{bcd} = \sum_{bc}\bigl(-iT^a_{\rm adj}\bigr)^{cd}\times\bigl(+iT^b_{\rm adj}\bigr)^{dc} = +\,\mathrm{tr}\bigl(T^a_{\rm adj}T^b_{\rm adj}\bigr) = \delta^{ab}\times R(\text{adjoint}) \tag{48}$$

where

$$R(\text{adjoint}) = C(\text{adjoint}), \quad \text{often denoted } C(G); \tag{49}$$

for an $SU(N)$ gauge group, $C(G)=N$.

Plugging the group factor into eq. (46) and assembling all the constant factors, we obtain

$$\bigl[\Sigma_2^{\mu\nu}(k)\bigr]^{ab} = \delta^{ab}\times\frac{g^2}{2}\,C(G)\times\int\!\frac{d^4p_1}{(2\pi)^4}\,\frac{-i\mathcal{N}_2^{\mu\nu}}{(p_1^2+i0)\times(p_2^2+i0)} \tag{50}$$

where the numerator is

$$\begin{aligned} \mathcal{N}_2^{\mu\nu} &= -V^{\mu\alpha\beta}(k,p_1,p_2)\times V^\nu{}_{\alpha\beta}(-k,-p_1,-p_2) = +V^{\mu\alpha\beta}(k,p_1,p_2)\times V^\nu{}_{\alpha\beta}(k,p_1,p_2) \\ &= D\times(p_1-p_2)^\mu(p_1-p_2)^\nu + g^{\mu\nu}\times(p_2-k)^2 + g^{\mu\nu}\times(k-p_1)^2 \\ &\qquad + (p_1-p_2)^{(\mu}(p_2-k)^{\nu)} + (p_2-k)^{(\mu}(k-p_1)^{\nu)} + (k-p_1)^{(\mu}(p_1-p_2)^{\nu)} \end{aligned} \tag{51}$$

The second line here has form

$$A^{(\mu}B^{\nu)} + B^{(\mu}C^{\nu)} + C^{(\mu}A^{\nu)} = (A{+}B{+}C)^\mu(A{+}B{+}C)^\nu - A^\mu A^\nu - B^\mu B^\nu - C^\mu C^\nu; \tag{52}$$

moreover,

$$A+B+C = (p_1-p_2) + (p_2-k) + (k-p_1) = 0. \tag{53}$$

Consequently, the numerator (51) simplifies to

$$\begin{aligned} \mathcal{N}_2^{\mu\nu} &= D\times(p_1-p_2)^\mu(p_1-p_2)^\nu + g^{\mu\nu}\times(p_2-k)^2 + g^{\mu\nu}\times(k-p_1)^2 \\ &\qquad - (p_1-p_2)^\mu(p_1-p_2)^\nu - (p_2-k)^\mu(p_2-k)^\nu - (k-p_1)^\mu(k-p_1)^\nu \\ &= g^{\mu\nu}\times\bigl[(p_2-k)^2+(k-p_1)^2\bigr] + (D-1)\times(p_1-p_2)^\mu(p_1-p_2)^\nu \\ &\qquad - (p_2-k)^\mu(p_2-k)^\nu - (k-p_1)^\mu(k-p_1)^\nu. \end{aligned} \tag{54}$$

As usual, the first step in evaluating the momentum integral like (50) is to simplify the denominator using the Feynman parameters. By momentum conservation $p_2 \equiv -k - p_1$, hence

$$\frac{1}{(p_1^2+i0)(p_2^2+i0)} = \int\limits_0^1 \frac{dx}{\bigl[(1-x)p_1^2 + x(p_1+k)^2 + i0\bigr]^2} = \int\limits_0^1 \frac{dx}{[\ell^2 - \Delta + i0]^2} \tag{55}$$

where

$$\ell = p_1 + xk \quad \text{and} \quad \Delta = -x(1-x)k^2. \tag{56}$$

Plugging this denominator int eq. (50) we get

$$\Sigma_2^{\mu\nu}(k) = -i\,\frac{g^2}{2}\,C(G) \times \int\limits_0^1 dx \int \frac{d^4\ell}{(2\pi)^4}\,\frac{\mathcal{N}_2^{\mu\nu}}{[\ell^2-\Delta+i0]^2}\,, \tag{57}$$

and now we need to re-express the numerator in terms of the shifted momentum ℓ. Using $p_1 = +\ell - xk$ and $p_2 = -\ell - (1-x)k$, we obtain

$$p_1 - p_2 = 2\ell - (2x-1)k, \quad p_2 - k = -\ell + (x-2)k, \quad k - p_1 = -\ell + (x+1)k, \tag{58}$$

and hence

$$\begin{aligned}\mathcal{N}_2^{\mu\nu} &= g^{\mu\nu} \times \bigl[(-\ell + (x-2)k)^2 + (-\ell + (x+1)k)^2\bigr] \\ &\quad + (D-1) \times \bigl(2\ell - (2x-1)k\bigr)^\mu \bigl(2\ell - (2x-1)k\bigr)^\nu \\ &\quad - (-\ell + (x-2)k)^\mu(-\ell + (x-2)k)^\nu - (-\ell + (x+1)k)^\mu(-\ell + (x-2)k)^\nu.\end{aligned} \tag{59}$$

This whole big mess is a quadratic polynomial in ℓ and k, but the mixed terms like (ℓk) or $\ell^\mu k^\nu$ are odd with respect to $\ell \to -\ell$ and hence cancel out from the momentum integral (57). Thus, keeping only the terms carrying two or zero ℓ's, we arrive at

$$\mathcal{N}_2^{\mu\nu} = g^{\mu\nu} \times \bigl[2\ell^2 + A \times k^2\bigr] + B \times \ell^\mu \ell^\nu + C \times k^\mu k^\nu \tag{60}$$

where

$$A = (x-2)^2 + (x+1)^2 = 5 - 2x(1-x), \tag{61}$$

$$
\begin{aligned}
B &= (D-1)\times 4 \;-\; 1 \;-\; 1 \;=\; 4D \;-\; 6, && (62)\\
C &= (D-1)\times(2x-1)^2 \;-\; (x-2)^2 \;-\; (x+1)^2 \\
&= (D-6) \;-\; (4D-6)\times x(1-x). && (63)
\end{aligned}
$$

Moreover, in the context of the momentum integral (57),

$$
\ell^\mu \ell^\nu \;\cong\; \frac{\ell^2}{D}\times g^{\mu\nu}, \tag{64}
$$

hence

$$
\begin{aligned}
\mathcal{N}_2^{\mu\nu} &\cong g^{\mu\nu}\ell^2 \times \left(2 + \frac{B}{D}\right) \;+\; (A+C)\times g^{\mu\nu}\,k^2 \;-\; C\times\big((g^{\mu\nu}k^2 \;-\; k^\mu k^\nu\big) \\
&= \big(k^2 g^{\mu\nu} \;-\; k^\mu k^\nu\big)\times \mathcal{N}_2^{\rm good} \;+\; g^{\mu\nu}\times\mathcal{N}_2^{\rm bad},
\end{aligned}
\tag{65}
$$

where

$$
\begin{aligned}
\mathcal{N}_2^{\rm good} &= -C \;=\; (6-D) \;+\; (4D-6)x(1-x), && (66)\\
\mathcal{N}_2^{\rm bad} &= \ell^2\times\left(2+\frac{B}{D}\right) \;+\; k^2\times(A+C) \\
&= \ell^2\times\left(6-\frac{6}{D}\right) \;+\; k^2\times(D-1)(1-4x+4x^2). && (67)
\end{aligned}
$$

Obviously, the $\mathcal{N}_2^{\rm good}$ has the right tensor structure for the gluon's self-energy corrections, while the $\mathcal{N}_2^{\rm bad}$ has the wrong tensor structure. In the context of the momentum integral (57), the bad term $\mathcal{N}_2^{\rm bad}$ does *not* integrate to zero. However, its integral cancels against integrals of the similar bad terms stemming from the two two remaining diagrams.

To see how the cancellation works, let us postpone taking the momentum integral (57) until we have evaluated the sideways gluon loop and the ghost loop diagrams and brought them to a similar form

$$
\big[\Sigma_3^{\mu\nu}(k)\big]^{ab} \;=\; \delta^{ab}\times\frac{g^2}{2}\,C(G)\times\int_0^1\!dx\int\frac{d^4\ell}{(2\pi)^4}\,\frac{-i\mathcal{N}_3^{\mu\nu}}{[\ell^2-\Delta+i0]^2}\,, \tag{68}
$$

$$
\big[\Sigma_4^{\mu\nu}(k)\big]^{ab} \;=\; \delta^{ab}\times\frac{g^2}{2}\,C(G)\times\int_0^1\!dx\int\frac{d^4\ell}{(2\pi)^4}\,\frac{-i\mathcal{N}_4^{\mu\nu}}{[\ell^2-\Delta+i0]^2}\,, \tag{69}
$$

for some numerators $\mathcal{N}_3^{\mu\nu}$ and $\mathcal{N}_4^{\mu\nu}$, then we are going to add up the numerators,

$$\mathcal{N}_{234}^{\mu\nu} = \mathcal{N}_2^{\mu\nu} + \mathcal{N}_3^{\mu\nu} + \mathcal{N}_4^{\mu\nu}, \tag{70}$$

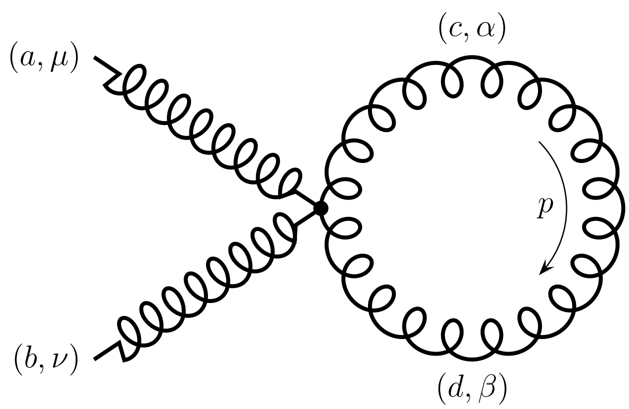
split the net numerator into the 'good' and the 'bad' tensor structures,

$$\mathcal{N}_{234}^{\mu\nu} = \mathcal{N}_{234}^{\text{good}} \times \left(g^{\mu\nu}k^2 - k^\mu k^\nu\right) + \mathcal{N}_{234}^{\text{bad}} \times g^{\mu\nu}, \tag{71}$$

and only then take the momentum integral.

⋆ ⋆ ⋆

For the sideways gluon loop



(72)

we have

$$\left[i\Sigma_3^{\mu\nu}(k)\right]^{ab} = \frac{1}{2}\int\frac{d^4p}{(2\pi)^4}\,\frac{-ig_{\alpha\beta}\delta^{cd}}{p^2+i0} \times -ig^2 \begin{bmatrix} f^{abe}f^{cde}(g^{\mu\alpha}g^{\nu\beta} - g^{\mu\beta}g^{\beta\alpha}) \\ + f^{ace}f^{bde}(g^{\mu\nu}g^{\alpha\beta} - g^{\mu\beta}g^{\alpha\nu}) \\ + f^{ade}f^{bce}(g^{\mu\nu}g^{\beta\alpha} - g^{\mu\alpha}g^{\beta\nu}) \end{bmatrix} \tag{73}$$

where the overall factor $\frac{1}{2}$ comes from the symmetry of the propagator. The group factors in this amplitude evaluate to

$$\begin{aligned} \delta^{cd} \times f^{abe}f^{cde} &= 0, \\ \delta^{cd} \times f^{ace}f^{bde} &= f^{ace}f^{bce} = C(G) \times \delta^{ab}, \\ \delta^{cd} \times f^{ade}f^{bce} &= f^{ace}f^{bce} = C(G) \times \delta^{ab}, \end{aligned} \tag{74}$$

— *cf.* eq. (48), — and consequently

$$\delta^{cd} \times [\cdots] \;=\; C(G)\delta^{ab} \times \Big(2g^{\mu\nu} g^{\alpha\beta} \;-\; g^{\mu(\alpha} g^{\beta)\nu}\Big) \tag{75}$$

and

$$g_{\alpha\beta}\delta^{cd} \times [\cdots] \;=\; C(G)\delta^{ab} \times (2D-2)g^{\mu\nu}. \tag{76}$$

Plugging this result into eq. (73), we obtain

$$\big[\Sigma_3^{\mu\nu}(k)\big]^{ab} \;=\; \delta^{ab} \times g^2 C(G) \times i(D-1)g^{\mu\nu} \int \frac{d^4p}{(2\pi)^4} \frac{1}{p^2+i0} \tag{77}$$

Instead of directly evaluating the momentum integral here, we are going to combine the integrand with the other one-loop diagrams. Since this diagram has only one propagator rather than two, we may identify the loop momentum p here as either p_1 or $p_2 = -p_1 - k$ — as long as our UV regulator allows constant shifts of the integration variable, both choices are equivalent. For symmetry's sake, let's take the average between the two choices and identify

$$\frac{1}{p^2+i0} \;\to\; \frac{1/2}{p_1^2+i0} + \frac{1/2}{p_2^2+i0} \;=\; \frac{p_1^2+p_2^2}{2(p_1^2+i0)(p_2^2+i0)} \;=\; \int_0^1 dx\, \frac{(\ell-xk)^2 \,+\, (-\ell-(1-x)k)^2}{2[\ell^2-\Delta+i0]^2} \tag{78}$$

Consequently, the amplitude (77) takes form (68) for the numerator

$$\mathcal{N}_3^{\mu\nu} \;=\; (1-D)g^{\mu\nu} \times \big[(\ell-xk)^2 + (-\ell-(1-x)k)^2\big] \;\cong\; (1-D)g^{\mu\nu} \times \big[2\ell^2 + (1-2x+2x^2)k^2\big]. \tag{79}$$

In terms of the 'good' and the 'bad' tensor structures along the lines of eq. (65), this whole numerator is 'bad', thus

$$\mathcal{N}_3^{\rm good} \;=\; 0, \tag{80}$$

$$\mathcal{N}_3^{\rm bad} \;=\; (1-D) \times \big[2\ell^2 \,+\, (1-2x+2x^2)k^2\big]. \tag{81}$$

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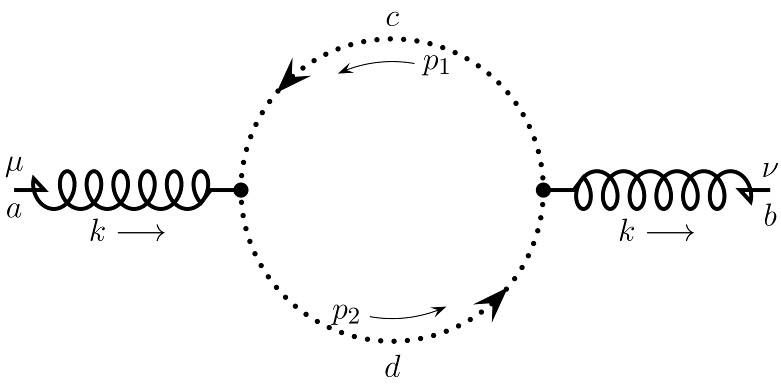
Finally, there is the ghost loop diagram



(82)

which evaluates to

$$\left[i\Sigma_4^{\mu\nu}(k)\right]^{ab} \;=\; -\int\!\frac{d^4p_1}{(2\pi)^4}\,\frac{i}{p_1^2+i0}\times\frac{i}{p_2^2+i0}\times -gf^{acd}p_2^\mu\times -gf^{bdc}p_1^\nu. \tag{83}$$

Note: the ghost propagators are oriented and go in opposite directions, so this diagram does not have the symmetry factor $\frac{1}{2}$. Instead, it carries an overall minis sign for the fermionic loop. Another minus sign hides in the group factor:

$$f^{acd}f^{bdc} \;=\; -f^{acd}f^{bcd} \;=\; -C(G)\times\delta^{bc} \tag{84}$$

Consequently,

$$\left[i\Sigma_4^{\mu\nu}(k)\right]^{ab} \;=\; \delta^{ab}\times\frac{g^2}{2}C(G)\times\int\!\frac{d^4p_1}{(2\pi)^4}\,\frac{+2i\,p_2^\mu p_1^\nu}{(p_1^2+i0)(p_2^2+i0)} \tag{85}$$

for $p_2 = +p_1 + k$. Combining the two denominator factors via the Feynman parameter integral, this amplitude takes form (69) for the numerator

$$\begin{aligned}
\mathcal{N}_4^{\mu\nu} \;&=\; -2p_2^\mu p_1^\nu \\
&=\; -2(\ell - xk + k)^\mu(\ell - xk)^\nu \\
&\cong\; -2\ell^\mu\ell^\nu \;+\; 2x(1-x)k^\mu k^\nu \\
&\cong\; -\frac{2}{D}\,\ell^2\times g^{\mu\nu} \;+\; 2x(1-x)k^\mu k^\nu\,,
\end{aligned} \tag{86}$$

or in terms of ‘good’ and ‘bad’ tensor structures,

$$\mathcal{N}_4^{\text{good}} \;=\; -2x(1-x), \tag{87}$$

$$\mathcal{N}_4^{\rm bad} \;=\; -\frac{2}{D}\times\ell^2 \;+\; 2x(1-x)\times k^2. \tag{88}$$

⋆ ⋆ ⋆

Now let's total up the numerators of the three diagrams according to the 'good' and 'bad' tensor structures:

$$\mathcal{N}_{234} \;=\; \mathcal{N}_{234}^{\rm good}\times\bigl(k^2g^{\mu\nu}\,-\,k^\mu k^\nu\bigr) \;+\; \mathcal{N}_{234}^{\rm bad}\times g^{\mu\nu}, \tag{89}$$

where

$$\begin{aligned}
\mathcal{N}_{234}^{\rm good} \;&=\; \mathcal{N}_2^{\rm good} \;+\; \mathcal{N}_3^{\rm good} \;+\; \mathcal{N}_4^{\rm good}\\
&=\; \bigl[(6-D)\;+\;(4D-6)\times x(1-x)\bigr] \;+\; 0 \;+\; \bigl[-2x(1-x)\bigr]\\
&=\; (6-D)\;+\;4(D-2)\times x(1-x),
\end{aligned} \tag{90}$$

$$\begin{aligned}
\mathcal{N}_{234}^{\rm bad} \;&=\; \mathcal{N}_2^{\rm bad} \;+\; \mathcal{N}_3^{\rm bad} \;+\; \mathcal{N}_4^{\rm bad}\\
&=\; \left[\left(6-\frac{6}{D}\right)\times\ell^2\;+\;(D-1)(1-4x+4x^2)\times k^2\right]\\
&\qquad+\;\left[-2(D-1)\times\ell^2\;-\;(D-1)(1-2x+2x^2)\times k^2\right]\\
&\qquad+\;\left[-\frac{2}{D}\times\ell^2\;+\;2x(1-x)\times k^2\right]\\
&=\; \ell^2\times\left(8-\frac{8}{D}-2D\right)\;+\;k^2\times(-2D+4)\,x(1-x)\\
&=\; \frac{2(D-2)}{D}\times\Bigl(D\times\Delta\;-\;(D-2)\times\ell^2\Bigr),
\end{aligned} \tag{91}$$

where $\Delta=-x(1-x)k^2$, exactly as in the denominator of the momentum integral.

The *net* bad-tensor-structure term in the net numerator does not vanish, but it integrates to zero. Or rather, the dimensionally regularized integral of the bad term integrates to zero in any dimension D for which the integral converges (which takes $D < 2$). Indeed,

$$\int\frac{d^D\ell}{(2\pi)^D}\,\frac{-i\mathcal{N}_{234}^{\rm bad}(\ell)}{[\ell^2-\Delta+i0]^2}\ = \tag{92}$$

$$= \frac{2(D-2)}{D}\times\int\frac{d^D\ell}{(2\pi)^D}\,\frac{-i\big(D\Delta\,-\,(D-2)\ell^2\big)}{[\ell^2-\Delta+i0]^2}$$

$$= \frac{2(D-2)}{D}\times\int\frac{d^D\ell_E}{(2\pi)^D}\,\frac{(D\Delta\,+\,(D-2)\ell_E^2}{(\ell_E^2+\Delta)^2}$$

$$= \frac{2(D-2)}{D}\times\int\frac{d^D\ell_E}{(2\pi)^D}\left(\frac{D-2}{\ell_E^2+\Delta}\,+\,\frac{2\Delta}{(\ell^2+E+\Delta)^2}\right)$$

$$= \frac{2(D-2)}{D}\int\frac{d^D\ell_E}{(2\pi)^D}\int\limits_0^\infty dt\,\big((D-2)\,+\,2\Delta\times t\big)\times\exp\!\big(-t(\ell_E^2+\Delta)\big)$$

$$= \frac{2(D-2)}{D}\int\limits_0^\infty dt\,\big((D-2)\,+\,2\Delta\times t\big)\times e^{-t\Delta}\times\left(\int\frac{d^D\ell_E}{(2\pi)^D}\,e^{-t\ell_E^2}\ =\ (4\pi t)^{-D/2}\right)$$

$$= \frac{2(D-2)}{D(4\pi)^{D/2}}\times\Big((D-2)\times\Gamma\left(1-\tfrac{D}{2}\right)\Delta^{\frac{D}{2}-1}\,+\,2\Delta\times\Gamma\left(2-\tfrac{D}{2}\right)\Delta^{\frac{D}{2}-2}\Big)$$

$$= \frac{4(D-2)}{D(4\pi)^{D/2}}\times\Delta^{\frac{D}{2}-2}\times\Big(\left(\tfrac{D}{2}-1\right)\Gamma\left(1-\tfrac{D}{2}\right)\,+\,\Gamma\left(2-\tfrac{D}{2}\right)\Big) \tag{92}$$

because

$$\left(\tfrac{D}{2}-1\right)\times\Gamma\left(1-\tfrac{D}{2}\right)\,+\,\Gamma\left(2-\tfrac{D}{2}\right)\ =\ 0. \tag{93}$$

Thus, the net vacuum polarization tensor for the gluons does have the right k dependence,

$$\big[\Sigma_{234}^{\mu\nu}(k)\big]^{ab}\ =\ \delta^{ab}\times\big(k^2g^{\mu\nu}\,-\,k^\mu k^\nu\big)\times\Pi_{234}(k^2) \tag{94}$$

where

$$\Pi_{234}\ =\ \frac{g^2\,C(G)}{2}\times\int\limits_0^1 dx\int\frac{d^4\ell}{(2\pi)^4}\,\frac{-i\mathcal{N}_{234}^{\rm good}}{(\ell^2-\Delta+i0)^2}\,. \tag{95}$$

Since the numerator $\mathcal{N}_{234}^{\rm good}$ does not depend on the loop momentum ℓ but only on the

Feynman parameter x, the momentum integral here becomes the familiar

$$\int\frac{d^4\ell}{(2\pi)^4}\,\frac{-i}{(\ell^2-\Delta+i0)^2}\;=\int\frac{d^4\ell_E}{(2\pi)^4}\,\frac{1}{(\ell_E^2+\Delta)^2}\;\xrightarrow[\mathrm{DR}]{}\;\frac{1}{16\pi^2}\left(\frac{1}{\epsilon}\,+\,\mathrm{finite}(x)\right). \tag{96}$$

Consequently

$$\begin{aligned}\Pi_{2+3+4}(k^2)\;&=\;+\frac{g^2C(G)}{32\pi^2}\times\int\limits_0^1\!dx\,\mathcal{N}_{\mathrm{good}}(x,D)\times\left(\frac{1}{\epsilon}\,+\,\mathrm{finite}(x,k^2)\right)\\ &=\;+\frac{g^2C(G)}{32\pi^2}\times\left(\frac{1}{\epsilon}\times\int\limits_0^1\!dx\,\mathcal{N}_{\mathrm{good}}(x,D=4)\;+\;\mathrm{finite}(k^2)\right).\end{aligned} \tag{97}$$

Note that the UV divergence here — the pole at $\epsilon\to0$ — obtains from the $\mathcal{N}_{234}^{\mathrm{good}}$ at $D=4$ (since the difference between the $\mathcal{N}$ at $D=4-2\epsilon$ and at $D=4$ is $O(\epsilon)$), thus

$$\mathcal{N}_{234}^{\mathrm{good}}(x)\;=\;(6-D)+4(D-2)\times x(1-x)\;\to\;2\,+\,8x(1-x)\quad\Longrightarrow\quad\int\limits_0^1\!dx\,\mathcal{N}_{234}^{\mathrm{good}}(x)\;\to\;\frac{10}{3}\,. \tag{98}$$

Therefore,

$$\Pi_{234}\;=\;+\frac{g^2}{16\pi^2}\times\frac{5C(G)}{3}\times\left(\frac{1}{\epsilon}\,+\,\mathrm{finite}(k^2)\right), \tag{99}$$

and the δ_3 counterterm which cancels this divergence is

$$\delta_3(2^{\mathrm{nd}}+3^{\mathrm{rd}}+4^{\mathrm{th}})\;=\;+\frac{g^2}{16\pi^2}\times\frac{5C(G)}{3}\times\frac{1}{\epsilon}\,. \tag{100}$$

Finally, adding the quark loops' contribution (43), we arrive at the complete one-loop δ_3 counterterm,

$$\delta_3\;=\;\frac{g^2}{16\pi^2}\times\left(\frac{5}{3}\,C(G)\;-\;\frac{4}{3}\,R_{\mathrm{net}}(\mathrm{quarks})\right)\times\frac{1}{\epsilon}\,. \tag{101}$$

The Beta Function

Now that we have the one-loop counterterms $\delta_2^{(q)}$, $\delta_1^{(q)}$, and δ_3, we may use them to obtain the one-loop β-function for the gauge coupling. According to eq. (3.1),

$$\begin{aligned}
\beta^{1\,\text{loop}} &= g\times\text{Res}\big[2\delta_1^{(q)}\,-\,2\delta_2^{(q)}\,-\,\delta_3\big]^{1\,\text{loop}}\\
&= g\times\left[2\times\frac{-g^2\big(C(Q)+C(G)\big)}{16\pi^2}\,-\,2\times\frac{-g^2C(Q)}{16\pi^2}\,-\,\frac{g^2\big(\frac{5}{3}C(G)-\frac{4}{3}R_{\text{net}}(Q)\big)}{16\pi^2}\right]\\
&= \frac{g^3}{16\pi^2}\times\left(-\frac{11}{3}\,C(G)\,+\,\frac{4}{3}\,R_{\text{net}}(Q)\right).
\end{aligned} \tag{102}$$

For the QCD and for QCD-like theories with an $SU(N_c)$ gauge group and N_f flavors of fundamental multiplets of quarks, $C(G)=N_c$, $R_{\text{net}}(Q)=N_f\times\frac{1}{2}$, hence

$$\beta^{1\,\text{loop}}(g)\;=\;\frac{g^3}{16\pi^2}\left(-\frac{11}{3}\,N_c\,+\,\frac{2}{3}\,N_f\right). \tag{103}$$

Note the negative coefficient of the N_c-dependent term. Consequently, for $N_f<\frac{11}{2}N_c$, the whole β-function is negative — or at least it's negative for a weak enough coupling g — which makes QCD or a QCD-like gauge theory *asymptotically free*. For more general gauge theories with fermions, the asymptotic freedom requires

$$R_{\text{net}}(\text{all the fermions})\;<\;\frac{11}{4}\,C(G). \tag{104}$$

More General Gauge Theories

For completeness sake, let me give you a formula for the one-loop beta function for any gauge theory coupled to several kinds of 'matter' fields: Dirac fermions like the quarks, but also chiral Weyl fermions (left-handed or right-handed only), Majorana fermions, complex scalars, or real scalars. In general, the Dirac fermions, the Weyl fermions, and the complex scalars can be in any multiplets of the gauge group G, while the Majorana fermions and the real scalars must be in real multiplets of G.

As required by the gauge coupling universality, for any kind of the matter multiplets —fermionic or scalar — coupled to G we should have the same difference $\delta_1 - \delta_2$ as for the quarks: To all loop orders

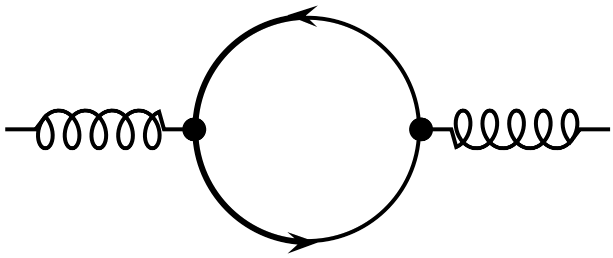
$$\text{same } \operatorname{Res}\left[\delta_1^{(m)} - \delta_2^{(m)}\right] \quad \forall \text{ matter multiplet } (m), \tag{105}$$

and specifically at one loop,

$$\delta_1^{(m)} - \delta_2^{(m)} = -\frac{g^2 C(G)}{16\pi^2} \times \frac{1}{\epsilon} \quad \forall \text{ matter multiplet } (m), \tag{106}$$

cf. eq. (34) for the multiplets of Dirac fermions. But let me skip the proof of these formulae and focus on the δ_3 counterterm.

At the one loop level, multiplets of Majorana or Weyl fermions affect the δ_3 counterterm similarly to the Dirac fermions, via the loop



(36.1)

However, for the Majorana fermions, the solid lines have no arrows, which gives the diagram an extra symmetry factor 1/2. Consequently, their contribution to the δ_3 counterterm is $\frac{1}{2}$ of what the Dirac fermions in the same multiplets would contribute,

$$\delta_3(\text{Majorana}) = -\frac{g^2}{16\pi^2} \times \frac{4/3}{2} R_{\text{net}}(\text{Majorana}) \times \frac{1}{\epsilon}\,. \tag{107}$$

For the Weyl fermions, the solid lines in the diagram (36.1) do have arrow — which avoids the $\frac{1}{2}$ symmetry factor, — but the vertices have extra chiral projection factors $\frac{1}{2}(1\mp\gamma^5)$ onto left or right chiralities, which changes the trace over the Dirac indices to

$$\operatorname{tr}\left((-ig\gamma^\mu)\,\frac{1\mp\gamma^5}{2} \times \frac{i}{\not{p}+i0} \times (-ig\gamma^\nu)\,\frac{1\mp\gamma^5}{2} \times \frac{i}{\not{p}+\not{k}+i0}\right). \tag{108}$$

To evaluate this trace we use the anticommutativity of the γ^5 matrix with the γ^μ and hence with the massless fermion propagators, and also $(1\mp\gamma^5)^2=2(1\mp\gamma^5)$. Consequently,

$$
\begin{aligned}
\mathrm{tr}\big(\mathrm{Weyl}\big) &= \frac{1}{4}\times\mathrm{tr}\left((1\pm\gamma^5)^2\times(-ig\gamma^\mu)\times\frac{i}{Dp+i0}\times(-ig\gamma^\nu)\times\frac{i}{\not p+\not k+i0}\right)\\
&= \frac{1}{2}\times\mathrm{tr}\left((-ig\gamma^\mu)\times\frac{i}{Dp+i0}\times(-ig\gamma^\nu)\times\frac{i}{\not p+\not k+i0}\right)\\
&\quad\pm\frac{1}{2}\times\mathrm{tr}\left(\gamma^5\times(-ig\gamma^\mu)\times\frac{i}{Dp+i0}\times(-ig\gamma^\nu)\times\frac{i}{\not p+\not k+i0}\right)\\
&= \frac{1}{2}\times\mathrm{tr}\big(Dirac\big)\ \pm\ \frac{1}{2}\times\mathrm{tr}\big(\mathrm{extra}\big),
\end{aligned}
\tag{109}
$$

where the 'extra' trace on the bottom line vanishes by the Lorentz symmetry after integrating over the loop momentum p. Indeed,

$$
\begin{aligned}
\mathrm{tr}\big(\mathrm{extra}\big) &= g^2\times\mathrm{tr}\left(\gamma^5\times\gamma^\mu\frac{1}{\not p+i0}\gamma^\nu\frac{1}{\not p+\not k+i0}\right)\\
&= \frac{g^2}{(p^2+i0)((p+k)^2+i0)}\times\left(\begin{array}{rl}\mathrm{tr}\big(\gamma^5\gamma^\mu\not p\gamma^\nu(\not p+\not k)\big) &= 4i\epsilon^{\mu\lambda\nu\rho}p_\lambda(p+k)_\rho\\ &= 4i\epsilon^{\mu\lambda\nu\rho}p_\lambda k_\rho\\ \multicolumn{2}{c}{\text{by the antisymmetry of }\epsilon^{\mu\lambda\nu\rho}}\end{array}\right)\\
&= 4ig^2\epsilon^{\mu\nu\rho\lambda}\,k_\rho\times\frac{p_\lambda}{(p^2+i0)((p+k)^2+i0)}\,.
\end{aligned}
\tag{110}
$$

By Lorentz, symmetry, when we integrate the p-dependent factor here over p, we obtain

$$
\int\!\frac{d^Dp}{(2\pi)^D}\,\frac{p_\lambda}{(p^2+i0)((p+k)^2+i0)}\ =\ (\mathrm{scalar})\times k_\lambda\,,
\tag{111}
$$

hence

$$
\int\!\frac{d^Dp}{(2\pi)^D}\ \mathrm{tr}\big(\mathrm{extra}\big)\ =\ 4ig^2\epsilon^{\mu\nu\rho\lambda}\,k_\rho\times(\mathrm{scalar})\times k_\lambda\ =\ 0
\tag{112}
$$

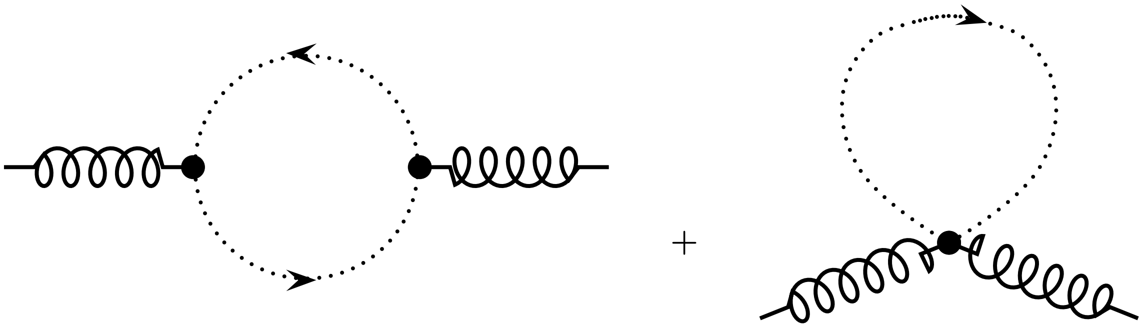
by the antisymmetry of the $\epsilon^{\mu\nu\rho\lambda}$ tensor. Consequently, eq. (109) for the trace over a loop of Weyl fermions simplifies to

$$
\mathrm{tr}\big(\mathrm{Weyl}\big)\ \cong\ \frac{1}{2}\times\mathrm{tr}\big(\mathrm{Dirac}\big),
\tag{113}
$$

so the whole loop diagram for a Weyl fermion multiplet is precisely $\frac{1}{2}$ of what we would get for a similar multiplet of Dirac fermions. Thus, just like for the Majorana fermions,

$$\delta_3(\text{Weyl}) \;=\; -\frac{g^2}{16\pi^2}\times\frac{4/3}{2}\,R_{\text{net}}(\text{Weyl})\times\frac{1}{\epsilon}\,. \tag{114}$$

For the complex scalars, we have two one-loop diagrams



(115)

These diagrams look exactly similar to the diagrams renormalizing the gauge coupling in the scalar QED, which you should have evaluated in homework set#16. The only difference from the scalar QED is that the scalar's electric charge^2 (in units of e^2) is replaced by the trace over the color indices

$$\text{tr}_{\text{colors}}\bigl(t^a t^b\bigr) \;=\; \delta^{ab}\times R(\text{scalar multiplet}). \tag{116}$$

All other aspects of the two diagrams — the Lorentz indexology and the momentum integrals — work exactly as for the scalar QED, so copying the formula for the scalar QED's δ_3 from the solutions to homework#16 and multiplying by the group factor (116), we obtain

$$\delta_3(\text{complex scalars}) \;=\; -\frac{g^2}{16\pi^2}\times\frac{1}{3}\times R_{\text{net}}(\text{complex scalars})\times\frac{1}{\epsilon}\,. \tag{117}$$

Finally, for the real scalars we have two one-loop diagrams similar to (115), but without the arrows on the dotted lines. Consequently, both diagrams get an overall symmetry factor 1/2, hence the δ_3 counterterm as in eq. (117) times $\frac{1}{2}$,

$$\delta_3(\text{real scalars}) \;=\; -\frac{g^2}{16\pi^2}\times\frac{1/3}{2}\times R_{\text{net}}(\text{real scalars})\times\frac{1}{\epsilon}\,. \tag{118}$$

Altogether, for a gauge theory with a simple gauge group G and matter (fermionic and/or scalar) in some multiplets (m), we have

$$\beta^{1\,\text{loop}}(g) \;=\; \frac{g^3}{16\pi^2}\times\sum_{\substack{\text{all physical}\\ \text{multiplets}}} R(\text{multiplet})\times\begin{cases}-\frac{11}{3} & \text{for the gauge fields,}\\ +\frac{4}{3} & \text{for Dirac fermions,}\\ +\frac{2}{3} & \text{for Majorana fermions,}\\ +\frac{2}{3} & \text{for chiral Weyl fermions,}\\ +\frac{1}{3} & \text{for complex scalar fields,}\\ +\frac{1}{6} & \text{for real scalar fields.}\end{cases} \tag{119}$$

In this formula, the gauge fields' contribution includes both the vector fields A^a_μ themselves as well as the ghosts c^a and $\bar{c}^a$, so please do not count the ghosts as separate multiplets.

Note that only the non-abelian gauge fields give negative contributions to the β function, all other fields' contributions are positive. Consequently, only the non-abelian gauge theories can be asymptotically free, and only when there are not too many fermionic or scalar fields coupled to the gauge fields. For example, the QCD-like theories are asymptotically free only for $N_f < \frac{11}{2}N_c$.

In a theory with a product gauge group $G = G_1 \otimes G_2 \otimes \cdots$, each component group G_i — abelian or non-abelian — has its own gauge coupling g_i. At the one-loop level, the beta functions of each g_i are independent from each other, and also from the other couplings like Yukawa of $\lambda\phi^4$, thus

$$\forall i,\ \beta_i \;=\; \frac{g_i^3}{16\pi^2}\times b_i + \frac{g_i^3}{(4\pi)^4}\times O(g_i^2,\,\text{other } g_j^2,\,\text{yukawa}^2,\,\lambda) \tag{120}$$

where b_i are the numerical factors which obtain exactly as the factor multiplying $g^3/16\pi^2$ in eq. (119). However, for each b_i you should count multiplets of the appropriate G_i without paying attention to the other gauge groups G_j. For example, a bi-fundamental $(\mathbf{m},\mathbf{n})$ multiplet of an $SU(m)\otimes SU(n)$ gauge group counts as m fundamental multiplets of $SU(n)$ when you calculate the β_n, — or as n fundamental multiplets of $SU(m)$ when you calculate the β_m, — thus

$$R_{SU(m)}((\mathbf{m},\mathbf{n})) \;=\; n\times R((\mathbf{m})) \;=\; m\times\tfrac{1}{2},\quad R_{SU(n)}((\mathbf{m},\mathbf{n})) \;=\; m\times R((\mathbf{n})) \;=\; n\times\tfrac{1}{2}. \tag{121}$$

And for the abelian $U(1)$ factors, the index of a charged singlet is simply its charge squared, while the index of a complete multiplet WRT all the other G_j factors is $R = (\text{multiplet size})\times$

(abelian charge)^2. For example, in the Standard Model $SU(3) \times SU(2) \times U(1)$, a multiplet ($\mathbf{m}$ of $SU(3)$; $\mathbf{n}$ of $SU(2)$; hypercharge $= y$) has

$$R_{U(1)}((\mathbf{m}, \mathbf{n}, y)) \;=\; mn \times y^2. \tag{122}$$