QED Feynman Rules in the Counterterm Perturbation Theory

The simplest version of QED (Quantum ElectroDynamics) has only 2 field types — the electromagnetic field A^μ and the electron field Ψ — and its physical Lagrangian is

$$\mathcal{L}_{\rm phys} \;=\; -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} \;+\; \overline{\Psi}(i\gamma^\mu D_\mu - m_e)\Psi \;=\; -\tfrac{1}{4}F_{\mu\nu}^2 \;+\; \overline{\Psi}(i\not{\partial} - m)\Psi +\; eA_\mu\overline{\Psi}\gamma^\mu\Psi. \quad (1)$$

The bare Lagrangian of the perturbation theory has a similar form, except for the bare coupling $e_{\rm bare}$ instead of the physical coupling e, the bare electron mass $m_{\rm bare}$ instead of the physical mass m, and the bare fields $A^\mu_{\rm bare}(x)$ and $\Psi_{\rm bare}(x)$ instead of the renormalized fields $A^\mu(x)$ and $\Psi(x)$. By convention, the fields strength[2] factors Z for the EM and the electron fields are called respectively the Z_3 and the Z_2, while the Z_1 is the electric charge renormalization factor. Thus,

$$A^\mu_{\rm bare}(x) \;=\; \sqrt{Z_3}\times A^\mu(x), \qquad \Psi_{\rm bare}(x) \;=\; \sqrt{Z_2}\times\Psi(x), \tag{2}$$

and plugging these bare fields into the bare Lagrangian we obtain

$$\mathcal{L}_{\rm bare} \;=\; -\frac{Z_3}{4}\,F_{\mu\nu}F^{\mu\nu} \;+\; Z_2\overline{\Psi}(i\not{\partial} - m_{\rm bare})\Psi \;+\; Z_1 e\times A_\mu\overline{\Psi}\gamma^\mu\Psi \tag{3}$$

where

$$Z_1\times e \;=\; Z_2\sqrt{Z_3}\times e_{\rm bare} \tag{4}$$

by definition of the Z_1.

As usual in the counterterm perturbation theory, we split

$$\mathcal{L}_{\rm bare} \;=\; \mathcal{L}_{\rm phys} \;+\; \mathcal{L}^{\rm counter}_{\rm terms} \tag{5}$$

where the physical Lagrangian $\mathcal{L}_{\rm phys}$ is exactly as in eq. (1) while the counterterms comprise the difference. Specifically,

$$\mathcal{L}^{\rm counter}_{\rm terms} \;=\; -\frac{\delta_3}{4}\times F_{\mu\nu}F^{\mu\nu} \;+\; \delta_2\times\overline{\Psi}i\not{\partial}\Psi \;-\; \delta_m\times\overline{\Psi}\Psi \;+\; e\delta_1\times A_\mu\overline{\Psi}\gamma^\mu\Psi \tag{6}$$

for

$$\delta_3 \;=\; Z_3 - 1, \quad \delta_2 \;=\; Z_2 - 1, \quad \delta_1 \;=\; Z_1 - 1, \quad \delta_m \;=\; Z_2 m_{\rm bare} - m_{\rm phys}\,. \tag{7}$$

Actually, the bare Lagrangian (5) is not the whole story, since in the quantum theory the EM field $A^\mu(x)$ needs to be gauge-fixed. In the Feynman gauge, or in similar Lorentz-invariang gauges, the gauge fixing amounts to adding an extra gauge-symmetry breaking term to the Lagrangian,

$$\mathcal{L}_{\text{bare}} \;=\; \mathcal{L}_{\text{phys}} \;+\; \mathcal{L}^{\text{gauge}}_{\text{fixing}} \;+\; \mathcal{L}^{\text{counter}}_{\text{terms}} \tag{8}$$

for

$$\mathcal{L}^{\text{gauge}}_{\text{fixing}} \;=\; -\frac{1}{2\xi}\,(\partial_\mu A^\mu)^2 \tag{9}$$

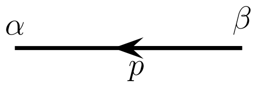
where ξ is a constant parametrixing a specific gauge. In the Feynman gauge $\xi = 1$.

In the counterterm perturbation theory, we take the free Lagrangian to be the quadratic part of the physical Lagrangian plus the gauge fixing term, thus

$$\mathcal{L}_{\text{free}} \;=\; \overline{\Psi}(i\,\partial\!\!\!/ - m)\Psi \;-\; \tfrac{1}{4}F_{\mu\nu}^2 \;-\frac{1}{2\xi}\,(\partial_\mu A^\mu)^2 \tag{10}$$

(where m is the physical mass of the electron), while all the other terms in the bare Lagrangian — the physical coupling $eA_\mu\overline{\Psi}\gamma^\mu\Psi$ and all the counterterms (6) — are treated as perturbations. Consequently, the QED Feynman rules have the following propagators and vertices:

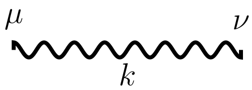
- The electron propagator



$$= \left[\frac{i}{p\!\!\!/ - m + i0}\right]_{\alpha\beta} \;=\; \frac{i(p\!\!\!/ + m)_{\alpha\beta}}{p^2 - m^2 + i0} \tag{11}$$

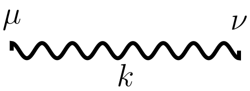
  where α and β are the Dirac indices, usually not written down.

- The photon propagator



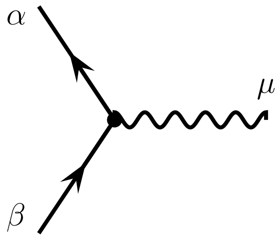
$$= \frac{-i}{k^2 + i0} \times \left(g^{\mu\nu} \;+\; (\xi - 1)\frac{k^\mu k^\nu}{k^2 + i0}\right). \tag{12}$$

  In the Feynman gauge $\xi = 1$ this propagator simplifies to



$$= \frac{-ig^{\mu\nu}}{k^2 + i0}\,. \tag{13}$$

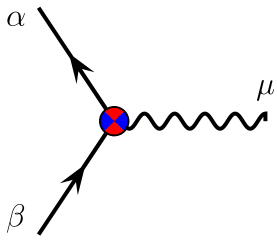
- The physical vertex



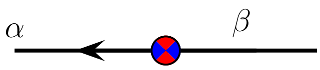
$$= \left(+ie\gamma^{\mu}\right)_{\alpha\beta}\,. \tag{14}$$

The Dirac indices α and β of the fermionic lines are usually not written down.

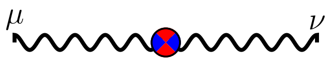
⋆ And then there are three kinds of the counterterm vertices:



$$= +ie\delta_1 \times (\gamma^{\mu})_{\alpha\beta}\,, \tag{15}$$



$$= +i\big(\delta_2 \times \not{p} - \delta_m\big)_{\alpha\beta}\,, \tag{16}$$



$$= -i\delta_3 \times \big(g^{\mu\nu}k^2 - k^{\mu}k^{\nu}\big). \tag{17}$$