Problem **1**:

In the Lagrangian

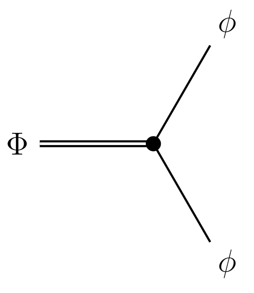
$$\mathcal{L} \;=\; \frac{1}{2}(\partial_\mu\Phi)^2 \;-\; \frac{M^2}{2}\,\Phi^2 \;+\; \frac{1}{2}(\partial_\mu\phi)^2 \;-\; \frac{m^2}{2}\,\phi^2 \;-\; \frac{\mu}{2}\,\Phi\phi^2, \tag{1}$$

the first 4 terms on the RHS describe two free scalar fields $\Phi(x)$ and $\phi(x)$, while the fifth term is the interaction that we treat as a perturbation. In Feynman rules, the propagators follow from the free part of the Lagrangian, so for the theory at hand there are two distinct propagators,

Φ ══════ Φ $\;=\; \dfrac{i}{q^2 - m^2 + i0}$ and ϕ ────── ϕ $\;=\; \dfrac{i}{q^2 - M^2 + i0}\,.$ (S.1)

Likewise, there are two kinds of external lines according to the species of the incoming or outgoing particles for the process in question.

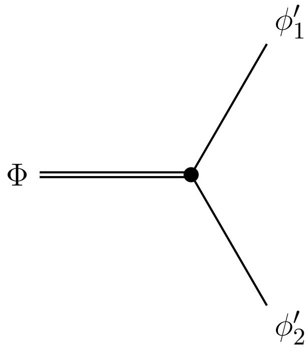
The Feynman vertices follow from the interaction part of the Lagrangian, which for the theory at hand is the cubic potential term $V_3 = \frac{\mu}{2}\,\Phi\phi^2$. Consequently, all vertices should be connected to three lines (net valence = 3), one double line for the one $\hat{\Phi}$ field, and two single lines for the two $\hat{\phi}$ fields,



$$= \; -i\frac{\mu}{2}\times 2! \;=\; -i\mu \tag{S.2}$$

where the 2! factor comes from the interchangeability of two identical $\hat{\phi}$ fields in the vertex.

Now consider the decay process $\Phi \to \phi + \phi$. To the lowest order of the perturbation theory, the decay amplitude follows from a single tree diagrams



(S.3)

This diagram has one vertex, one incoming double line, two outgoing single lines and no internal lines of either kind, hence

$$\langle \phi'_1 + \phi'_2 | \, i\hat{T} \, | \Phi \rangle \;\equiv\; i\mathcal{M} \times (2\pi)^4 \delta^{(4)}(p - p'_1 - p'_2) \;=\; -i\mu \times (2\pi)^4 \delta^{(4)}(p - p'_1 - p'_2), \quad \text{(S.4)}$$

or in other words

$$\mathcal{M}(\Phi \to \phi'_1 + \phi'_2) \;=\; -\mu. \quad \text{(S.5)}$$

This amplitude is related to the $\Phi \to \phi\phi$ decay rate as

$$\Gamma \;=\; \int |\mathcal{M}|^2 \, d\mathcal{P} \quad \text{(S.6)}$$

where the phase space factor for 1 particle $\to$ 2 particles decays is

$$\begin{aligned} d\mathcal{P} \;&=\; \frac{1}{2E} \times \frac{d^3\mathbf{p}'_1}{(2\pi)^3 \, 2E'_1} \times \frac{d^3\mathbf{p}'_2}{(2\pi)^3 \, 2E'_2} \times (2\pi)^4 \delta^{(4)}(p - p'_1 - p'_2) \\ &=\; \frac{1}{32\pi^2 \, E E'_1 E'_2} \times d^3\mathbf{p}'_1 \, \delta(E - E'_1 - E'_2) \quad \text{for} \quad \mathbf{p}'_2 \;=\; \mathbf{p} - \mathbf{p}'_1 \quad \text{and on-shell energies,} \\ &=\; \frac{|\mathbf{p}'|}{32\pi^2 M^2} \times d\Omega_{\mathbf{p}'_1} \qquad \text{in the rest frame of the decaying particle,} \end{aligned} \quad \text{(S.7)}$$

cf. my notes on phase space. For decays to two particles of equal masses $m < \frac{M}{2}$,

$$E'_1 \;=\; E'_2 \;=\; \frac{M}{2} \quad\Longrightarrow\quad |\mathbf{p}'| \;=\; \sqrt{E'^2_1 - m^2} \;=\; \frac{M}{2} \times \sqrt{1 - \frac{4m^2}{M^2}}\,, \tag{S.8}$$

hence

$$d\mathcal{P} \;=\; \sqrt{1 - \frac{4m^2}{M^2}} \times \frac{1}{64\pi^2 M} \times d\Omega, \tag{S.9}$$

and therefore the partial decay rate is

$$\frac{d\Gamma}{d^2\Omega} \;=\; \sqrt{1 \,-\, \frac{4m^2}{M^2}} \times \frac{|\mathcal{M}|^2}{64\pi^2 M}\,. \tag{S.10}$$

For the problem at hand, $\mathcal{M}_{\text{tree}} = -\mu$ regardless of directions of final particles, hence

$$\frac{d\Gamma_{\text{tree}}}{d^2\Omega} \;=\; \sqrt{1 \,-\, \frac{4m^2}{M^2}} \times \frac{\mu^2}{64\pi^2 M}\,. \tag{S.11}$$

Integrating this partial decay rate over the directions of $\mathbf{p}'$ we must remember that the two final particles are identical bosons, so we cannot tell $\mathbf{p}'_1$ from $\mathbf{p}'_2 = -\mathbf{p}'_1$. Consequently, $\int d^2\Omega = 4\pi/2$ and therefore

$$\Gamma \;=\; \sqrt{1 \,-\, \frac{4m^2}{M^2}} \times \frac{\mu^2}{32\pi M}\,. \tag{S.12}$$

Problem **2**:

Similar to the previous problem, the Feynman propagators of a theory follow from the free part of its Lagrangian. This time, we have N scalar fields $\phi^i(x)$ of similar mass m, hence in momentum space

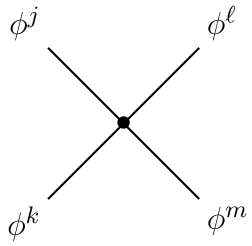
$$\phi^j \;\rule[0.5ex]{6em}{0.4pt}\; \phi^k \;=\; \frac{i\delta^{jk}}{q^2 - m^2 + i0}\,. \tag{S.13}$$

Note the δ^{jk} factor — the two fields connected by a propagator must be of the same species. Graphically, this means that both ends of the propagator carry the same species label $j = k$. Likewise, the external lines should also carry a species label of the incoming or outgoing particle in question. For the external lines, these labels are fixed (for a particular process), while for the internal lines we sum over $j = 1, 2, \ldots, N$.

The Feynman vertices follow from the interactions between the fields; for the theory in question, they come form the quartic potential

$$V_4 \;=\; \frac{\lambda}{8}\Bigl(\phi\cdot\phi = \sum_j \phi^j\phi^j\Bigr)^2 \;=\; \sum_j \frac{\lambda}{8}\,(\hat\phi^j)^4 \;+\; \sum_{j<k} \frac{\lambda}{4}\,(\hat\phi^j)^2(\hat\phi^k)^2. \qquad \text{(S.14)}$$

Consequently, all vertices have net valence = 4, but there are two vertex types with different indexologies: (1) a vertex involving 4 lines of the same field species ϕ^j, with the vertex factor $-i(\lambda/8)\times 4! = -3i\lambda$; and (2) a vertex involving 2 lines of one field species ϕ^j and 2 lines of a different species ϕ^k, with the vertex factor $-i(\lambda/4)\times(2!)^2 = -i\lambda$. (The combinatorial factors arise from the interchanges of the identical fields in the same vertex, thus 4! for the first vertex type and $(2!)^2$ for the second type.) Equivalently, we may use a single vertex type involving 4 fields of whatever species, with the species-dependent vertex factor



$$=\; -i\lambda\bigl(\delta^{jk}\delta^{\ell m} + \delta^{j\ell}\delta^{km} + \delta^{jm}\delta^{k\ell}\bigr). \qquad \text{(S.15)}$$

Now consider the scattering process $\phi^j + \phi^k \to \phi^\ell + \phi^m$. At the lowest order of the perturbation theory, there is just one Feynman diagram for this process; it has one vertex, 4 external legs and no internal lines. Consequently, at the lowest order,

$$\mathcal{M}(\phi^j + \phi^k \to \phi^\ell + \phi^m) \;=\; -\lambda\bigl(\delta^{jk}\delta^{\ell m} + \delta^{j\ell}\delta^{km} + \delta^{jm}\delta^{k\ell}\bigr) \qquad \text{(S.16)}$$

independent of the particles' momenta. Specifically,

$$\begin{aligned}
\mathcal{M}(\phi^1 + \phi^2 \to \phi^1 + \phi^2) &= -\lambda,\\
\mathcal{M}(\phi^1 + \phi^1 \to \phi^2 + \phi^2) &= -\lambda,\\
\mathcal{M}(\phi^1 + \phi^1 \to \phi^1 + \phi^1) &= -3\lambda.
\end{aligned} \qquad \text{(S.17)}$$

The *partial* cross sections in the CM frame follow from these amplitudes via eq. (4.85) of the textbook or eq. (17) of my notes on phase space: For elastic scattering,

$$\frac{d\sigma}{d\Omega_{\rm c.m.}} = \frac{|\mathcal{M}|^2}{64\pi^2 E_{\rm c.m.}^2}], \tag{S.18}$$

hence

$$\begin{aligned} \frac{d\sigma(\phi^1+\phi^2\to\phi^1+\phi^2)}{d\Omega_{\rm c.m.}} &= \frac{\lambda^2}{64\pi^2 E_{\rm c.m.}^2}, \\ \frac{d\sigma(\phi^1+\phi^1\to\phi^2+\phi^2)}{d\Omega_{\rm c.m.}} &= \frac{\lambda^2}{64\pi^2 E_{\rm c.m.}^2}, \\ \frac{d\sigma(\phi^1+\phi^1\to\phi^1+\phi^1)}{d\Omega_{\rm c.m.}} &= \frac{9\lambda^2}{64\pi^2 E_{\rm c.m.}^2}. \end{aligned} \tag{S.19}$$

To calculate the total cross sections, we integrate over $d\Omega$, which gives the factor of 4π when the two final particles are of distinct species, but for the same species, we only get 2π because of Bose statistics. Thus,

$$\begin{aligned} \sigma_{\rm tot}(\phi^1+\phi^2\to\phi^1+\phi^2) &= \frac{\lambda^2}{16\pi E_{\rm c.m.}^2}, \\ \sigma_{\rm tot}(\phi^1+\phi^1\to\phi^2+\phi^2) &= \frac{\lambda^2}{32\pi E_{\rm c.m.}^2}, \\ \sigma_{\rm tot}(\phi^1+\phi^1\to\phi^1+\phi^1) &= \frac{9\lambda^2}{32\pi E_{\rm c.m.}^2}. \end{aligned} \tag{S.20}$$

Problem **3**(a):

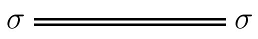
In perturbation theory, the Feynman propagators follow from the quadratic part of the Lagrangian (and hence free Hamiltonian), while the vertices follow from the cubic, quartic, *etc.*, terms treated as perturbation. For the linear sigma model's Lagrangian (3),

$$\mathcal{L} = \mathcal{L}_{\rm free} - V_{\rm pert}, \tag{S.21}$$

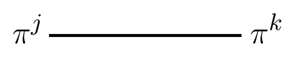
$$\mathcal{L}_{\rm free} = \frac{1}{2}\sum\nolimits_i (\partial_\mu\phi_i)^2 + \frac{1}{2}(\partial_\mu\sigma)^2 - \frac{M_\sigma^2 = \lambda f^2}{2} \times \sigma^2, \tag{S.22}$$

$$V_{\rm pert} \;=\; \frac{3\lambda f}{6}\times\sigma^3 \;+\; \frac{\lambda f}{2}\times\sum\nolimits_i \sigma\phi_i^2 \\ \;+\; \frac{3\lambda}{24}\times\sigma^4 \;+\; \frac{\lambda}{4}\times\sum\nolimits_i \sigma^2\phi_i^2 \;+\; \frac{\lambda}{8}\times\Big(\sum\nolimits_i \phi_i^2\Big)^2. \tag{S.23}$$

The free Lagrangian (S.22) describes one massive field σ plus N massless fields π_i, hence two types of scalar propagators,



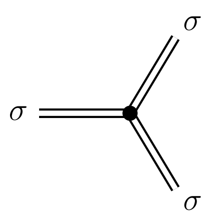
$$= \frac{i}{q^2 - 2\mu^2 + i0} \qquad \text{and}$$



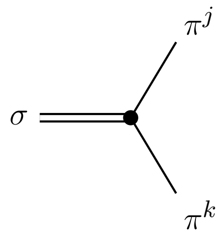
$$= \frac{i\delta^{jk}}{q^2 + i0}\,, \tag{S.24}$$

and the $\pi\pi$ propagator carries a label $j = k = 1, 2 \ldots, N$ specifying a particular species of the pion field.

As to the perturbation (S.23), it has two cubic terms and 3 quartic terms, hence two types of valence $= 3$ vertices and three types of valence $= 4$ vertices: the $\sigma\sigma\sigma$ and $\sigma\pi\pi$ vertices

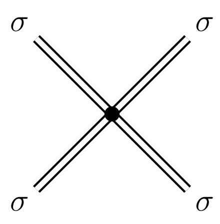


$$= -3i\lambda f \qquad \text{and}$$

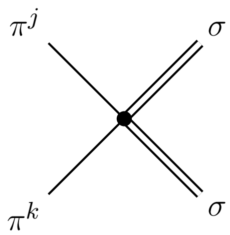


$$= -i\lambda f\delta^{jk}\,, \tag{S.25}$$

the $\sigma\sigma\sigma\sigma$ and $\sigma\sigma\pi\pi$ vertices

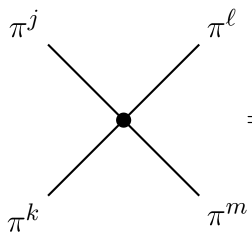


$$= -3i\lambda \qquad \text{and}$$



$$= -i\lambda\delta^{ik} \tag{S.26}$$

and finally the $\pi\pi\pi\pi$ vertex similar to what we had in problem (2),



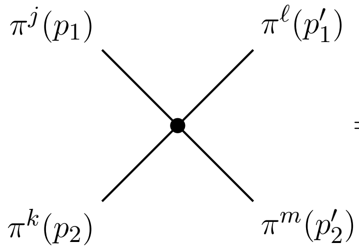
$$= -i\lambda\big(\delta^{jk}\delta^{\ell m} + \delta^{j\ell}\delta^{km} + \delta^{jm}\delta^{k\ell}\big). \tag{S.27}$$

This completes the Feynman rules of the linear sigma model.

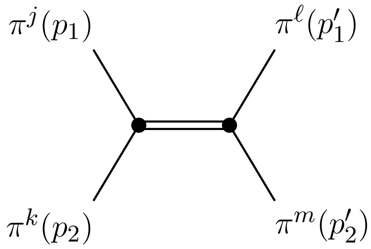
Problem **3**(b):

As explained in class, a tree diagram ($L = 0$) with $E = 4$ external lines has either (A) one valence $= 4$ vertex and no propagators, or else (B) two valence $= 3$ vertices and one propagator. Topologically, there are three diagrams of type (B) with different arrangements of incoming versus outgoing external lines, so altogether there are 4 tree diagrams.

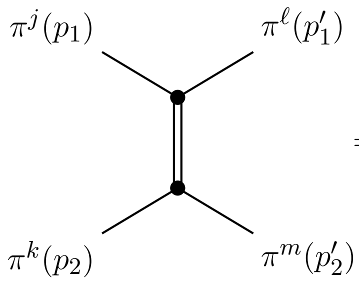
Specifically for the $\pi^j + \pi^k \to \pi^\ell + \pi^m$ scattering, the diagrams are



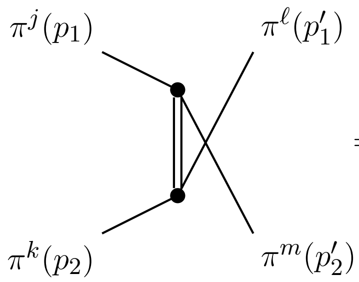
$$= \; -i\lambda\big(\delta^{jk}\delta^{\ell m} + \delta^{j\ell}\delta^{km} + \delta^{jm}\delta^{k\ell}\big), \tag{S.28}$$



$$= \; (-i\lambda f\delta^{jk})\,\frac{i}{s - M_\sigma^2}\,(-i\lambda f\delta^{\ell m}), \tag{S.29}$$



$$= \; (-i\lambda f\delta^{j\ell})\,\frac{i}{t - M_\sigma^2}\,(-i\lambda f\delta^{km}), \tag{S.30}$$



$$= \; (-i\lambda f\delta^{jm})\,\frac{i}{u - M_\sigma^2}\,(-i\lambda f\delta^{k\ell}), \tag{S.31}$$

where s, t, u are the Mandelstam variables

$$\begin{aligned} s &\stackrel{\text{def}}{=} (p_1 + p_2)^2 \;\equiv\; (p_1' + p_2')^2, \\ t &\stackrel{\text{def}}{=} (p_1' - p_1)^2 \;\equiv\; (p_2' - p_2)^2, \\ u &\stackrel{\text{def}}{=} (p_1' - p_2)^2 \;\equiv\; (p_2' - p_1')^2. \end{aligned} \tag{S.32}$$

Note that in the diagrams (S.29), (S.30), and (S.31), the internal line belongs to the σ field rather than to any π^i fields since there are no $\pi\pi\pi$ vertices but only $\pi\pi\sigma$.

Also, each of the diagrams (S.29), (S.30), and (S.31) yields a different combination of Kronecker $\delta\delta$ for the j, k, ℓ, m indices of the four pions, while the first diagram (S.28) yields all three combinations. So when we total up the four tree diagrams' amplitudes, it's convenient to reorganize the net tree amplitude by the j, k, ℓ, m indexology, thus

$$\begin{aligned} \mathcal{M}\big(\pi^j(p_1) + \pi^k(p_2) \to \pi^\ell(p_1') + \pi^m(p_2')\big) \;=\; & -\delta^{jk}\delta^{\ell m}\left(\lambda + \frac{\lambda^2 f^2}{s - M_\sigma^2}\right) \\ & - \delta^{j\ell}\delta^{km}\left(\lambda + \frac{\lambda^2 f^2}{t - M_\sigma^2}\right) \\ & - \delta^{jm}\delta^{k\ell}\left(\lambda + \frac{\lambda^2 f^2}{u - M_\sigma^2}\right). \end{aligned} \tag{S.33}$$

Problem **3**(c):

The Lagrangian (3) of the linear sigma models has a very important relation between the quartic coupling λ, the cubic coupling $\mu = \lambda f$, and the σ particle's mass $M_\sigma^2 = \lambda f^2$, thus

$$\big(M_\sigma^2 = \lambda f^2\big) \times \lambda \;=\; \big(\mu = \lambda f\big)^2. \tag{S.34}$$

Thanks to this relation,

$$\lambda + \frac{(\lambda f)^2}{s - M_\sigma^2} \;=\; \frac{\lambda s - \cancel{\lambda M_\sigma^2} + \cancel{(\lambda f)^2}}{s - M_\sigma^2} \;=\; \frac{\lambda s}{s - M_\sigma^2} \tag{S.35}$$

and likewise

$$\lambda + \frac{(\lambda f)^2}{t - M_\sigma^2} \;=\; \frac{\lambda t}{t - M_\sigma^2} \quad \text{and} \quad \lambda + \frac{(\lambda f)^2}{u - M_\sigma^2} \;=\; \frac{\lambda u}{u - M_\sigma^2}\,. \tag{S.36}$$

Thanks to these formulae, the scattering amplitude (S.33) simplifies to

$$\mathcal{M} \;=\; -\lambda\Big(\delta^{jk}\delta^{\ell m}\times\frac{s}{s-M_\sigma^2}\;+\;\delta^{j\ell}\delta^{km}\times\frac{t}{t-M_\sigma^2}\;+\;\delta^{jm}\delta^{k\ell}\times\frac{u}{u-M_\sigma^2}\Big). \qquad \text{(S.37)}$$

Now consider the low-energy limit of this amplitude. In the CM frame, all 4 pions have the same energy E, hence

$$s \;=\; (E_{\rm cm}^{\rm tot})^2 \;=\; 4E^2,\quad t \;=\; -4E^2\times\sin^2(\theta/2),\quad u \;=\; -4E^2\times\cos^2(\theta/2), \qquad \text{(S.38)}$$

and therefore

$$s,t,u \;=\; O(E^2). \qquad \text{(S.39)}$$

Consequently, when the pion's energies are much smaller than the σ particle's mass, the denominators in the amplitude (S.37) may be approximated as

$$\frac{1}{s-M_\sigma^2}\;\approx\;\frac{1}{t-M_\sigma^2}\;\approx\;\frac{1}{u-M_\sigma^2}\;\approx\;\frac{-1}{m_\sigma^2}\;=\;\frac{-1}{\lambda f^2}\,. \qquad \text{(S.40)}$$

Consequently, the scattering amplitude (S.37) simplifies to

$$\mathcal{M} \;=\; \left(\frac{+\lambda}{M_\sigma^2}\;=\;\frac{+1}{f^2}\right)\times\left(\delta^{jk}\delta^{\ell m}\times s\;+\;\delta^{j\ell}\delta^{km}\times t\;+\;\delta^{jm}\delta^{k\ell}\times u\;+\;O\left(\frac{E^4}{M_\sigma^2}\right)\right). \qquad \text{(S.41)}$$

The magnitude of this amplitude is generally $O(E^2/v^2)$, so in the low-energy limit it becomes quite small.

Now consider the $\pi\pi\to\pi\pi$ scattering in a completely general frame of reference. Since the pions are massless, Mandelstam's s,t,u variables may be written as

$$\begin{aligned} s &\stackrel{\rm def}{=} (p_1+p_2)^2 \;\equiv\; (p_1'+p_2')^2 \;=\; +2(p_1p_2) \;=\; +2(p_1'p_2'),\\ t &\stackrel{\rm def}{=} (p_1'-p_1)^2 \;\equiv\; (p_2'-p_2)^2 \;=\; -2(p_1'p_1) \;=\; -2(p_2'p_2),\\ u &\stackrel{\rm def}{=} (p_1'-p_2)^2 \;\equiv\; (p_2'-p_1')^2 \;=\; -2(p_1p_2') \;=\; -2(p_1'p_2), \end{aligned} \qquad \text{(S.42)}$$

so whenever any one of the four momenta becomes small, all 3 of the s,t,u become small. In particular, when 3 of the momenta are $O(M_\sigma)$ or smaller while the fourth momentum becomes much smaller, we have

$$s,t,u \;=\; O(M_\sigma\times p_{\rm smallest}) \;\ll\; M_\sigma^2. \tag{S.43}$$

Consequently, the scattering amplitude becomes as in eq. (S.41), and its magnitude is generally

$$\mathcal{M}\;\sim\;(s,t,u)\times\frac{\lambda}{M_\sigma^2}\;\lesssim\;P_{\rm smallest}\times\frac{\lambda}{M_\sigma}\,. \tag{S.44}$$

The physical reason for this behavior is the Goldstone theorem: Among other things, it says that *all scattering amplitudes involving Goldstone particles — such as the pions in this problem — become small as $O(p_\pi)$ when* **any** *Goldstone particle's momentum p_π becomes small.* A few lines above we saw how this works for the tree-level $\langle\pi,\pi|\,\mathcal{M}\,|\pi,\pi\rangle$ amplitude (S.37); the same behavior persists at all the higher orders of the perturbation theory, but seeing how *that* works is waaay beyond the scope of this exercise.

Problem **3**(d):

In the low-energy limit $E\ll M_\sigma$, the tree-level $\pi\pi\to\pi\pi$ amplitudes may be approximated as in eq. (S.41). In particular,

$$\begin{aligned}
\mathcal{M}(\pi^1+\pi^2\to\pi^1+\pi^2) \;&=\; \frac{\lambda t}{M_\sigma^2}\;+\;O\left(\frac{\lambda E^4}{M_\sigma^4}\right)\;\approx\;\frac{t}{f^2}\,,\\
\mathcal{M}(\pi^1+\pi^1\to\pi^2+\pi^2) \;&=\; \frac{\lambda s}{M_\sigma^2}\;+\;O\left(\frac{\lambda E^4}{M_\sigma^4}\right)\;\approx\;\frac{s}{f^2}\,,\\
\mathcal{M}(\pi^1+\pi^1\to\pi^1+\pi^1) \;&=\; \frac{\lambda(s+t+u)}{M_\sigma^2}\;+\;O\left(\frac{\lambda E^4}{M_\sigma^4}\right)\;=\;O\left(\frac{\lambda E^4}{M_\sigma^4}\right)\\
&\qquad\langle\!\langle\,\text{since } s+t+u=4m_\pi^2=0\,\rangle\!\rangle
\end{aligned} \tag{S.45}$$

Translating these amplitudes into the partial and total scattering cross-sections, we obtain

$$\begin{aligned}
\frac{d\sigma(\pi^1+\pi^2\to\pi^1+\pi^2)}{d\Omega_{\text{c.m.}}} &= \frac{1}{64\pi^2 s}\times\frac{t^2}{f^4} = \frac{E^2_{\text{c.m.}}}{64\pi^2 f^4}\times\sin^4\frac{\theta_{\text{c.m.}}}{2}\,, \\
\sigma_{\text{tot}}(\pi^1+\pi^2\to\pi^1+\pi^2) &= \frac{E^2_{\text{c.m.}}}{48\pi f^4}\,, \\
\frac{d\sigma(\pi^1+\pi^1\to\pi^2+\pi^2)}{d\Omega_{\text{c.m.}}} &= \frac{1}{64\pi^2 s}\times\frac{s^2}{f^4} = \frac{E^2_{\text{c.m.}}}{64\pi^2 f^4}\,, \\
\sigma_{\text{tot}}(\pi^1+\pi^1\to\pi^2+\pi^2) &= \frac{E^2_{\text{c.m.}}}{32\pi f^4}\,, \\
\frac{d\sigma(\pi^1+\pi^1\to\pi^1+\pi^1)}{d\Omega_{\text{c.m.}}} &= \frac{1}{64\pi^2 s}\times O\left(\frac{\lambda^2 E^8}{M_\sigma^8}\right) = O\left(\frac{E^6_{\text{c.m.}}}{f^4 M_\sigma^4}\right), \\
\sigma(\pi^1+\pi^1\to\pi^1+\pi^1) &= O\left(\frac{E^6_{\text{c.m.}}}{f^4 M_\sigma^4}\right) \ll \frac{E^2_{\text{c.m.}}}{f^4}\,.
\end{aligned} \tag{S.46}$$

For a more accurate approximation to the same-species scattering like $\pi^1+\pi^2\to\pi^1+\pi^1$, we need to go back to the amplitude (S.37) and expand it to second powers in s,t,u. Thus,

$$\begin{aligned}
-\frac{\lambda s}{s-M_\sigma^2} &\approx \frac{\lambda s}{M_\sigma^2} + \frac{\lambda s^2}{M_\sigma^4}\,, \\
-\frac{\lambda t}{t-M_\sigma^2} &\approx \frac{\lambda t}{M_\sigma^2} + \frac{\lambda t^2}{M_\sigma^4}\,, \\
-\frac{\lambda u}{u-M_\sigma^2} &\approx \frac{\lambda u}{M_\sigma^2} + \frac{\lambda u^2}{M_\sigma^4}\,,
\end{aligned} \tag{S.47}$$

and therefore

$$\begin{aligned}
\mathcal{M}(\pi^1+\pi^2\to\pi^1+\pi^1) &= -\frac{\lambda s}{s-M_\sigma^2} - \frac{\lambda t}{t-M_\sigma^2} - \frac{\lambda u}{u-M_\sigma^2} \\
&\approx \frac{\lambda}{M_\sigma^2}\times(s+t+u=0) + \frac{\lambda}{M_\sigma^4}\times(s^2+t^2+u^2).
\end{aligned} \tag{S.48}$$

In the center of mass frame,

$$\begin{aligned}
s^2+t^2+u^2 &= 16E^4 + 16E^4\times\sin^4(\theta/2) + 16E^4\times\cos^4(\theta/2) \\
&= 8E^4\times(3+\cos^2\theta) = \left(E_{\text{cm}}^{\text{tot}}\right)^4\times\frac{3+\cos^2\theta}{2}\,,
\end{aligned} \tag{S.49}$$

hence

$$\frac{d\sigma(\pi^1+\pi^1\to\pi^1+\pi^1)}{d\Omega_{\rm c.m.}} \approx \frac{\lambda^2 E_{\rm cm}^6}{256\pi^2 M_\sigma^8}\times(3+\cos^2\theta)^2\,, \tag{S.50}$$

and therefore

$$\sigma(\pi^1+\pi^1\to\pi^1+\pi^1) \approx \frac{7\lambda^2 E_{\rm cm}^6}{80\pi M_\sigma^8}\,. \tag{S.51}$$

Problem **4**(a):

As explained in my notes on phase space — and in much more detail in §4.5 of the *Peskin & Schroeder* textbook, — the partial rate of a decay process (in the rest frame of the initial particle) is given by

$$d\Gamma = \frac{1}{2M_0}\times\overline{|\mathcal{M}|^2}\times d\mathcal{P} \tag{S.52}$$

where $\mathcal{M}$ is the decay's amplitude, $\overline{|\mathcal{M}|^2}$ is $|\mathcal{M}|^2$ averaged over the unknown initial spins and summed over the unmeasured final spins, and $d\mathcal{P}$ is the infinitesimal phase space factor for the final particles. For three final particles,

$$d\mathcal{P} = \frac{d^3\mathbf{p}_1}{(2\pi)^3(2E_1)}\,\frac{d^3\mathbf{p}_2}{(2\pi)^3(2E_2)}\,\frac{d^3\mathbf{p}_3}{(2\pi)^3(2E_3)}\times(2\pi)^3\delta^{(3)}(\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3)\times(2\pi)\delta(E_1+E_2+E_3-M_0) \tag{S.53}$$

where the energy-momentum conservation laws apply in the rest frame, thus $\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3=\mathbf{p}_{\rm tot}=\mathbf{0}$ and $E_1+E_2+E_3=E_{\rm tot}=M_0$.

We start by using the momentum-conservation δ–function to eliminate the $\mathbf{p}_3$ as independent variable, thus

$$d\mathcal{P} = \frac{d^3\mathbf{p}_1\,d^3\mathbf{p}_2}{256\pi^5}\times\frac{\delta(E_1+E_2+E_3-E_{\rm tot})}{E_1E_2E_3}\bigg|_{\mathbf{p}_3=-(\mathbf{p}_1+\mathbf{p}_2)}\,. \tag{S.54}$$

Next, we use spherical coordinates for the two remaining momenta,

$$d^3\mathbf{p}_1 = p_1^2\,dp_1\,d^2\Omega_1\,,\qquad d^3\mathbf{p}_2 = p_2^2\,dp_2\,d^2\Omega_2\,, \tag{S.55}$$

and then replace the $d^2\Omega_2$ describing the direction of the second particle's momentum relative to the fixed external frame with

$$d^2\Omega_2^{(1)} \;=\; d\theta_{12}\,\sin\theta_{12}\,d\phi_2^{(1)}$$

describing the same direction of $\mathbf{p}_2$ relative to the frame centered on the $\mathbf{p}_1$. Consequently,

$$d^2\Omega_1\,d^2\Omega_2 \;=\; d^2\Omega_1\,d^2\Omega_2^{(1)} \;=\; \left[d^2\Omega_1\,d\phi_2^{(1)}\right]\,d\theta_{12}\,\sin\theta_{12} \;\equiv\; d^3\Omega\times d(\cos\theta_{12}) \tag{S.56}$$

and hence

$$d\mathcal{P} \;=\; \frac{d^3\Omega}{256\pi^5}\times\frac{p_1^2p_2^2}{E_1E_2E_3}\,dp_1\,dp_2\times d(\cos\theta_{12})\,\delta(E_1+E_2+E_3-E_{\rm tot})\Big|_{\mathbf{p}_3=-(\mathbf{p}_1+\mathbf{p}_2)}\,. \tag{S.57}$$

Next, we use the cosine theorem

$$p_3^2 \;=\; (\mathbf{p}_1+\mathbf{p}_2)^2 \;=\; p_1^2\,+\,p_2^2\,+\,2p_1p_2\cos\theta_{12}$$

which gives

$$d(\cos\theta_{12}) \;=\; \frac{p_3\,dp_3}{p_1\,p_2}$$

(for fixed p_1, p_2), and therefore

$$d\mathcal{P} \;=\; \frac{d^3\Omega}{256\pi^5}\times\frac{p_1p_2p_3}{E_1E_2E_3}\times dp_1\,dp_2\,dp_3\times\delta(E_1+E_2+E_3-E_{\rm tot}). \tag{S.58}$$

Finally, we notice that for a relativistic particle of any mass, $pdp = EdE$, hence

$$\frac{p_1p_2p_3}{E_1E_2E_3}\times dp_1\,dp_2\,dp_3 \;=\; dE_1\,de_2\,dE_3 \tag{S.59}$$

and therefore

$$d\mathcal{P} \;=\; \frac{d^3\Omega}{256\pi^5}\times dE_1\,dE_2\,dE_3\,\delta(E_1+E_2+E_3-E_{\rm tot}). \tag{S.60}$$

Plugging this formula into eq. (S.52) immediately gives us eq. (5) for the partial 3-body decay rate, *quod erat demonstrandum.*

Problem **4**(b):

The kinematic limits on the final particles' energies follow from the triangle inequalities for the magnitudes of three momentum vectors which add up to zero:

$$\mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3 = \mathbf{0} \quad \Longrightarrow \quad p_1 \le p_2 + p_3 \quad \textbf{and} \quad p_2 \le p_1 + p_3 \quad \textbf{and} \quad p_3 \le p_1 + p_2 . \tag{S.61}$$

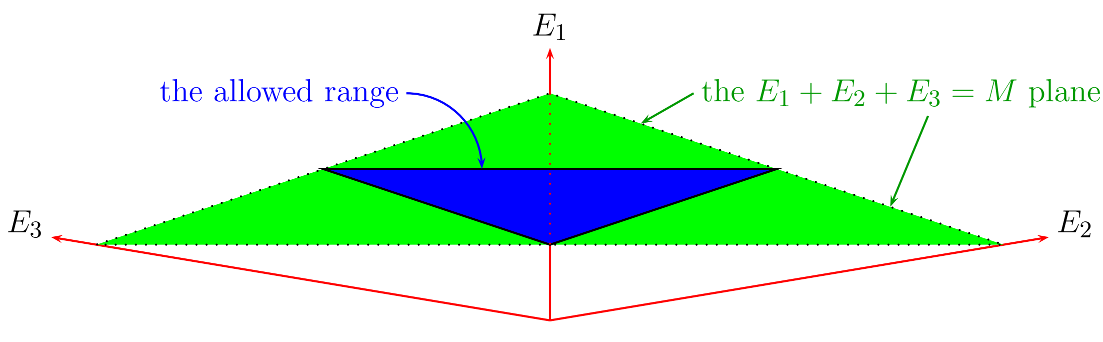
These inequalities look simple in terms of momenta but generally produce rather complicated inequalities for the energies $E_1 = \sqrt{p_1^2 + m_1^2}$, $E_2 = \sqrt{p_2^2 + m_2^2}$, and $E_3 = \sqrt{p_3^2 + m_3^2}$. However, when all three final particles are massless, the kinematic restrictions become simply

$$\begin{aligned} E_1 &\le E_2 + E_3 = M - E_1 , \\ E_2 &\le E_1 + E_3 = M - E_2 , \\ E_3 &\le E_1 + E_2 = M - E_3 , \end{aligned} \tag{S.62}$$

where the second expression on each right hand side follows from the net energy conservation $E_1 + E_2 + E_3 = M$. In other words, the kinematically allowed energies of the three final particles' range over

$$0 \le E_1, E_2, E_3 \le \tfrac{1}{2} M_0 , \qquad \text{while} \quad E_1 + E_2 + E_3 = M_0 . \tag{21}$$

The picture below shows this range in the (E_1, E_2, E_3) space:



(S.63)

Problem **4**(c):

In the muon's rest frame

$$(p_\mu \cdot p_{\bar\nu}) \;=\; M_\mu E_{\bar\nu} \tag{S.64}$$

while

$$\begin{aligned}
(p_e \cdot p_\nu) \;&=\; E_e E_\nu \;-\; p_e p_\nu \cos\theta_{e\nu} \\
\langle\!\langle \text{ by the cosine theorem } \rangle\!\rangle \\
&=\; E_e E_\nu \;+\; \tfrac{1}{2}p_e^2 \;+\; \tfrac{1}{2}p_\nu^2 \;-\; \tfrac{1}{2}p_{\bar\nu}^2 \\
\langle\!\langle \text{ neglecting } m_e, m_\nu, m_{\bar\nu} \rangle\!\rangle \\
&\approx\; E_e E_\nu \;+\; \tfrac{1}{2}E_e^2 \;+\; \tfrac{1}{2}E_\nu^2 \;-\; \tfrac{1}{2}E_{\bar\nu}^2 \qquad \text{(S.65)} \\
&=\; \tfrac{1}{2}(E_e + E_\nu)^2 \;-\; \tfrac{1}{2}E_{\bar\nu}^2 \\
\langle\!\langle \text{ using } E_e + E_\nu = M_\mu - E_{\bar\nu} \rangle\!\rangle \\
&=\; \tfrac{1}{2}(M_\mu - E_{\bar\nu})^2 \;-\; \tfrac{1}{2}E_{\bar\nu}^2 \\
&=\; \tfrac{1}{2}M_\mu(M_\mu - 2E_{\bar\nu}).
\end{aligned}$$

Consequently, the spin-averaged muon decay amplitude^2 (4) becomes

$$\overline{|\mathcal{M}|^2} \;=\; 32 G_F^2 M_\mu^2 E_{\bar\nu}(M_\mu - 2E_{\bar\nu}). \tag{S.66}$$

Plugging this formula into eq. (19) for the decay rate gives us

$$d\Gamma(\mu^- \to e^- \nu_\mu \bar\nu_e) \;=\; \frac{G_F^2}{16\pi^5}\, M_\mu E_{\bar\nu}(M_\mu - 2E_{\bar\nu}) \times dE_e\, dE_\nu\, dE_{\bar\nu}\, d^3\Omega\, \delta(E_e + E_\nu + E_{\bar\nu} - M_\mu), \tag{S.67}$$

and all we need to do now is to integrate this formula over the final-state variables.

The integration variables comprise 3 angles $d^3\Omega$ — which integrate to $\int d^3\Omega = 8\pi^2$ — and 3 particles' energies subject to the constraint $E_e + E_\nu + E_{\bar\nu} = M_\mu$ and the kinematic limits (20). Integrating the decay rate (S.67) over these variables, we have

$$
\begin{aligned}
\Gamma &= \frac{G_F^2 M_\mu}{2\pi^3} \int_0^{M_\mu/2} dE_e \int_0^{M_\mu/2} dE_{\bar\nu} E_{\bar\nu}(M_\mu - 2E_{\bar\nu}) \times \int_0^{M_\mu/2} dE_\nu\, \delta(E_e + E_\nu + E_{\bar\nu} - M_\mu) \\
&= \frac{G_F^2 M_\mu}{2\pi^3} \int_0^{M_\mu/2} dE_e \int_0^{M_\mu/2} dE_{\bar\nu} E_{\bar\nu}(M_\mu - 2E_{\bar\nu}) \times \text{restrict to}(E_\nu = M - E_e - E_{\bar\nu} \le \tfrac{1}{2}M) \\
&= \frac{G_F^2 M_\mu}{2\pi^3} \int_0^{\frac{1}{2}M_\mu} dE_e \int_{\frac{1}{2}M_\mu - E_e}^{\frac{1}{2}M_\mu} dE_{\bar\nu}\, E_{\bar\nu}(M_\mu - 2E_{\bar\nu}) \\
&\qquad \langle\!\langle \text{ where the lower limit of the } \textstyle\int dE_{\bar\nu} \text{ comes from } E_\nu \le \tfrac{1}{2}M_\mu \Longrightarrow E_e + E_{\bar\nu} \ge \tfrac{1}{2}M_\mu \,\rangle\!\rangle \\
&= \frac{G_F^2 M_\mu}{2\pi^3} \int_0^{\frac{1}{2}M_\mu} dE_e \left(\tfrac{1}{2}M_\mu E_e^2 E_e^2 \;-\; \tfrac{2}{3}E_e^3\right).
\end{aligned}
\tag{S.68}
$$

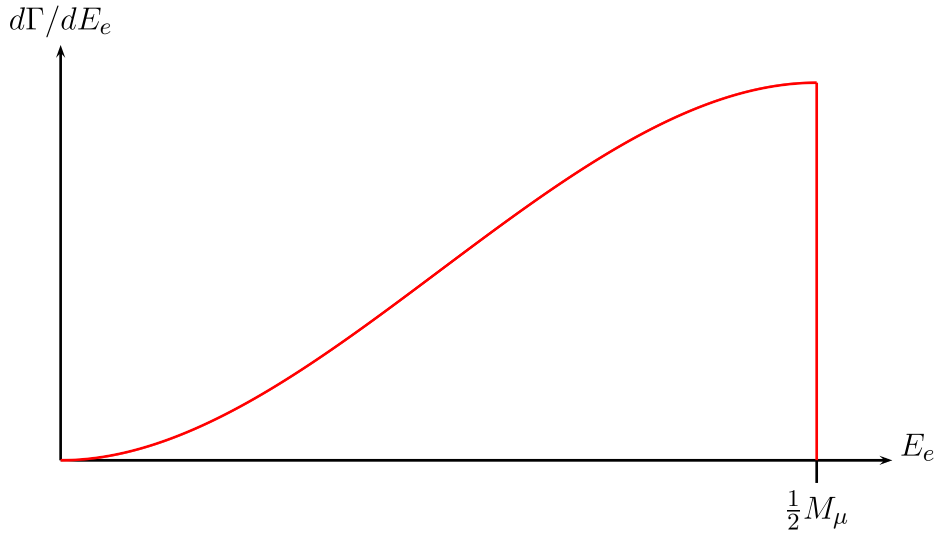
In other words, the partial muon decay rate with respect to the final electron's energy is given by

$$
\frac{d\Gamma}{dE_e} \;=\; \frac{G_F^2 M_\mu}{12\pi^3} \times E_e^2(3M_\mu - 4E_e) \tag{S.69}
$$

or rather

$$
\frac{d\Gamma}{dE_e} \;\approx \begin{cases} \frac{G_F^2}{12\pi^3}\, M_\mu E_e^2(3M_\mu - 4E_e) & \text{for } E_e < \tfrac{1}{2}M_\mu, \\ 0 & \text{for } E_e > \tfrac{1}{2}M_\mu. \end{cases} \tag{S.70}
$$

Graphically,



Note how this curve smoothly reaches its maximum at $E_e = \frac{1}{2}M_\mu$ and then abruptly falls down to zero.

It remains to calculate the total decay rate of the muon by integrating the partial rate (S.70) over the electron's energy. The result is

$$\Gamma_{\rm tot}(\mu \to e\nu\bar{\nu}) \;=\; \frac{G_F^2 M_\mu}{12\pi^3} \times \int\limits_0^{\frac{1}{2}M_\mu} dE_e\, E_e^2(3M_\mu - 4E_e) \;=\; \frac{G_F^2 M_\mu^5}{192\pi^3}\,. \qquad \text{(S.71)}$$