Problem **1**(a):

In the first diagram (1), the virtual photon has momentum $q = p_1' - p_1 = p_2 - p_2'$, hence $q^2 = t$. In the second diagram, the virtual photon's momentum is $\tilde{q} = p_1 + p_2 = p_1' + p_2'$, hence $\tilde{q}^2 = s$. Accordingly, the two diagrams are called the s-channel diagram and the t-channel diagram.

The t-channel diagram evaluates to

$$\begin{aligned} i\mathcal{M}_1 &= -\Big(\bar{v}(e^+)(ie\gamma_\mu)v(e^{+\prime})\Big) \times \Big(\bar{u}(e^{-\prime})(ie\gamma_\nu)u(e^-)\Big) \times \frac{-ig^{\mu\nu}}{q^2} \\ &= \frac{-ie^2}{t} \times \bar{v}(e^+)\gamma_\mu v(e^{+\prime}) \times \bar{u}(e^{-\prime})\gamma^\mu u(e^-) \end{aligned} \tag{S.1}$$

where the overall minus sign is due to the positron-out to positron-in fermionic line. And the s-channel diagram evaluates to

$$\begin{aligned} i\mathcal{M}_2 &= +\Big(\bar{v}(e^+)(ie\gamma_\mu)u(e^-)\Big) \times \Big(\bar{u}(e^{-\prime})(ie\gamma_\nu)v(e^{+\prime}\Big) \times \frac{-ig^{\mu\nu}}{\tilde{q}^2} \\ &= \frac{+ie^2}{s} \times \bar{v}(e^+)\gamma_\mu u(e^-) \times \bar{u}(e^{-\prime})\gamma^\mu v(e^{+\prime}) \end{aligned} \tag{S.2}$$

where the overall sign is plus because both fermionic lines have an incoming or outgoing electron at one end.

Problem **1**(b):

Summing /averaging the $|\mathcal{M}_2|^2$ over spins works exactly as for the muon pair production discussed in class:

$$\begin{aligned} \sum_{\text{spins}} |\mathcal{M}_2|^2 &= \left(\frac{e^2}{s}\right)^2 \sum_{\text{spins}} \Big[\bar{v}(e^+)\gamma_\mu u(e^-) \times \bar{u}(e^-)\gamma_\nu v(e^+)\Big] \times \Big[\bar{u}(e^{-\prime})\gamma^\mu v(e^{+\prime}) \times \bar{v}(e^{+\prime})\gamma^\nu u(e^{-\prime})\Big] \\ &= \left(\frac{e^2}{s}\right)^2 \operatorname{tr}\left[(\not{p}_2 - m)\gamma_\mu(\not{p}_1 + m)\gamma_\nu\right] \times \operatorname{tr}\left[(\not{p}_1' - m)\gamma^\mu(\not{p}_2' - m)\gamma^\nu\right] \\ &\qquad \langle\!\langle \text{ neglecting the mass relative to the momenta } \rangle\!\rangle \\ &\approx \left(\frac{e^2}{s}\right)^2 \operatorname{tr}\left[\not{p}_2\gamma_\mu \not{p}_1\gamma_\nu\right] \times \operatorname{tr}\left[\not{p}_1'\gamma^\mu \not{p}_2'\gamma^\nu\right] \end{aligned} \tag{S.3}$$

$$
\begin{aligned}
&= \left(\frac{e^2}{s}\right)^2 \times 4\left[p_{2\mu}p_{1\nu} \,+\, p_{2\nu}p_{1\mu} \,-\, g_{\mu\nu}(p_2p_1)\right] \times 4\left[p_2'^{\mu}p_1'^{\nu} \,+\, p_2'^{\nu}p_1'^{\mu} \,-\, g^{\mu\nu}(p_2'p_1')\right] \\
&= 16\left(\frac{e^2}{s}\right)^2 \Big[2(p_2'p_2)(p_1'p_1) \,+\, 2(p_2'p_1)(p_1'p_2). \\
&\qquad\qquad -\, 2(p_2'p_1')(p_2p_1) \,-\, 2(p_2'p_1')(p_2p_1) \,+\, 4(p_2'p_1')(p_2p_1)\Big] \\
&= 32\left(\frac{e^2}{s}\right)^2 \left[(p_2'p_2)(p_1'p_1) \,+\, (p_2'p_1)(p_1'p_2)\right] \\
&= 8\left(\frac{e^2}{s}\right)^2 \left[t^2 \,+\, u^2\right] \qquad\qquad \text{(S.3)}
\end{aligned}
$$

where the last equality follows from the kinematic relations (4). Altogether,

$$
\frac{1}{4}\sum_{\text{spins}} |\mathcal{M}_2|^2 \;=\; 2e^4 \times \frac{t^2+u^2}{s^2}\,. \qquad (5)
$$

Problem **1**(c):

The two diagrams for Bhabha scattering are related by the *crossing symmetry*, so the amplitudes $\mathcal{M}_1$ and $\mathcal{M}_2$ are related to each other via analytic continuation of particle's momenta. In terms of the spin-summed $|\mathcal{M}|^2$ and Mandelstam variables,

$$
\sum_{\text{spins}} |\mathcal{M}_1(s,t,u)|^2 \;=\; \sum_{\text{spins}} |\mathcal{M}_2(t,s,u)|^2, \qquad \text{(S.4)}
$$

hence eq. (5) for the second amplitude implies a similar equation for the first amplitude, but with s and t exchanged with each other — *i.e.*, eq. (6).

Alternatively, we may sum the $|\mathcal{M}_1|^2$ over all the spins in the same way as we summed the $|\mathcal{M}_2|^2$ in part (b):

$$
\begin{aligned}
\sum_{\text{spins}} |\mathcal{M}_1|^2 &= \left(\frac{e^2}{t}\right)^2 \sum_{\text{spins}} \left[\bar u(e^{-\prime})\gamma^\mu u(e^-) \times \bar u(e^-)\gamma^\nu u(e^{-\prime})\right] \times \left[\bar v(e^+)\gamma_\mu v(e^{+\prime}) \times \bar v(e^{+\prime})\gamma_\nu v(e^+)\right] \\
&= \left(\frac{e^2}{t}\right)^2 \operatorname{tr}\left[(\not p_1' + m)\gamma^\mu(\not p_1 + m)\gamma^\nu\right] \times \operatorname{tr}\left[(\not p_2 - m)\gamma_\mu(\not p_2' - m)\gamma_\nu\right]
\end{aligned}
$$

$$\approx \left(\frac{e^2}{t}\right)^2 \operatorname{tr}\left[\not{p}_1'\gamma^\mu \not{p}_1\gamma^\nu\right] \times \operatorname{tr}\left[\not{p}_2\gamma_\mu \not{p}_2'\gamma_\nu\right] \tag{S.5}$$

$$= \left(\frac{e^2}{t}\right)^2 \times 4\left[p_1'^\mu p_1^\nu + p_1'^\nu p_1^\mu - g^{\mu\nu}(p_1'p_1)\right] \times 4\left[p_{2\mu}'p_{2\nu} + p_{2\nu}'p_{2\mu} - g_{\mu\nu}(p_2'p_2)\right]$$

$$= 16\left(\frac{e^2}{t}\right)^2 \Big[2(p_1'p_2')(p_1p_2) + 2(p_1'p_2)(p_1p_2')$$

$$- 2(p_1'p_1)(p_2'p_2) - 2(p_1'p_1)(p_2'p_2) + 4(p_1'p_1)(p_2'p_2)\Big]$$

$$= 32\left(\frac{e^2}{t}\right)^2 \left[(p_1'p_2')(p_1p_2) + (p_1'p_2)(p_1p_2')\right]$$

$$= 8\left(\frac{e^2}{t}\right)^2 \left[s^2 + u^2\right] \tag{S.5}$$

and hence

$$\frac{1}{4}\sum_{\text{spins}} |\mathcal{M}_1|^2 = 2e^4 \times \frac{s^2+u^2}{t^2}\,. \tag{6}$$

Problem **1**(d):

The interference term between the two diagrams is more complicated:

$$\mathcal{M}_1^* \times \mathcal{M}_2 = -\frac{e^2}{t}\Big(\bar{u}(e^-)\gamma^\nu u(e^{-\prime}) \times \bar{v}(e^{+\prime})\gamma_\nu v(e^+)\Big) \times$$

$$\times \frac{e^2}{s}\Big(\bar{v}(e^+)\gamma_\mu u(e^-) \times \bar{u}(e^{-\prime})\gamma^\mu v(e^{+\prime})\Big)$$

$$= -\frac{e^4}{st} \times \bar{u}(e^-)\gamma^\nu u(e^{-\prime}) \times \bar{u}(e^{-\prime})\gamma^\mu v(e^{+\prime}) \times \bar{v}(e^{+\prime})\gamma_\nu v(e^+) \times \bar{v}(e^+)\gamma_\mu u(e^-) \tag{S.6}$$

where on the last line I have re-ordered the factors so that each $\bar{u}$ is followed by u of the same electron and each $\bar{v}$ is followed by v for the same positron. After summing over all the spins, each $u \times \bar{u}$ becomes $(\not{p} + m)$, each $v \times \bar{v}$ becomes $(\not{p} - m)$, and the whole product becomes a single big trace rather than a product of two traces,

$$\sum_{\text{spins}} \mathcal{M}_1^* \times \mathcal{M}_2 = -\frac{e^4}{st} \times \operatorname{tr}\Big[(\not{p}_1 + m)\gamma^\nu(\not{p}_1' + m)\gamma^\mu(\not{p}_2' - m)\gamma_\nu(\not{p}_2 - m)\gamma_\mu\Big]$$

$$\approx -\frac{e^4}{st} \times \operatorname{tr}\Big[\not{p}_1\gamma^\nu \not{p}_1'\gamma^\mu \not{p}_2'\gamma_\nu \not{p}_2\gamma_\mu\Big]. \tag{S.7}$$

This trace looks more complicated than it is, and we may drastically simplify it by summing over ν and μ before taking the trace. Back in homework#6 we saw that

$$\gamma^\alpha \not{a}\,\not{b}\,\not{c}\gamma_\alpha \;=\; -2\,\not{c}\,\not{b}\,\not{a} \quad\text{and}\quad \gamma^\alpha \not{a}\,\not{b}\gamma_\alpha \;=\; 4(ab). \tag{S.8}$$

For the problem at hand, this gives us $\gamma^\nu \not{p}_1'\gamma^\mu \not{p}_2'\gamma_\nu = -2\,\not{p}_2'\gamma^\mu \not{p}_1'$ and hence

$$\begin{aligned}
\operatorname{tr}\Bigl[\not{p}_1\times\gamma^\nu\not{p}_1'\gamma^\mu\not{p}_2'\gamma_\nu\times\not{p}_2\gamma_\mu\Bigr] &= -2\operatorname{tr}\Bigl[\not{p}_1\times\not{p}_2'\gamma^\mu\not{p}_1'\times\not{p}_2\gamma_\mu\Bigr] \;=\; -2\operatorname{tr}\Bigl[\not{p}_1\not{p}_2'\times\gamma^\mu\not{p}_1'\not{p}_2\gamma_\mu\Bigr]\\
&= -2\operatorname{tr}\Bigl[\not{p}_1\not{p}_2'\times 4(p_1'p_2)\Bigr] \;=\; -8(p_1'p_2)\times\operatorname{tr}\Bigl[\not{p}_1\not{p}_2'\Bigr]\\
&= -8(p_1'p_2)\times 4(p_1p_2')\\
&= -8u^2.
\end{aligned} \tag{S.9}$$

Plugging this trace back into eq. (S.6), we arrive at

$$\frac{1}{4}\sum_{\text{spins}}\mathcal{M}_1^*\times\mathcal{M}_2 \;=\; +2e^4\times\frac{u^2}{st}\,. \tag{7}$$

<u>Problem **1**(e):</u>

Assembling the spin sums / averages (5–7) together according to eq. (3), we get

$$\begin{aligned}
\overline{|\mathcal{M}|^2} &\stackrel{\text{def}}{=} \frac{1}{4}\sum_{\text{spins}}|\mathcal{M}_1+\mathcal{M}_2|^2\\
&= \frac{1}{4}\sum_{\text{spins}}\Bigl(|\mathcal{M}_1|^2\,+\,|\mathcal{M}_2|^2\,+\,2\operatorname{Re}\mathcal{M}_1^*\mathcal{M}_2\Bigr)\\
&= 2e^4\times\frac{s^2+u^2}{t^2}\;+\;2e^4\times\frac{t^2+u^2}{s^2}\;+\;4e^4\times\frac{u^2}{st}\\
&= 2e^4\left(\frac{s^2}{t^2}\,+\,\frac{t^2}{s^2}\,+\,\frac{u^2}{s^2t^2}\times\Bigl(s^2+t^2+2st\;=\;(s+t)^2\;=\;u^2\Bigr)\right)\\
&= 2e^4\times\frac{s^4+t^4+u^4}{s^2\times t^2}\,.
\end{aligned} \tag{S.10}$$

Consequently, the un-polarized partial cross-section for the Bhabha scattering is

$$\frac{d\sigma}{d\Omega_{\text{c.m.}}} \;=\; \frac{\overline{|\mathcal{M}|^2}}{64\pi^2\,s} \;=\; \frac{\alpha^2}{2s}\times\frac{s^4+t^4+u^4}{s^2\times t^2}\,. \tag{8.a}$$

To complete the problem, let's work out the kinematics in the center of mass frame:

$$\begin{aligned} s &= 4E^2 \approx 4\mathbf{p}^2, \\ t &= -(\mathbf{p}'_1 - \mathbf{p}_1)^2 = -2\mathbf{p}^2(1-\cos\theta), \\ u &= -(\mathbf{p}'_2 - \mathbf{p}_1)^2 = -2\mathbf{p}^2(1+\cos\theta), \end{aligned} \tag{S.11}$$

hence

$$\begin{aligned} \frac{s^4+t^4+u^4}{s^2\,t^2} &= \frac{(4\mathbf{p}^2)^4 + (2\mathbf{p}^2)^4 \times (1-\cos\theta)^4 + (2\mathbf{p}^2)^4 \times (1+\cos\theta)^4}{(4\mathbf{p}^2)^2 \times (2\mathbf{p}^2)^2 (1-\cos\theta)^2} \\ &= \frac{16 + (1-\cos\theta)^4 + (1+\cos\theta)^4}{4\times(1-\cos\theta)^2} = \frac{18 + 12\cos^2\theta + 2\cos^4\theta}{4\times(1-\cos\theta)^2} \\ &= \frac{(3+\cos^2\theta)^2}{2(1-\cos\theta)^2}\,. \end{aligned} \tag{S.12}$$

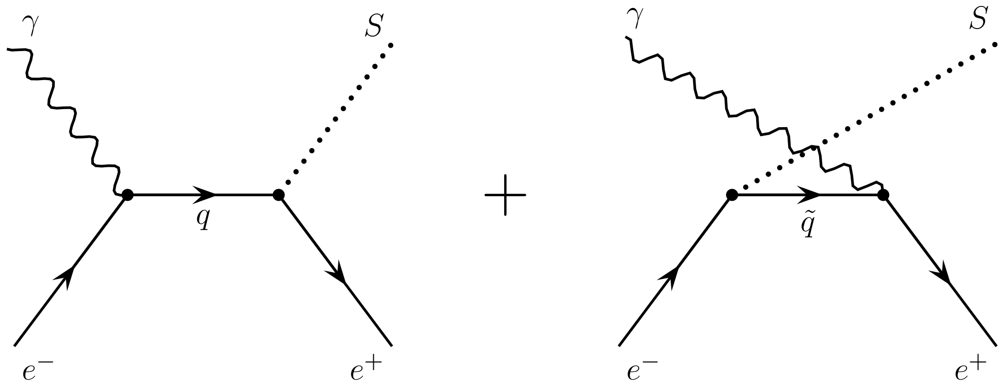
Plugging this formula into eq. (8.a), we finally obtain

$$\frac{d\sigma^{\rm Bhabha}}{d\Omega_{\rm cm}} = \frac{\alpha^2}{4s} \times \frac{(3+\cos^2\theta)^2}{(1-\cos\theta)^2}\,. \tag{8.b}$$

Quod erat demonstrandum.

Problem **3**(a):

There are two tree diagrams for the $e^-e^+ \to S\gamma$ process, namely



(S.13)

These two diagrams are related by $t \leftrightarrow u$ crossing, and also by the charge conjugation (which exchanges the initial e^- and e^+). The net tree-level amplitude is

$$\mathcal{M}_{\rm tree} = \mathcal{E}^{*\mu}_{\mathbf{k},\lambda}(\gamma) \times \mathcal{M}_\mu\,, \tag{S.14.a}$$

$$\mathcal{M}^\mu = \mathcal{M}_1^\mu + \mathcal{M}_2^\mu, \tag{S.14.b}$$

$$\mathcal{M}_1^\mu = -i\,\bar{v}(e^+)(-ig)\frac{i}{\not{q}-m_e}\,(ie\gamma^\mu)\,u(e^-)$$

$$= \frac{eg}{t-m^2}\times\bar{v}(\not{q}+m_e)\gamma^\mu u\,, \tag{S.14.c}$$

$$\mathcal{M}_2^\mu = -i\,\bar{v}(e^+)\,(ie\gamma^\mu)\,\frac{i}{\tilde{\not{q}}-m_e}(-ig)u(e^-)$$

$$= \frac{eg}{u-m^2}\times\bar{v}\gamma^\mu(\tilde{\not{q}}+m_e)u\,, \tag{S.14.d}$$

where

$$\begin{aligned} q &= p_- - k_\gamma = k_s - p_+\,, \quad q^2 = t, \\ \text{and}\quad \tilde{q} &= p_- - k_s = k_\gamma - p_+\,, \quad \tilde{q}^2 = u. \end{aligned} \tag{S.15}$$

Problem **3**(b):

The Ward identity for the one-photon amplitude (S.14.a) says $k_\gamma^\mu\times\mathcal{M}_\mu = 0$. To verify it, let's start with the first diagram:

$$\begin{aligned} k_\gamma^\mu\times\bar{v}(\not{q}+m_e)\gamma_\mu u &= \bar{v}(\not{q}+m_e)\,\not{k}_\gamma u \\ &= \bar{v}(\not{p}_- - \not{k}_\gamma + m_e)\,\not{k}_\gamma u \\ &= \bar{v}(\not{p}_- + m_e)\,\not{k}_\gamma u \qquad \langle\!\langle\, \text{because } \not{k}_\gamma\,\not{k}_\gamma = k_\gamma^2 = 0\,\rangle\!\rangle \\ &= \bar{v}\Big(2(p_-k_\gamma) - \not{k}_\gamma(\not{p}_- - m_e)\Big)u \quad \langle\!\langle\, \text{anticommuting } \not{p}_- \text{ and } \not{k}_\gamma\,\rangle\!\rangle \\ &= 2(p_-k_\gamma)\times\bar{v}u - 0 \qquad \langle\!\langle\, \text{because } (\not{p}_- - m_e)\times u(e^-) = 0\,\rangle\!\rangle \\ &= (m_e^2 - t)\times\bar{v}u\,, \end{aligned} \tag{S.16}$$

and hence

$$k_\gamma^\mu\times\mathcal{M}_{1\mu} = -eg\times\bar{v}u\,. \tag{S.17}$$

We see that *by itself*, the first diagram does not satisfy the Ward entity. Instead, we need to add the second diagram's contribution

$$\begin{aligned}
k_\gamma^\mu \times \bar{v}\gamma_\mu(\not{q}+m_e)u \ &= \ \bar{v}\,\not{k}_\gamma(\not{q}+m_e)u \\
&= \ \bar{v}\,\not{k}_\gamma(\not{k}_\gamma - \not{p}_+ + m_e)u \\
&= \ \bar{v}\,\not{k}_\gamma(-\not{p}_+ + m_e)u \qquad \langle\!\langle \text{ because } \not{k}_\gamma\not{k}_\gamma = k_\gamma^2 = 0 \,\rangle\!\rangle \\
&= \ \bar{v}\Big(-2(p_+k_\gamma) + \not{k}_\gamma(\not{p}_+ + m_e)\Big)u \quad \langle\!\langle \text{ anticommuting } \not{p}_+ \text{ and } \not{k}_\gamma \,\rangle\!\rangle \\
&= \ -2(p_+k_\gamma)\times\bar{v}u \ + \ 0 \qquad \langle\!\langle \text{ because } \bar{v}(e^+)\times(\not{p}_+ + m_e) = 0 \,\rangle\!\rangle \\
&= \ (u-m_e^2)\times\bar{v}u\,,
\end{aligned} \tag{S.18}$$

and hence

$$k_\gamma^\mu \times \mathcal{M}_{2\mu} \ = \ +eg\times\bar{v}u\,. \tag{S.19}$$

Again, the second diagram does not satisfy the Ward identity *by itself*, but the net amplitude does:

$$k_\gamma^\mu \times (\mathcal{M}_\mu \ = \ \mathcal{M}_{1\mu} \ + \ \mathcal{M}_{2\mu}) \ = \ 0. \tag{S.20}$$

Problem **3**(c):

Thanks to the Ward identity, summing $|\mathcal{M}|^2$ over the photon's polarizations is easy:

$$\begin{aligned}
\sum_\lambda |\mathcal{M}|^2 \ &= \ -\mathcal{M}^\mu\mathcal{M}^*_\mu \qquad \langle\!\langle \text{ see my notes on Ward identities } \rangle\!\rangle \\
&= \ -\mathcal{M}_1^\mu\mathcal{M}^*_{1\mu} \ - \ \mathcal{M}_2^\mu\mathcal{M}^*_{2\mu} \ - \ 2\,\mathrm{Re}\left(\mathcal{M}_1^\mu\mathcal{M}^*_{2\mu}\right) \\
&= \ -\frac{e^2g^2}{(t-m_e^2)^2}\times\bar{v}(\not{q}+m_e)\gamma^\mu u\times\bar{u}\gamma_\mu(\not{q}+m_e)v \\
&\quad - \ \frac{e^2g^2}{(u-m_e^2)^2}\times\bar{v}\gamma^\mu(\not{q}+m_e)u\times\bar{u}(\not{q}+m_e)\gamma_\mu v \\
&\quad - \ \frac{2e^2g^2}{(t-m_e^2)(u-m_e^2)}\times\mathrm{Re}\Big(\bar{v}(\not{q}+m_e)\gamma^\mu u\times\bar{u}(\not{q}+m_e)\gamma_\mu v\Big).
\end{aligned} \tag{S.21}$$

Consequently, averaging this formula over the electron's and the positron's spins yields

$$\overline{|\mathcal{M}|^2} \ = \ \frac{1}{4}\sum_{s_-,s_+}\sum_\lambda |\mathcal{M}|^2 \ = \ e^2g^2\left(\frac{A_{11}}{(t-m_2^2)^2} \ + \ \frac{A_{22}}{(u-m_e^2)^2} \ + \ \frac{2\,\mathrm{Re}\,A_{12}}{(t-m_e^2)(u-m_e^2)}\right) \tag{10}$$

where

$$
\begin{aligned}
A_{11} &= \frac{1}{4}\sum_{s_-,s_+} \bar{v}(\not{q}+m_e)\gamma^\mu u \times \bar{u}\gamma_\mu(\not{q}+m_e)v, \\
A_{22} &= \frac{1}{4}\sum_{s_-,s_+} \bar{v}\gamma^\mu(\tilde{\not{q}}+m_e)u \times \bar{u}(\tilde{\not{q}}+m_e)\gamma_\mu v, \\
A_{12} &= \frac{1}{4}\sum_{s_-,s_+} \bar{v}(\not{q}+m_e)\gamma^\mu u \times \bar{u}(\tilde{\not{q}}+m_e)\gamma_\mu v.
\end{aligned}
\tag{S.22}
$$

At this point, we use the spin sums

$$
\sum_{s_-} u \times \bar{u} = (\not{p}_- + m_e), \quad \sum_{s_+} v \times \bar{v} = (\not{p}_+ - m_e) \tag{S.23}
$$

to convert eqs. (S.22) to Dirac traces (11):

$$
\begin{aligned}
A_{11} &= \frac{1}{4}\,\mathrm{Tr}\left(\left(\sum\nolimits_{s_+} v\times\bar{v}\right)(\not{q}+m_e)\gamma^\mu\left(\sum\nolimits_{s_-} u\times\bar{u}\right)\gamma_\mu(\not{q}+m_e)\right) \\
&= \frac{1}{4}\,\mathrm{Tr}\Big((\not{p}_+ - m_e)(\not{q}+m_e)\gamma^\mu(\not{p}_- + m_e)\gamma_\mu(\not{q}+m_e)\Big),
\end{aligned}
\tag{S.24}
$$

and likewise

$$
\begin{aligned}
A_{22} &= \frac{1}{4}\,\mathrm{Tr}\left(\left(\sum\nolimits_{s_+} v\times\bar{v}\right)\gamma^\mu(\tilde{\not{q}}+m_e)\left(\sum\nolimits_{s_-} u\times\bar{u}\right)(\tilde{\not{q}}+m_e)\gamma_\mu\right) \\
&= \frac{1}{4}\,\mathrm{Tr}\Big((\not{p}_+ - m_e)\gamma^\mu(\tilde{\not{q}}+m_e)(\not{p}_- + m_e)(\tilde{\not{q}}+m_e)\gamma_\mu\Big),
\end{aligned}
\tag{S.25}
$$

$$
\begin{aligned}
A_{12} &= \frac{1}{4}\,\mathrm{Tr}\left(\left(\sum\nolimits_{s_+} v\times\bar{v}\right)(\not{q}+m_e)\gamma^\mu\left(\sum\nolimits_{s_-} u\times\bar{u}\right)(\tilde{\not{q}}+m_e)\gamma_\mu\right) \\
&= \frac{1}{4}\,\mathrm{Tr}\Big((\not{p}_+ - m_e)(\not{q}+m_e)\gamma^\mu(\not{p}_- + m_e)(\tilde{\not{q}}+m_e)\gamma_\mu\Big).
\end{aligned}
\tag{S.26}
$$

Quod erat demonstrandum.

Problem **3**(d):

Evaluating the Dirac traces (11) is straightforward but tedious. Fortunately, it becomes much simpler when we neglect the electron's mass. In that limit, the first trace becomes

$$\begin{aligned} A_{11} &\approx -\tfrac{1}{4}\,\mathrm{Tr}(\not{p}_+\,\not{q}\gamma^\mu\,\not{p}_-\gamma_\mu\,\not{q}) \\ &= +\tfrac{1}{2}\,\mathrm{Tr}(\not{p}_+\,\not{q}\,\not{p}_-\,\not{q}) \qquad \langle\!\langle\,\text{using } \gamma^\mu\,\not{p}_-\gamma_\mu = -2\,\not{p}_-\,\rangle\!\rangle \\ &= 4(p_+q)(p_-q)\;-\;2(p_+p_-)\,q^2 \\ &\approx (M_s^2-t)\times t\;-\;s\times t\;=\;(M_s^2-t-s)\times t \\ &\approx u\times t\,, \end{aligned} \tag{S.27}$$

where the last two lines follow from

$$\begin{aligned} q^2 &= t, \\ p_+p_- &= \tfrac{1}{2}(p_-+p_+)^2 - \cancel{m_e^2} \approx \frac{s}{2}\,, \\ p_-q &= p_-(p_--k_\gamma) = \tfrac{1}{2}(p_--k_\gamma)^2 + \cancel{\tfrac{1}{2}m_e^2} \approx \frac{t}{2}\,, \\ p_+q &= p_+(k_S-p_+) = -\tfrac{1}{2}(p_+-k_S)^2 + \tfrac{1}{2}M_s^2 - \cancel{\tfrac{1}{2}m_e^2} \approx \frac{M_s^2-t}{2}\,, \\ s+t+u &= M_s^2 + \cancel{2m_e^2} \approx M_s^2\,. \end{aligned} \tag{S.28}$$

Likewise, the second trace becomes

$$\begin{aligned} A_{22} &\approx -\tfrac{1}{4}\,\mathrm{Tr}(\not{p}_+\gamma^\mu\,\tilde{\not{q}}\,\not{p}_-\,\tilde{\not{q}}\gamma_\mu) \\ &= -\tfrac{1}{4}\,\mathrm{Tr}(\gamma_\mu\,\not{p}_+\gamma^\mu\,\tilde{\not{q}}\,\not{p}_-\,\tilde{\not{q}}) \\ &= +\tfrac{1}{2}\,\mathrm{Tr}(\not{p}_+\,\tilde{\not{q}}\,\not{p}_-\,\tilde{\not{q}}) \qquad \langle\!\langle\,\text{using } \gamma_\mu\,\not{p}_+\gamma^\mu = -2\,\not{p}_+\,\rangle\!\rangle \\ &= 4(p_+\tilde{q})(p_-\tilde{q})\;-\;2(p_+p_-)\,\tilde{q}^2 \\ &\approx (M_s^2-u)\times u\;-\;s\times u\;=\;(M_s^2-u-s)\times u \\ &\approx t\times u\,, \end{aligned} \tag{S.29}$$

where the last two lines follow from (S.28) and

$$\begin{aligned} \tilde{q}^2 &= u, \\ p_+\tilde{q} &= p_+(k_\gamma-p_+) = -\tfrac{1}{2}(k_\gamma-p_+)^2 - \cancel{\tfrac{1}{2}m_e^2} \approx -\frac{u}{2}\,, \\ p_-\tilde{q} &= p_-(p_--k_s) = \tfrac{1}{2}(p_--k_s)^2 - \tfrac{1}{2}M_s^2 + \cancel{\tfrac{1}{2}m_e^2} \approx \frac{u-M_s^2}{2}\,. \end{aligned} \tag{S.30}$$

Finally, the third trace becomes

$$\begin{aligned} A_{22} &\approx -\tfrac{1}{4}\,\mathrm{Tr}(\not{p}_+\,\not{q}\gamma^\mu\,\not{p}_-\,\tilde{\not{q}}\gamma_\mu) \\ &= -(p_-\tilde{q})\times\mathrm{Tr}(\not{p}_+\,\not{q}) \qquad \langle\!\langle \text{ using } \gamma^\mu\,\not{p}_-\,\tilde{\not{q}}\gamma^\mu = +4(p_-\tilde{q})\,\rangle\!\rangle \\ &= -4(p_-\tilde{q})(p_+q) \\ &\approx +(u-M_s^2)(t-M_s^2). \end{aligned} \tag{S.31}$$

Quad erat demonstrandum.

Problem **3**(e):

Now let's evaluate eq. (10) for the spin summed/averaged $\overline{|\mathcal{M}|^2}$. Neglecting the m_e^2 terms in the denominators and plugging in eqs. (12) for the A_{11}, A_{22}, and A_{12}, we have

$$\begin{aligned} \overline{|\mathcal{M}|^2} &= e^2g^2\left(\frac{tu}{t^2}\,+\,\frac{ut}{u^2}\,+\,\frac{2(t-M_s^2)(u-M_s^2)}{tu}\right) \\ &= \frac{e^2g^2}{tu}\times\Big(u^2\,+\,t^2\,+\,2(t-M_s^2)(u-M_s^2)\Big) \\ &= \frac{e^2g^2}{tu}\times\Big((t+u-M_s^2)^2\,+\,M_s^4\Big) \\ &= e^2g^2\times\frac{s^2+M_s^4}{tu}\,. \end{aligned} \tag{S.32}$$

Now let's work out the kinematics in the center of mass frame. The initial electron and positron have 4–momenta $p_\mp^\mu = (E_e,\pm\mathbf{p})$ where $E_e \approx |\mathbf{p}|$. But since the scalar and the photon produced in the collision have different masses, they have equal and opposite 3-momenta (in the CM frame) but different energies: $k_\gamma^\mu = (\omega,+\mathbf{k})$ while $k_S^\mu = (E_s,-\mathbf{k})$, where $\omega = |\mathbf{k}| \neq E_s = \sqrt{\mathbf{k}^2+M_s^2}\,$. By energy conservation

$$\omega\,+\,E_s \;=\; 2E_e \;=\; \sqrt{s}\,. \tag{S.33}$$

To solve this equation, we rewrite it as

$$\omega^2\,+\,M_s^2 \;=\; E_s^2 \;=\; (\sqrt{s}\,-\omega)^2 \;=\; s\,-\,2\sqrt{s}\times\omega\,+\omega^2, \tag{S.34}$$

which gives us

$$\omega = \frac{s - M_s^2}{2\sqrt{s}} \implies E_s = \frac{s + M_s^2}{2\sqrt{s}} . \tag{S.35}$$

Given all these momenta, Mandelstam's t and u obtain as

$$\begin{aligned} t &\approx -2(p_- k_\gamma) = -2E_e\omega + 2\mathbf{p}\cdot\mathbf{k} \approx -2E_e\omega \times (1 - \cos\theta) \\ &= -\tfrac{1}{2}(s - M_s^2) \times (1 - \cos\theta), \end{aligned} \tag{S.36}$$

$$\begin{aligned} u &\approx -2(p_+ k_\gamma) = -2E_e\omega - 2\mathbf{p}\cdot\mathbf{k} \approx -2E_e\omega \times (1 + \cos\theta) \\ &= -\tfrac{1}{2}(s - M_s^2) \times (1 + \cos\theta). \end{aligned} \tag{S.37}$$

Hence, plugging these values into eq. (S.32) gives us

$$\overline{|\mathcal{M}|^2} = 4e^2 g^2 \times \frac{s^2 + M_S^4}{(s - M_S^2)^2} \times \frac{1}{\sin^2\theta} . \tag{S.38}$$

Finally, the partial cross-section for a 2 *particles* $\to$ 2 *particles* inelastic scattering in the CM frame is given by

$$\frac{d\sigma}{d\Omega_{\rm cm}} = \frac{|\mathcal{M}|^2}{64\pi^2 s} \times \frac{|\mathbf{p}'|}{|\mathbf{p}|} . \tag{S.39}$$

For the problem at hand, the inelasticity factor $|\mathbf{p}'|/|\mathbf{p}|$ is

$$\frac{|\mathbf{k}|}{|\mathbf{p}|} \approx \frac{\omega}{E_e} = \frac{s - M_s^2}{s} . \tag{S.40}$$

Combining this factor with eq. (S.38), we finally arrive at the following formula for the partial cross-section:

$$\frac{d\sigma(e^- e^+ \to \gamma S)}{d\Omega_{\rm c.m.}} = \frac{\alpha g^2}{4\pi} \times \frac{s^2 + M_s^4}{s^2(s - M_s^2)} \times \frac{1}{\sin^2\theta} . \tag{S.41}$$

Note the forward-backward symmetry $\theta \leftrightarrow \pi - \theta$ of this cross section. Physically, it is due to the charge-conjugation symmetry which exchanges the initial electron and positron.

As usual for annihilation processes in the ultra-relativistic limit, the cross-section (S.41) has divergent peaks in forward and backward directions, $\theta \to 0$ or $\theta \to \pi$. The divergence here is an artefact of the $m_e^2 = 0$ approximation, which becomes inaccurate at very small angles $\theta \lesssim (m_e/E)$ (or $\pi - \theta \lesssim (m_e/E)$).

A more careful analysis — which was not a required part of this homework — leads to

$$\text{for } \theta \lesssim \gamma^{-1}, \quad \frac{d\sigma(e^-e^+ \to \gamma S)}{d\Omega_{\text{c.m.}}} \approx \frac{\alpha g^2}{4\pi s} \times \left(\frac{s - M_s^2}{s} \times \frac{1}{\theta^2 + \gamma^{-2}} + \frac{M_s^2}{s - M_s^2} \times \frac{2\theta^2}{(\theta^2 + \gamma^{-2})^2} \right) \tag{S.42}$$

— where $\gamma^{-1} = m_e/E \ll 1$ — instead of eq. (S.41). Consequently, the total cross-section turns out to be finite rather then divergent, namely

$$\sigma_{\text{tot}}(e^-e^+ \to \gamma S) = \alpha g^2 \times \frac{(s^2 + M_s^4)}{s^2(s - M_s^2)} \left(\log \frac{2E_e}{m_e} - \frac{sM_s^2}{s^2 + M_s^4} + O\left(\frac{m_e^2}{E_e^2}\right) \right). \tag{S.43}$$

Problem **4**(a):

The scalar potential part of the linear sigma model's Lagrangian (13) is

$$V(\phi) = \frac{\lambda}{8} \left(\sum_i \phi_i^2 - f^2 \right)^2 - \beta\lambda f^2 \times \phi_{N+1}, \tag{S.44}$$

where the last term explicitly breaks the $O(N+1)$ symmetry of the first term down to $O(N)$. To find the minimum of this potential, let's first find the stationary points where all the first derivatives $\partial V/\partial\phi_i$ are zero:

$$\text{for } i = 1, \ldots, N, \quad \frac{\partial V}{\partial \phi_i} = \frac{\lambda}{2} \left(\sum_j \phi_j^2 - f^2 \right) \times \phi_i = 0, \tag{S.45}$$

$$\text{and} \quad \frac{\partial V}{\partial \phi_{N+1}} = \frac{\lambda}{2} \left(\sum_j \phi_j^2 - f^2 \right) \times \phi_{N+1} - \beta\lambda f^2 = 0. \tag{S.46}$$

From eq. (S.46) we immediately see that at any stationary point $\left(\sum \phi^2 - f^2\right) \neq 0$, hence eqs. (S.45) tell us that $\phi_1 = \cdots = \phi_N = 0$. In other words, all the stationary points lie on the ϕ_{N+1} axis in the $(N+1)$ dimensional space of the scalar field values. And in this space, eq. (S.46) becomes a simple cubic equation

$$\phi_{N+1}^3 \;-\; f^2 \times \Phi_{N+1} \;-\; 2\beta f^2 \;=\; 0. \tag{S.47}$$

For small $\beta \ll f$, this cubic equation has 3 real solutions, approximately

$$\langle\phi_{N+1}\rangle_1 \;\approx\; -2\beta, \qquad \langle\phi_{N+1}\rangle_2 \;\approx\; -f \;+\; \beta, \qquad \langle\phi_{N+1}\rangle_3 \;\approx\; +f \;+\; \beta. \tag{S.48}$$

Now let's find out which of the three stationary points is a minimum (or at least a local minimum) by looking at the second derivatives of the potential (S.44). Along the ϕ_{N+1} axis in the field space, the second derivatives amount to

$$\frac{\partial^2 V}{\partial\phi_i \partial\phi_j} \;=\; \frac{\lambda}{2} \times \begin{cases} \left(3\phi_{N+1}^2 - f^2\right) & \text{for } i = j = N+1, \\ 0 & \text{for } i \leq N,\ j = N+1 \text{ or } j \leq N,\ i = N+1, \\ \left(\phi_{N+1}^2 - f^2\right) \times \delta_{ij} & \text{for } i, j \leq N. \end{cases} \tag{S.49}$$

Evaluating these derivatives for the 3 stationary points (S.48) — while assuming small $\beta > 0$ — gives us

$$\begin{array}{llll} @\ \langle\phi_{N+1}\rangle_1 : & \dfrac{\partial^2 V}{(\partial\phi_{N+1})^2} \;<\; 0 \;\; \text{while other } \dfrac{\partial^2 V}{(\partial\phi_i)^2} \;<\; 0 & \Longrightarrow & \text{maximum,} \\ @\ \langle\phi_{N+1}\rangle_2 : & \dfrac{\partial^2 V}{(\partial\phi_{N+1})^2} \;>\; 0 \;\; \text{while other } \dfrac{\partial^2 V}{(\partial\phi_i)^2} \;<\; 0 & \Longrightarrow & \text{saddle point,} \\ @\ \langle\phi_{N+1}\rangle_3 : & \dfrac{\partial^2 V}{(\partial\phi_{N+1})^2} \;>\; 0 \;\; \text{while other } \dfrac{\partial^2 V}{(\partial\phi_i)^2} \;>\; 0 & \Longrightarrow & \text{minimum.} \end{array} \tag{S.50}$$

Thus, the potential (S.44) has a unique minimum at

$$\langle\phi_1\rangle \;=\; \cdots \;=\; \langle\phi_N\rangle \;=\; 0, \quad \langle\phi_{N+1}\rangle \;=\; +f \;+\; \beta \;+\; O(\beta^2/f). \tag{14}$$

Quod erat demonstrandum.

Problem **4**(b):

Let's shift the fields as in eq. (15). In terms of the shifted fields,

$$T \stackrel{\text{def}}{=} \sum_i \phi_i^2 - f^2 \;=\; \underset{\sim}{\pi}^2 + \bigl(\sigma + \langle\phi_{N+1}\rangle\bigr)^2 - f^2 \;=\; \underset{\sim}{\pi}^2 + \sigma^2 + 2\langle\phi_{N+1}\rangle\times\sigma + \bigl(\langle\phi_{N+1}\rangle^2 - f^2\bigr), \tag{S.51}$$

where $\underset{\sim}{\pi}$ is a short-hand for N-vector $(\pi^1,\ldots,\pi^N)$ of the pion fields, thus $\underset{\sim}{\pi}^2 = (\pi^1)^2 + \cdots + (\pi^N)^2$. Therefore, expanding the scalar potential (S.44) into powers of the shifted fields, we obtain

$$\begin{aligned} V \;&=\; \frac{\lambda}{8}\times T^2 \;-\; \beta\lambda f^2 \times \bigl(\sigma + \langle\phi_{N+1}\rangle\bigr) \\ &=\; \frac{\lambda}{8}\times\bigl(\underset{\sim}{\pi}^2 + \sigma^2\bigr)^2 \;+\; \frac{\lambda\langle\phi_{N+1}\rangle}{2}\times\sigma\times\bigl(\underset{\sim}{\pi}^2 + \sigma^3\bigr) \\ &\quad+\; \frac{\lambda\langle\phi_{N+1}\rangle^2}{2}\times\sigma^2 \;+\; \frac{\lambda(\langle\phi_{N+1}\rangle^2 - f^2)}{4}\times\bigl(\underset{\sim}{\pi}^2 + \sigma^2\bigr) \\ &\quad+\; \left(\frac{\lambda\langle\phi_{N+1}\rangle}{2}\times\bigl(\langle\phi_{N+1}\rangle^2 - f^2\bigr) \;-\; \beta\lambda f^2\right)\times\sigma \;+\; \text{const.} \end{aligned} \tag{S.52}$$

On the last line here, the coefficient of σ vanishes thanks to $\langle\phi_{N+1}\rangle$ obeying the cubic equation (S.47). For the same reason, the coefficient of $(\underset{\sim}{\pi}^2 + \sigma^2)$ on the line before the last may be simplified as

$$\frac{\lambda(\langle\phi_{N+1}\rangle^2 - f^2)}{4} \;=\; \frac{\beta\lambda f^2}{2\langle\phi_{N+1}\rangle}\,. \tag{S.53}$$

Altogether, we have

$$V(\sigma,\underset{\sim}{\pi}) \;=\; \frac{\lambda}{8}\times\bigl(\underset{\sim}{\pi}^2 + \sigma^2\bigr)^2 \;+\; \frac{\kappa}{2}\times\bigl(\sigma^3 + \sigma\underset{\sim}{\pi}^2\bigr) \;+\; \frac{M_\sigma^2}{2}\times\sigma^2 \;+\; \frac{M_\pi^2}{2}\times\underset{\sim}{\pi}^2 \;+\; \text{const}, \tag{S.54}$$

where

$$\begin{aligned} \text{quartic coupling}\quad \lambda \;&=\; \lambda, \\ \text{cubic coupling}\quad \kappa \;&=\; \lambda\times\langle\phi_{N+1}\rangle \;\approx\; \lambda(f+\beta), \\ \text{pion mass}^2\quad M_\pi^2 \;&=\; \frac{\beta\lambda f^2}{\langle\phi_{N+1}\rangle} \;\approx\; \beta\lambda f, \\ \text{sigma mass}^2\quad M_\sigma^2 \;&=\; M_\pi^2 \;+\; \lambda\langle\phi_{N+1}\rangle^2 \;\approx\; \lambda f(f+3\beta). \end{aligned} \tag{S.55}$$

Let's take a closer look at the pion's mass^2, $M_\pi^2 \approx \beta\times\lambda f$. In the $\beta = 0$ limit, the pions are massless in accordance with the Goldstone theorem. Indeed, for $\beta = 0$ the sigma model's Lagrangian has an exact $SO(N+1)$ symmetry which is spontaneously broken down to an $SO(N)$ subgroup; there are N spontaneously broken generators, so there should be N massless Goldstone bosons. But for $\beta \neq 0$, the $SO(N+1)$ symmetry of the Lagrangian is only approximate, and its *explicit* breaking by the $\beta\lambda f^2 \times \phi_{N+1}$ term spoils the Goldstone theorem. Thus, instead of exactly massless Goldstone bosons we should get light but not quite massless pseudo-Goldstone bosons; to the first order in β, their mass^2 should be proportional to β. And indeed, in the linear sigma model $M_\pi^2 \approx \beta \times \lambda f$.

Still, for $\beta \ll f$, the pions should be much lighter than the sigma particle. And indeed, according to eqs. (S.55),

$$\frac{M_\pi^2}{M_\sigma^2} \approx \frac{\beta\lambda f}{\lambda f^2} = \frac{\beta}{f} \ll 1. \tag{S.56}$$

Problem **4**(c): Back in homework#9 (problem 3), we had a very similar setup for the *shifted* fields of the linear sigma models: $N+1$ scalar fields $\sigma(x)$ and $\pi^i(x)$, with the Lagrangian

$$\begin{aligned}
\mathcal{L} &= \tfrac{1}{2}(\partial_\mu\sigma)^2 + \tfrac{1}{2}(\partial_\mu\underset{\sim}{\pi})^2 - V, \\
V(\sigma,\underset{\sim}{\pi}) &= \frac{\lambda}{8}\Big(\underset{\sim}{\pi}^2 + \sigma^2 + 2f\times\sigma\Big)^2 \\
&= \frac{\lambda f^2}{2}\times\sigma^2 + \frac{\lambda f}{2}\times\left(\sigma^3 + \sigma\underset{\sim}{\pi}^2\right) + \frac{\lambda}{8}\times\left(\sigma^2 + \underset{\sim}{\pi}^2\right)^2.
\end{aligned} \tag{9.2}$$

In particular, there is a mass term for the σ field but not for the pions, which are exactly massless — exactly as in the present sigma model with $\beta = 0$. The cubic and quartic terms in the potential (9.2) also have exactly the same form as in eq. (S.54), and the quartic coupling λ and the cubic coupling $\kappa = \lambda f$ are related to the σ field's mass as

$$\kappa^2 = \lambda \times M_\sigma^2\,. \tag{S.57}$$

For the present sigma model, we have exactly similar relation for $\beta = 0$. Indeed, according to eq. (S.55),

$$\text{for } \beta = 0, \quad \kappa = \lambda f, \quad M_\sigma^2 = \lambda f^2 \quad \Longrightarrow \quad \kappa^2 = \lambda \times M_\sigma^2\,. \tag{S.58}$$

Therefore, the $\pi\pi \to \pi\pi$ scattering amplitudes in the linear sigma model for $\beta = 0$ come out to be exactly as homework#9: At the tree level,

$$\begin{aligned}\mathcal{M}(\pi^j+\pi^k\to\pi^\ell+\pi^m)\ =\ &-\left(\lambda\,+\,\frac{\kappa^2}{s-M_\sigma^2}\right)\times\delta^{jk}\delta^{\ell m}\ -\ \left(\lambda\,+\,\frac{\kappa^2}{t-M_\sigma^2}\right)\times\delta^{j\ell}\delta^{km}\\ &-\ \left(\lambda\,+\,\frac{\kappa^2}{u-M_\sigma^2}\right)\times\delta^{jm}\delta^{k\ell},\end{aligned}\tag{S.59}$$

which in light of the relation (S.57) becomes

$$\begin{aligned}\mathcal{M}(\pi^j+\pi^k\to\pi^\ell+\pi^m)\ =\ &-\frac{\lambda s}{s-M_\sigma^2}\times\delta^{jk}\delta^{\ell m}\ -\ \frac{\lambda t}{t-M_\sigma^2}\times\delta^{j\ell}\delta^{km}\\ &-\ \frac{\lambda u}{u-M_\sigma^2}\times\delta^{jm}\delta^{k\ell}.\end{aligned}\tag{S.60}$$

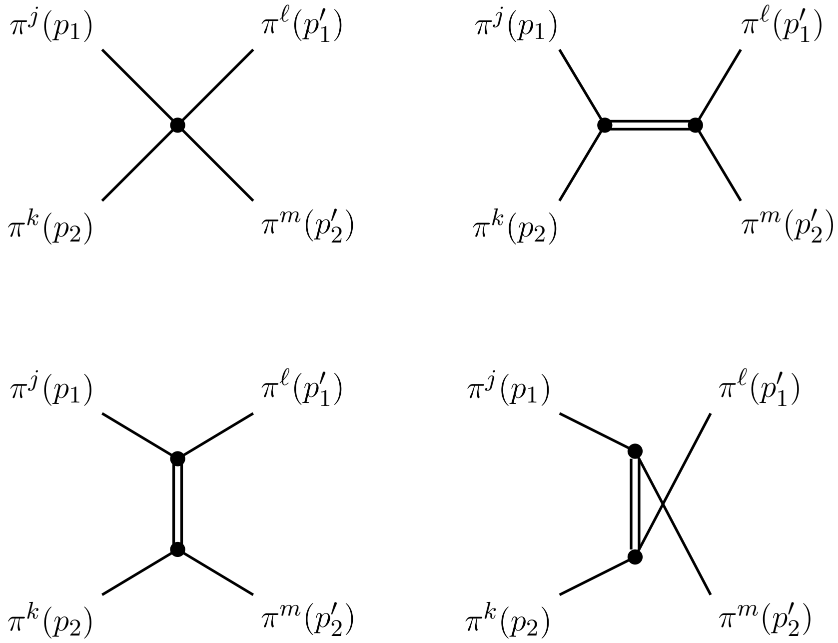
When any of the 4 pions' energy becomes small, we get $s,t,u\ll M_\sigma^2$, and the scattering amplitude becomes small as

$$\mathcal{M}\ \approx\ \frac{\lambda}{M_\sigma^2}\times\Big(s\times\delta^{jk}\delta^{\ell m}\,+\,t\times\delta^{j\ell}\delta^{km}\,+\,u\times\delta^{jm}\delta^{k\ell}\Big)\ =\ O\left(\frac{\lambda E_{\rm cm}^2}{M_\sigma^2}\right).\tag{S.61}$$

Problem **4**(d): For $\beta\neq0$, the quartic and the cubic couplings of the σ and π^i to each other has similar overall form to what we had back in homework#9, but the overall coefficients λ and κ of those couplings are no longer related to the σ particle's mass by eq. (S.57). Instead, eq. (S.55) gives us

$$\kappa\ =\ \lambda\,\langle\phi_{N+!}\rangle\,,\quad M_\sigma^2\ =\ M_\pi^2\ +\ \lambda\,\langle\phi_{N+!}\rangle^2\quad\Longrightarrow\quad \kappa^2\ =\ \lambda\times\big(M_\sigma^2-M_\pi^2\big).\tag{S.62}$$

Now consider the $\pi\pi\to\pi\pi$ scattering. At the tree level, we have exactly the same 4 diagrams for such scattering as in the homework#9, namely



(S.63)

Altogether, these diagrams yield the scattering amplitude exactly as in eq. (S.59), but for the β-modified couplings and masses. Consequently, in light of eq. (S.62) instead of (S.57), we have

$$\lambda\ +\ \frac{\kappa^2}{s-M_\sigma^2}\ =\ \frac{\lambda s-\lambda M_\sigma^2+\kappa^2}{s-M_\sigma^2}\ =\ \frac{\lambda s-\lambda M_\pi^2}{s-M_\sigma^2} \tag{S.64}$$

and likewise

$$\lambda\ +\ \frac{\kappa^2}{t-M_\sigma^2}\ =\ \frac{\lambda(t-M_\pi^2)}{t-M_\sigma^2}\quad\text{and}\quad\lambda\ +\ \frac{\kappa^2}{u-M_\sigma^2}\ =\ \frac{\lambda(u-M_\pi^2)}{u-M_\sigma^2}\,, \tag{S.65}$$

so the amplitude (S.59) becomes

$$\begin{aligned}\mathcal{M}(\pi^j+\pi^k\to\pi^\ell+\pi^m)\ =\ -\frac{\lambda(s-M_\pi^2)}{s-M_\sigma^2}\times\delta^{jk}\delta^{\ell m}\ &-\ \frac{\lambda(t-M_\pi^2)}{t-M_\sigma^2}\times\delta^{j\ell}\delta^{km}\\ &-\ \frac{\lambda(u-M_\pi^2)}{u-M_\sigma^2}\times\delta^{jm}\delta^{k\ell}.\end{aligned} \tag{S.66}$$

When the pions' energies become low compared to M_σ — or in Lorentz-invariant terms, when $s,t,u\ll M_\sigma^2$ — we may simplify this amplitude by approximating all the denominators as $-M_\sigma^2$, thus

$$\mathcal{M}(\pi^j+\pi^k\to\pi^\ell+\pi^m)\;\approx\;\left(\frac{\lambda}{M_\sigma^2}\;\approx\;\frac{1}{f^2}\right)\times\left(\begin{array}{r}(s-M_\pi^2)\times\delta^{jk}\delta^{\ell m}\;+\;(t-M_\pi^2)\times\delta^{j\ell}\delta^{km}\\ +\;(u-M_\pi^2)\times\delta^{jm}\delta^{k\ell}\end{array}\right). \tag{16}$$

What happens to this amplitude when one of the pions' momentum becomes very small? Alas, for $\beta\neq 0$ the pions are massive, so we cannot take all 4 components of a pion's p^μ to zero. The best we can do is to take $\mathbf{p}\to 0$ while $p^0\to m$, which is the *non-relativistic limit*. However, if only one pion is non-relativistic while the other 3 pions have $E\gg M_\pi$ (but $E\ll M_\sigma$), we generally have $s,t,u=O(E\times M_\pi)\gg M_\pi^2$ (albeit $s,t,u\ll M_\sigma^2$), and the scattering amplitude becomes

$$\mathcal{M}\;=\;O\left(\frac{E\times M_\pi}{f^2}\right)\;\not\to\;0. \tag{S.67}$$

The strongest low-energy limit we an take for massive pions is to make all four pions non-relativistic. In this limit, $s=E_{\rm cm}^2\approx 4M_\pi^2$ while $u,t=O(\mathbf{p}^2)\ll M_\pi^2$, so the scattering amplitude (5) becomes

$$\mathcal{M}(\pi^j+\pi^k\to\pi^\ell+\pi^m)\;\approx\;\left(\frac{\lambda M_\pi^2}{M_\sigma^2}\approx\frac{\beta\lambda}{f}\right)\times\Bigl(3\delta^{jk}\delta^{\ell m}\;-\;\delta^{j\ell}\delta^{km}\;-\;\delta^{jm}\delta^{k\ell}\Bigr). \tag{17}$$

This amplitude is suppressed by the factor β/f, but it does not vanish! And even if all 4 pions belong to the same species, the scattering amplitude does not vanish in the non-relativistic limit,

$$\mathcal{M}(\pi^1+\pi^1\to\pi^1+\pi^1)\;\approx\;\frac{\lambda\beta}{f}\;\neq\;0, \tag{S.68}$$

unlike what we had back in homework#9.

Problem **5**(a):

$$\begin{aligned}
\text{given}\quad \Phi\ &\to\ e^{+i\theta}U_L\Phi U_R^\dagger\,, && (20)\\
\text{we have}\quad \Phi^\dagger\ &\to\ e^{-i\theta}U_R\Phi^\dagger U_L^\dagger\,, && (S.69)\\
\text{hence}\quad \Phi^\dagger\Phi\ &\to\ U_R(\Phi^\dagger\Phi)U_R^\dagger\,, && (S.70)\\
\big(\Phi^\dagger\Phi\big)^2\ &\to\ U_R\Phi^\dagger\Phi U_R^\dagger U_R\Phi^\dagger\Phi U_R^\dagger\ =\ U_R\big(\Phi^\dagger\Phi\big)^2U_R^\dagger\,, && (S.71)\\
\text{likewise}\quad \big(\Phi^\dagger\Phi\big)^n\ &\to\ U_R\big(\Phi^\dagger\Phi\big)^nU_R^\dagger\quad \forall n=1,2,3,\ldots, && (S.72)
\end{aligned}$$

and therefore

$$\text{all traces}\quad \mathrm{tr}\Big(\big(\Phi^\dagger\Phi\big)^n\Big)\quad \text{are invariant under symmetries (20),} \tag{S.73}$$

thanks to the cyclic invariance rule for traces, $\mathrm{tr}\big(U_RXU_R^\dagger\big)=\mathrm{tr}\big(XU_R^\dagger U_R\big)=\mathrm{tr}\big(X\big)$ for any $X=\big(\Phi^\dagger\Phi\big)^n$. Consequently, the scalar potential (19) is invariant under the symmetries (20).

For the global symmetries where $e^{i\theta}$, U_L, and U_R do not depend on x, the kinetic term in the Lagrangian (18) is also invariant. Indeed,

$$\begin{aligned}
&\text{for constant } e^{i\theta},\ U_L,\ U_R,\\
&\qquad \partial_\mu\Phi\ \to\ e^{+i\theta}U_L(\partial_\mu\Phi)U_R^\dagger\,,\\
&\qquad \partial_\mu\Phi^\dagger\ \to\ e^{-i\theta}U_R(\partial_\mu\Phi^\dagger)U_L^\dagger\,,\\
&\qquad \partial^\mu\Phi^\dagger\,\partial_\mu\Phi\ \to\ U_R(\partial^\mu\Phi^\dagger\,\partial_\mu\Phi)U_R^\dagger\,,\\
&\text{and}\quad \mathrm{tr}\big(\partial^\mu\Phi^\dagger\,\partial_\mu\Phi\big)\quad \text{is invariant.}
\end{aligned} \tag{S.74}$$

Altogether, the whole Lagrangian (18) is invariant, $\mathcal{Q.E.D.}$

Problem **5**($\star$):

The kinetic term in (10) and the last two terms in the potential (11) have a much bigger symmetry than $G=SU(N)\times SU(N)\times U(1)$, namely the $SO(2N^2)$ which does not care for the matrix structure of the $\Phi(x)$ and treats it as $2N^2$ real component fields. Indeed,

$$\mathrm{tr}\big(\Phi^\dagger\Phi\big)\ =\ \sum_{i,j}\left|\Phi_i^{\ j}\right|^2\ =\ \sum_{i,j}\left(\left(\mathrm{Re}\,\Phi_i^{\ j}\right)^2\ +\ \left(\mathrm{Im}\,\Phi_i^{\ j}\right)^2\right) \tag{S.75}$$

is invariant under all $SO(2N^2)$ "rotations" of the components, and so is the kinetic term.

On the other hand, the $\mathrm{tr}\left(\Phi^\dagger\Phi\,\Phi^\dagger\Phi\right)$ in the potential does depend on the packing of $2N^2$ real components into a complex $N\times N$ matrix, and it is this term which reduces the internal symmetry group of the theory to $G=SU(N)\times SU(N)\times U(1)$.

Proving that all the $SO(2N^2)/G$ symmetries are broken by the quartic trace term is a non-trivial exercise in group theory rather than field theory. You do not have to do it as part of this homework set, and I am not writing down the proof here.

Problem **5**(b):

Given the eigenvalues $(\kappa_1,\ldots,\kappa_N)$ of the $\Phi^\dagger\Phi$ matrix, the invariant traces (S.73) obtain as

$$\mathrm{tr}\left(\left(\Phi^\dagger\Phi\right)^n\right) \;=\; \sum_{i=1}^N \kappa_i^n\,. \tag{S.76}$$

Consequently, the scalar potential is

$$V \;=\; \frac{\alpha}{2}\sum_i \kappa_i^2 \;+\; \frac{\beta}{2}\left(\sum_i \kappa_i\right)^2 \;+\; m^2\sum_i \kappa_i\,. \tag{S.77}$$

Now let's minimize this potential. Since the matrix $\Phi^\dagger\Phi$ cannot have any negative eigenvalues, we are looking for a minimum of $V(\kappa_1,\ldots,\kappa_N)$ *under constraints* $\kappa_i\geq 0$. This requires

$$\forall i=1,\ldots,N,\quad \textbf{either }\kappa_i\geq 0\textbf{ and }\frac{\partial V}{\partial\kappa_i} \;=\; 0,\quad \textbf{or else }\kappa_i=0\textbf{ and }\frac{\partial V}{\partial\kappa_i}\;>\;0, \tag{S.78}$$

where

$$\frac{\partial V}{\partial\kappa_i} \;=\; \alpha\kappa_i \;+\; m^2 \;+\; \beta\sum_j \kappa_j\,. \tag{S.79}$$

These derivatives are linear functions of the eigenvalues κ_i, so all the non-zero eigenvalues must obey the same linear equation

$$\alpha\times\kappa_i \;=\; -m^2 \;-\; \beta\times\sum_j \kappa_j\,,\quad \text{same for all }\kappa_i\neq 0,$$

which means that all non-zero κ_i have the same value. Thus, up to a permutation of eigenvalues,

$$\kappa_1 = \cdots = \kappa_k = C^2, \quad \kappa_{k+1} = \cdots = \kappa_N = 0, \tag{S.80}$$

for some $k = 0, 1, 2, \ldots, N$, and C^2 obtains from

$$\alpha \times C^2 + m^2 + \beta \times kC^2 = 0 \quad \longrightarrow \quad C^2 = \frac{-m^2}{\alpha + k\beta}\,. \tag{S.81}$$

To make sure that the solution (S.80) is a minimum rather that a maximum or a saddle point, we need

$$\begin{aligned} C^2 &= \frac{-m^2}{\alpha + k\beta} > 0 \quad \text{unless } k = 0, \\ m^2 + \beta k C^2 &= \frac{\alpha m^2}{\alpha + k\beta} > 0 \quad \text{unless } k = N. \end{aligned} \tag{S.82}$$

Depending on the signs of α, β and m^2 parameters, this limits the solutions to the following:

- For $\alpha > 0$, $\beta > 0$, and $m^2 > 0$, the only solution is $k = 0$, which means $\kappa_1 = \cdots \kappa_N = 0$ and hence $\langle\Phi\rangle = 0$.
- For $\alpha > 0$, $\beta > 0$, and $m^2 < 0$, the only solutions is $k = N$, which means

  $$\kappa_1 = \cdots = \kappa_N = C^2 = \frac{-m^2}{\alpha + N\beta} > 0, \tag{21}$$

  and hence $\langle\Phi\rangle = C \times$ a unitary matrix as in eq. (22). We shall focus on this regime through the rest of this problem.
- For $\alpha < 0$ or $\beta < 0$, the situation is more complicated:
  - ∗ When $\alpha+\beta < 0$ or $\alpha+N\beta < 0$, the scalar potential (19) is unbounded from below and the theory is sick.
  - ∗ When $\alpha > 0$ and $\beta < 0$ but $\alpha + N\beta > 0$, the solutions are similar to the $\beta > 0$ case: For $m^2 > 0$ all $\kappa_i = 0$, while for $m^2 < 0$ the κ_i are as in eq. (21).
  - ∗ When $\beta > 0$ and $\alpha < 0$ but $\alpha + \beta > 0$: for $m^2 > 0$ the only solution is $k = 0$, meaning $\langle\Phi\rangle = 0$, but for $m^2 < 0$ all the solutions (S.80) with $k = 1, 2, \ldots, N$ are good local minima.

To find the global minimum, we compare the potentials at the local minima,

$$\begin{aligned} V(\text{minimum}\#k) &= \frac{\alpha}{2}\times kC^4 \;+\; \frac{\beta}{2}\times(kC^2)^2 \;+\; m^2\times kC^2 \\ &= \frac{k\alpha+k^2\beta}{2}\times\frac{m^4}{(\alpha+k\beta)^2} \;+\; km^2\times\frac{-m^2}{(\alpha+k\beta)} \\ &= -\frac{m^4}{2}\times\frac{k}{k\beta+\alpha}\,. \end{aligned} \tag{S.83}$$

Since $\alpha<0$ but $\alpha+\beta>0$, the deepest minimum obtains for $k=1$, thus

$$\kappa_1 \;=\; \frac{-m^2}{\alpha+\beta}\,,\quad \kappa_2 \;=\; \cdots \;=\; \kappa_N \;=\; 0. \tag{S.84}$$

Problem **5**(c):

Let's act with some $SU(N)_L\times SU(N)_R\times U(1)$ symmetry (20) on the vacuum expectation values (22):

$$\langle\Phi\rangle \;=\; C\times\mathbf{1}_{N\times N} \;\to\; e^{i\theta}U_L\,\langle\Phi\rangle\,U_R^\dagger \;=\; C\times e^{i\theta}U_LU_R^\dagger\,. \tag{S.85}$$

Clearly, to keep the VEVs $\langle\Phi\rangle$ invariant, we need

$$e^{i\theta}U_LU_R^\dagger \;=\; \mathbf{1}_{N\times N} \tag{S.86}$$

and hence

$$U_R \;=\; e^{i\theta}\times U_L\,. \tag{S.87}$$

Moreover, since the U_L and U_R matrices have unit determinants, this requires

$$\det\Big(e^{i\theta}\times\mathbf{1}_{N\times N}\Big) \;=\; 1 \quad\Longrightarrow\quad N\times\theta \;=\; 0 \pmod{2\pi}. \tag{S.88}$$

Such a phase can be absorbed into the $U_L\in SU(N)$, so without loss of generality we need

$$e^{i\theta} \;=\; 1 \quad\text{and}\quad U_L \;=\; U_R \;\in\; SU(N). \tag{S.89}$$

In other words, the *unbroken* symmetry group is $SU(N)_V$ which acts on the scalar fields as

$$\Phi(x) \;\to\; U\Phi(x)\,U^\dagger\,,\quad U\in SU(N). \tag{S.90}$$

Problem **5**(d):

In terms of the shifted fields (23),

$$\partial_\mu\Phi \;=\; \frac{1}{\sqrt{2}}\left(\partial_\mu\varphi_1 \,+\, i\partial_\mu\varphi_2\right), \qquad \partial_\mu\Phi^\dagger \;=\; \frac{1}{\sqrt{2}}\left(\partial_\mu\varphi_1 \,-\, i\partial_\mu\varphi_2\right), \tag{S.91}$$

hence the kinetic term in the Lagrangian becomes

$$\mathrm{tr}\big(\partial_\mu\Phi^\dagger\,\partial^\mu\Phi\big) \;=\; \tfrac{1}{2}\,\mathrm{tr}(\partial_\mu\varphi_1\,\partial^\mu\varphi_1) \;+\; \tfrac{1}{2}\,\mathrm{tr}(\partial_\mu\varphi_2\,\partial^\mu\varphi_2). \tag{S.92}$$

As to the potential terms, we have

$$\begin{aligned}\Phi^\dagger\Phi \;&=\; C^2\times\mathbf{1}_{N\times N} \;+\; C\times\big(\delta\Phi^\dagger+\delta\Phi\big) \;+\; \delta\Phi^\dagger\delta\Phi\\ &=\; C^2\times\mathbf{1}_{N\times N} \;+\; \sqrt{2}C\times\varphi_1 \;+\; \tfrac{1}{2}\varphi_1^2 \;+\; \tfrac{1}{2}\varphi_2^2 \;+\; \tfrac{i}{2}[\varphi_1,\varphi_2]\end{aligned} \tag{S.93}$$

and consequently

$$\mathrm{tr}\big(\Phi^\dagger\Phi\big) \;=\; NC^2 \;+\; \sqrt{2}C\,\mathrm{tr}(\varphi_1) \;+\; \tfrac{1}{2}\,\mathrm{tr}(\varphi_1^2) \;+\; \tfrac{1}{2}\,\mathrm{tr}(\varphi_2^2), \tag{S.94}$$

$$\begin{aligned}\mathrm{tr}^2\big(\Phi^\dagger\Phi\big) \;&=\; N^2C^4 \;+\; 2\sqrt{2}NC^3\,\mathrm{tr}(\varphi_1) \;+\; 2C^2\,\mathrm{tr}^2(\varphi_1) \;+\; NC^2\,\big(\mathrm{tr}(\varphi_1^2) \;+\; \mathrm{tr}(\varphi_2^2)\big)\\ &\quad+\; \sqrt{2}C\,\mathrm{tr}(\varphi_1)\times\big(\mathrm{tr}(\varphi_1^2) \;+\; \mathrm{tr}(\varphi_2^2)\big) \;+\; \tfrac{1}{4}\,\big(\mathrm{tr}(\varphi_1^2) \;+\; \mathrm{tr}(\varphi_2^2)\big)^2\end{aligned} \tag{S.95}$$

$$\begin{aligned}\mathrm{tr}\Big(\big(\Phi^\dagger\Phi\big)^2\Big) \;&=\; NC^4 \;+\; 2\sqrt{2}C^3\,\mathrm{tr}(\varphi_1) \;+\; 3C^2\,\mathrm{tr}(\varphi_1^2) \;+\; C^2\,\mathrm{tr}(\varphi_2^2)\\ &\quad+\; \sqrt{2}C\,\mathrm{tr}(\varphi_1^3) \;+\; \sqrt{2}C\,\mathrm{tr}(\varphi_1\varphi_2^2)\\ &\quad+\; \tfrac{1}{4}\,\mathrm{tr}(\varphi_1^4) \;+\; \tfrac{1}{4}\,\mathrm{tr}(\varphi_2^4) \;+\; \tfrac{3}{2}\,\mathrm{tr}(\varphi_1^2\varphi_2^2) \;-\; \tfrac{1}{2}\,\mathrm{tr}(\varphi_1\varphi_2\varphi_1\varphi_2).\end{aligned} \tag{S.96}$$

Plugging all these formulae into the potential (12) and expanding in powers of φ_1 and φ_2, we obtain

$$V \;=\; \mathrm{const} \;+\; V_1 \;+\; V_2 \;+\; V_3 \;+\; V_4 \tag{S.97}$$

where

$$V_1 \;=\; \sqrt{2}C\times(m^2 \;+\; \beta NC^2 \;+\; \alpha C^2)\times\mathrm{tr}(\varphi_1) \;=\; 0 \tag{S.98}$$

$$\langle\!\langle\,\text{because } m^2 \;+\; (\alpha+N\beta)C^2 \;=\; 0\,\rangle\!\rangle$$

$$V_2 \;=\; \beta C^2\times\mathrm{tr}^2(\varphi_1)\;+\;\tfrac12(m^2\,+\,\beta NC^2\,+\,3\alpha C^2)\times\mathrm{tr}(\varphi_1^2)$$
$$\qquad\qquad +\;\tfrac12(m^2\,+\,\beta NC^2\,+\,\alpha C^2)\times\mathrm{tr}(\varphi_2^2)$$
$$=\;\beta C^2\times\mathrm{tr}^2(\varphi_1)\;+\;\alpha C^2\,\mathrm{tr}(\varphi_1^2)\;+\;0, \tag{S.99}$$

$$V_3 \;=\; \frac{\beta C}{\sqrt2}\times\mathrm{tr}(\varphi_1)\times\Bigl(\mathrm{tr}(\varphi_1^2)\,+\,\mathrm{tr}(\varphi_2^2)\Bigr)\;+\;\frac{\alpha C}{\sqrt2}\times\Bigl(\mathrm{tr}(\varphi_1^3)\,+\,\mathrm{tr}(\varphi_1\varphi_2^2)\Bigr), \tag{S.100}$$

$$V_4 \;=\; \frac{\beta}{8}\Bigl(\mathrm{tr}(\varphi_1^2)\,+\,\mathrm{tr}(\varphi_2^2)\Bigr)^2$$
$$+\,\frac{\alpha}{8}\Bigl(\mathrm{tr}(\varphi_1^4)\,+\,\mathrm{tr}(\varphi_2^4)\,+\,6\,\mathrm{tr}(\varphi_1^2\varphi_2^2)\,-\,2\,\mathrm{tr}(\varphi_1\varphi_2\varphi_1\varphi_2)\Bigr). \tag{S.101}$$

Combining the quadratic part (S.99) of this potential with the kinetic terms (S.92), we arrive at

$$\mathcal{L}_2 \;=\; \tfrac12\,\mathrm{tr}(\partial_\mu\varphi_1\,\partial^\mu\varphi_1)\;-\;\beta C^2\times\mathrm{tr}^2(\varphi_1)\;-\;\alpha C^2\,\mathrm{tr}(\varphi_1^2)+\;\tfrac12\,\mathrm{tr}(\partial_\mu\varphi_2\,\partial^\mu\varphi_2) \tag{S.102}$$

which gives us the mass spectrum of the theory: the $\varphi_1(x)$ matrix of fields is massive and the $\varphi_2(x)$ matrix is massless. Each matrix is $N\times N$ and hermitian, so it contains N^2 independent real scalar fields, which give rise to N^2 particles. Altogether, the spectrum comprises:

- N^2 massless particles from the $\varphi_2(x)$ matrix.
- N^2-1 massive particles with $M^2=2\alpha C^2$ from the traceless part of the $\varphi_1(x)$ matrix.
- One more massive particle with $M^2=2(\alpha+N\beta)C^2=-2m^2$ from the trace of $\varphi_1(x)$.

To see where the values of the masses come from, let's decompose the φ_1 matrix into the pure trace plus the traceless part,

$$\xi(x)\;\stackrel{\mathrm{def}}{=}\;\frac{\mathrm{tr}(\varphi_1(x))}{\sqrt N}\quad\text{and}\quad\tilde\varphi_1(x)\;\stackrel{\mathrm{def}}{=}\;\varphi_1(x)\;-\;\frac{\xi(x)}{\sqrt N}\times\mathbf{1}_{N\times N}\quad\Longrightarrow\quad\mathrm{tr}(\tilde\varphi_1)\;\equiv\;0. \tag{S.103}$$

Consequently,

$$\mathrm{tr}^2(\varphi_1)\;=\;N\times\xi^2,\qquad\mathrm{tr}(\varphi_1^2)\;=\;\xi^2\,+\,\mathrm{tr}(\tilde\varphi_1^2), \tag{S.104}$$

likewise

$$\operatorname{tr}\bigl(\partial_\mu\varphi_1\,\partial^\mu\varphi_1\bigr)\ =\ \partial_\mu\xi\,\partial^\mu\xi\ +\ \operatorname{tr}\bigl(\partial_\mu\tilde\varphi_1\,\partial^\mu\tilde\varphi_1\bigr),$$

so the free Lagrangian (S.102) becomes

$$\begin{aligned}\mathcal{L}_2\ =\ &\tfrac12(\partial_\mu\xi)^2\ -\ 2(\beta C^2N\ +\ \alpha C^2)\times\tfrac12\xi^2\\ &+\ \tfrac12\operatorname{tr}\bigl((\partial_\mu\tilde\varphi_1)^2\bigr)\ -\ 2\alpha C^2\times\tfrac12\operatorname{tr}(\tilde\varphi_1^2)\\ &+\ \tfrac12\operatorname{tr}\bigl((\partial_\mu\varphi_2)^2\bigr)\ -\ 0\times\tfrac12\operatorname{tr}(\varphi_2^2)\end{aligned}\tag{S.105}$$

where all the masses are manifest.

Problem **5**(e):

The unbroken $SU(N)_V$ symmetry acts on the scalar fields according to

$$\Phi(x)\ \to\ U\times\Phi(x)\times U^\dagger.\tag{S.90}$$

and since the VEV (22) is invariant, the shifted fields $\delta\Phi(x)=\Phi(x)-\langle\Phi\rangle$ also transform according to

$$\delta\Phi(x)\ \to\ U\times\delta\Phi(x)\times U^\dagger.\tag{S.106}$$

Moreover, unitary transforms like these preserve hermiticity, so when we decompose $\delta\Phi(x)$ into a hermitian matrix $\varphi_1(x)$ and an antihermitian matrix $i\varphi_2(x)$, the transforms (S.106) do not mix the φ_1 and φ_2 with each other. Instead, they transform like

$$\varphi_1(x)\ \to\ U\varphi_1(x)\,U^\dagger,\qquad \varphi_2(x)\ \to\ U\varphi_2(x)\,U^\dagger,\tag{S.107}$$

which means that φ_1 and φ_2 comprise separate $SU(N)$ multiplets. Furthermore, the transforms (S.107) preserve the traces $\operatorname{tr}(\varphi_1)$ and $\operatorname{tr}(\varphi_2)$, so to make the $SU(N)$ multiplet structure manifest, let's decompose both φ_1 and φ_2 into their traceless parts and pure traces along the lines of eq. (S.103),

$$\varphi_1(x) \;=\; \frac{\xi_1(x)}{\sqrt{N}}\times\mathbf{1}_{N\times N} + \tilde\varphi_1(x), \quad \varphi_2(x) \;=\; \frac{\xi_2(x)}{\sqrt{N}}\times\mathbf{1}_{N\times N} + \tilde\varphi_2(x), \quad \operatorname{tr}(\tilde\varphi_1) \equiv \operatorname{tr}(\tilde\varphi_2) \equiv 0. \tag{S.108}$$

With this decomposition, the ξ_1 and the ξ_2 are both invariant under the $SU(N)$ — which puts each of them into its own singlet multiplet — while the each of the traceless parts $\tilde\varphi_1$ and $\tilde\varphi_2$ makes its own *adjoint multiplet.*

This multiplet structure agrees with the masses we obtained in part (d). Indeed, all N^2-1 members of the adjoint multiplet $\tilde\varphi_1$ have the same mass $2\alpha C^2$, while the singlet ξ_1 has a different mass $2(\alpha+N\beta)C^2$.

On the other hand, both the adjoint multiplet $\tilde\varphi_2$ and the singlet ξ_2 are massless. The reason for this degeneracy goes beyond the un-broken $SU(N)$ symmetry; instead, both the $\tilde\varphi_2$ and the ξ_2 are Goldstone bosons of the spontaneously broken symmetries in

$$G/H \;=\; \Big(SU(N)_L\times SU(N)_R\times U(1)\Big)\Big/\,SU(N)\,. \tag{S.109}$$

Specifically, the singlet ξ_2 is the Goldstone boson of the broken $U(1)$ symmetry. Indeed, the $U(1)$'s generator commutes with all the other generators, so it belongs in its own singlet of the symmetry, and the corresponding Goldstone particle should also be a singlet.

Now consider the non-abelian generators. Generators T^a_L of the $SU(N)_L$ form an adjoint multiplet of the $SU(N)_L$, but are invariant under the $SU(N)_R$. Likewise, generators T^a_R of the $SU(N)_R$ form an adjoint multiplet of the $SU(N)_R$, but are invariant under the $SU(N)_L$. In other words, under an $(U_L,U_R)\in SU(N)_L\times SU(N)_R$ they transform as

$$T^a_L \;\to\; U_LT^a_LU^\dagger_L\,, \qquad T^a_R \;\to\; U_RT^a_RU^\dagger_R\,. \tag{S.110}$$

When the $SU(N)_L\times SU(N)_R$ is broken down to a single $SU(N)$ spanning $U_L=U_R=U$, both T^a_L and T^a_R transform as

$$T^a_L \;\to\; UT^a_LU^\dagger, \qquad T^a_R \;\to\; UT^a_RU^\dagger, \tag{S.111}$$

which puts them into *two adjoint multiplets* of the unbroken $SU(N)$. Equivalently, we may form two adjoint multiplets out of

$$T_V^a \;=\; T_L^a + T_R^a \qquad \text{and} \qquad T_A^a \;=\; T_L^a - T_R^a\,, \tag{S.112}$$

which act on the scalar fields according to

$$T_V^a\Phi \;=\; \tfrac{i}{2}[\lambda^a,\Phi], \qquad T_A^a\Phi \;=\; \tfrac{i}{2}\{\lambda^a,\Phi\}. \tag{S.113}$$

The T_V^a generate the unbroken $SU(N)_V$ symmetry, *cf.* eq. (S.90). The T_A^a generators are spontaneously broken, hence there should be an adjoint multiplet of massless Goldstone bosons. And indeed there is — the $\tilde{\varphi}_2$.