Fermions of the ElectroWeak Theory

THE QUARKS, THE LEPTONS, AND THEIR MASSES.

This is my second set of notes on the Glashow–Weinberg–Salam theory of weak and electromagnetic interactions. The first set was about the bosonic fields of the theory — the gauge fields of the $SU(2) \times U(1)$ gauge theory and the Higgs fields that give mass to the $W^{\pm}_{\mu}$ and Z^{0}_{μ} vector particles. This set is about the fermionic fields — the quarks and the leptons.

From the fermionic point of view, the electroweak gauge symmetry $SU(2)_W \times U(1)_Y$ is *chiral* — the left-handed and the right-handed fermions form different types of multiplets — and consequently, the weak interactions do not respect the parity or the charge-conjugation symmetries. Specifically, all the left-handed quarks and leptons form doublets of the $SU(2)_W$ while the all right-handed quarks and leptons are singlets, so the charged weak currents are purely left-handed,

$$J^{\mu}_{\pm} \;=\; \tfrac{1}{2}\big(V^{\mu} \,-\, A^{\mu}\big) \;=\; \overline{\Psi}\gamma^{\mu}\frac{1-\gamma^{5}}{2}\Psi \;=\; \psi^{\dagger}_{L}\bar{\sigma}^{\mu}\psi_{L} \;\text{ without a } \psi_{R} \text{ term.} \tag{1}$$

The left-handed and the right-handed fermions also have different $U(1)$ hypercharges, which is needed to give them similar electric charges $Q = Y + T^3$. For example, the LH up and down quarks — which form an $SU(2)_W$ doublet — have $Y = +\frac{1}{6}$, while the RH quarks are $SU(2)$ singlets and have $Y_u = +\frac{2}{3}$ and $Y_d = -\frac{1}{3}$. Consequently, their electric charges come out to be

$$\left.\begin{aligned} Q(u,L) \;&=\; Y(u,L) \,+\, T^3(u,L) \;&=\; +\tfrac{1}{6} \,+\, \tfrac{1}{2} \;&=\; +\tfrac{2}{3} \\ Q(u,R) \;&=\; Y(u,R) \,+\, T^3(u,R) \;&=\; +\tfrac{2}{3} \,+\, 0 \;&=\; +\tfrac{2}{3} \end{aligned}\right\} \text{ same,}$$

$$\left.\begin{aligned} Q(d,L) \;&=\; Y(d,L) \,+\, T^3(d,L) \;&=\; +\tfrac{1}{6} \,-\, \tfrac{1}{2} \;&=\; -\tfrac{1}{3} \\ Q(d,R) \;&=\; Y(d,R) \,+\, T^3(d,R) \;&=\; -\tfrac{1}{3} \,+\, 0 \;&=\; -\tfrac{1}{3} \end{aligned}\right\} \text{ same.} \tag{2}$$

In light of different quantum numbers for the LH and RH quarks, their Lagrangian cannot have any mass terms $\psi^{\dagger}_{L}\psi_{R}$ or $\psi^{\dagger}_{R}\psi_{L}$. Instead, the physical quark masses arise from the Yukawa couplings of the quarks to the Higgs scalars H_i. In general, the Yukawa couplings of fermions

to scalars (or pseudoscalars) have form

$$g\phi \times \overline{\Psi}\Psi \quad \text{or} \quad g\phi \times \overline{\Psi}(i\gamma^5)\Psi, \tag{3}$$

or in terms of the Weyl fermions,

$$g\phi \times \psi_L^\dagger \psi_R \; + \; g^*\phi^* \times \psi_R^\dagger \psi_L \, . \tag{4}$$

The theories with multiple fermionic and scalar fields may have different Yukawa couplings for different scalar and fermionic species, as long as they are invariant under all the required symmetries. For the electroweak symmetry at hand, the ψ_L are $SU(2)$ doublets while the ψ_R are singlets, so the bi-linears $\psi_L^\dagger \psi_R$ and $\psi_R^\dagger \psi_L$ are $SU(2)$ doublets, which may couple to the $SU(2)$ doublet of scalars such as the Higgs fields H^i or their conjugates H_i^*. Taking the $U(1)$ hypercharges into account, the allowed Yukawa terms for the up and down quarks comprise

$$\begin{aligned} \mathcal{L}_{\text{Yukawa}} \; = \; & -g_d H_i^* \times \psi_R^{d\dagger} \psi_L^i \; - \; g_d H^i \times \psi_{L,i}^\dagger \psi_R^d \\ & - \; g_u \epsilon_{ij} H^i \times \psi_R^{u\dagger} \psi_L^j \; - \; g_u \epsilon^{ij} H_i^* \times \psi_{L,j}^\dagger \psi_R^u \, . \end{aligned} \tag{5}$$

When the Higgs develops a non-zero Vacuum Expectation Value

$$\langle H \rangle \; = \; \frac{v}{\sqrt{2}} \times \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad v \approx 247 \text{ GeV}, \tag{6}$$

the Yukawa couplings of the fermions to this VEV give rise to fermionic mass terms,

$$\mathcal{L}_{\text{Yukawa}} \; \longrightarrow \; \mathcal{L}_{\text{mass}} \; + \; \text{couplings to the physical Higgs field}, \tag{7}$$

$$\begin{aligned} \mathcal{L}_{\text{mass}} \; &= \; \mathcal{L}_{\text{Yukawa}} \quad \text{for} \quad H \to \langle H \rangle \\ &= \; -g_d \frac{v}{\sqrt{2}} \times \left(\psi_R^{d\dagger} \psi_L^2 \; + \; \psi_L^{2\dagger} \psi_R^d \right) \; - \; g_u \frac{v}{\sqrt{2}} \times \left(\psi_R^{u\dagger} \psi_L^1 \; + \; \psi_L^{1\dagger} \psi_R^u \right) \qquad (8) \\ &\equiv \; -m_d \times \overline{\Psi}^d \Psi^d \; - \; m_u \overline{\Psi}^u \Psi^u, \qquad (9) \end{aligned}$$

where the Dirac fermions Ψ^u and Ψ^d comprise

$$\Psi^u \;=\; \begin{pmatrix} \psi^1_L \\ \psi^u_R \end{pmatrix}, \qquad \Psi^d \;=\; \begin{pmatrix} \psi^2_L \\ \psi^d_R \end{pmatrix} \tag{10}$$

and their masses follow from the Higgs VEV and the Yukawa couplings as

$$m_u \;=\; g_u \times \frac{v}{\sqrt{2}}\,, \qquad m_d \;=\; g_d \times \frac{v}{\sqrt{2}}\,. \tag{11}$$

The other 4 quark flavors — charm, strange, top, and bottom — have similar quantum numbers to the up and down quarks. The left-handed quarks form $SU(2)$ doublets $(c,s)_L$ and $(t,b)_L$ with $Y = +\frac{1}{6}$ while the right-handed quarks are singlets with hypercharges $Y(c_R) = Y(T_R) = +\frac{2}{3}$ and $Y(s_R) = Y(b_R) = -\frac{1}{3}$, which lead to non-chiral electric charges

$$\begin{aligned} Q(c_{L\,\mathrm{or}\,R}) \;&=\; Q(t_{L\,\mathrm{or}\,R}) \;=\; Q(u_{L\,\mathrm{or}\,R}) \;=\; +\tfrac{2}{3}, \\ Q(s_{L\,\mathrm{or}\,R}) \;&=\; Q(b_{L\,\mathrm{or}\,R}) \;=\; Q(d_{L\,\mathrm{or}\,R}) \;=\; -\tfrac{1}{3}. \end{aligned} \tag{12}$$

Again, the $SU(2) \times U(1)$ quantum numbers of these quarks forbid any mass terms $\psi^\dagger_L \psi_R$ or $\psi^\dagger_R \psi_L$ in the Lagrangian, but they allow the Yukawa couplings to the Higgs fields similar to (5). The physical masses obtain from those Yukawa couplings when the Higgs scalar develops a non-zero VEV and breaks the $SU(2)\times U(1)$ symmetry down to the $U(1)_{\mathrm{EM}}$; similar to eq. (11),

$$m_s \;=\; g_s \times \frac{v}{\sqrt{2}}\,, \qquad m_c \;=\; g_c \times \frac{v}{\sqrt{2}}\,, \qquad m_b \;=\; g_b \times \frac{v}{\sqrt{2}}\,, \qquad m_t \;=\; g_t \times \frac{v}{\sqrt{2}}\,. \tag{13}$$

Note that the charge $= +\frac{2}{3}$ quarks u, c, t have exactly similar electroweak quantum numbers but very different values of the Yukawa couplings, $g_u \ll g_c \ll g_t$, and hence very different physical masses, $m_u \ll m_c \ll m_t$. Likewise, the charge $= -\frac{1}{3}$ quarks d, s, t have exactly similar electroweak quantum numbers but different Yukawa couplings, $g_d \ll g_s \ll g_b$, and hence different physical masses, $m_d \ll m_s \ll m_b$. Experimentally

$$m_u \approx 2.15 \text{ MeV} \;\ll\; m_c \approx 1.28 \text{ GeV} \;\ll\; m_t \approx 173 \text{ GeV}, \tag{14}$$

$$m_d \approx 4.7 \text{ MeV} \;\ll\; m_s \approx 94 \text{ MeV} \;\ll\; m_b \approx 4.2 \text{ GeV}, \tag{15}$$

but we do not have a good explanation of this hierarchical pattern. In the Standard Model, the Yukawa couplings are arbitrary parameters to be determined experimentally. Beyond the

Standard Model, there have been all kinds of speculative explanations over the last 40+ years, but none of them can be supported by any experimental evidence whatsoever.

Besides the quarks, there are 3 species of charged leptons — the electron e^-, the muon μ^-, and the tau τ^- — and 3 species of neutrinos, ν_e, ν_μ, ν_τ. The left-handed fermions of these 6 species form three $SU(2)$ doublets $(\nu_e, e^-)_L$, $(\nu_\mu, \mu^-)_L$, and $(\nu_\tau, \tau^-)_L$ with $Y = -\frac{1}{2}$, so the bottom halves of these doublets have electric charges

$$Q(e_L^-) \;=\; Q(\mu_L^-) \;=\; Q(\tau_L^-) \;=\; Y - \tfrac{1}{2} \;=\; -1 \tag{16}$$

while the top-halves — the neutrinos — are electrically neutral,

$$Q(\nu_{eL}) \;=\; Q(\nu_{\mu L}) \;=\; Q(\nu_{\tau L}) \;=\; Y + \tfrac{1}{2} \;=\; 0. \tag{17}$$

The right-handed electron, muon, and tau are $SU(2)$ singlets with $Y = -1$, so their electric charge $Q = Y + 0 = -1$ is the same as for the left-handed e, μ, τ.

As to the right-handed neutrino fields, there are two theories: In one theory, the neutrino fields are left-handed Weyl spinors $\psi_{_L}(\nu)$ rather than Dirac spinors, so the $\psi_{_R}(\nu)$ simply do not exist! In the other theory, the $\psi_{_R}(\nu)$ do exist, but they are $SU(2)$ singlets with $Y = 0$ and so do not have any in weak interactions. Since they also do not have strong or EM interactions, this makes the RH neutrinos completely invisible to the experiment — that's why we do not know if they exist or not. For the moment, let me focus on the simplest version without the $\psi_{_R}(\nu)$; I'll come back to the other theory later in these notes when I discuss the neutrino masses.

Similar to the quarks, the $SU(2) \times U(1)$ quantum numbers of the leptons do not allow any mass terms in the Lagrangian, but they do allow the Yukawa couplings of leptons to the Higgs fields,

$$\begin{aligned}
\mathcal{L}_{\text{Yukawa}} \;=\; & -g_e H_i^* \times \psi_{_R}^\dagger(e)\psi_{_L}^i(\nu_e, e) \;-\; g_e H^i \times \psi_{L,i}^\dagger(\nu_e, e)\psi_{_R}(e) \\
& -\; g_\mu H_i^* \times \psi_{_R}^\dagger(\mu)\psi_{_L}^i(\nu_\mu, \mu) \;-\; g_\mu H^i \times \psi_{L,i}^\dagger(\nu_\mu, \mu)\psi_{_R}(\mu) \\
& -\; g_\tau H_i^* \times \psi_{_R}^\dagger(\tau)\psi_{_L}^i(\nu_\tau, \tau) \;-\; g_\tau H^i \times \psi_{L,i}^\dagger(\nu_\tau, \tau)\psi_{_R}(\tau).
\end{aligned} \tag{18}$$

When the Higgs field H_2 develop non-zero VEV $\frac{v}{\sqrt{2}}$, these Yukawa couplings give rise to the

lepton masses; similar to the quarks,

$$m_e \;=\; g_e \times \frac{v}{\sqrt{2}}\,, \qquad m_\mu \;=\; g_\mu \times \frac{v}{\sqrt{2}}\,, \qquad m_\tau \;=\; g_\tau \times \frac{v}{\sqrt{2}}\,. \tag{19}$$

Experimentally, these masses are

$$m_e = 0.511 \text{ MeV} \;\ll\; m_\mu = 106 \text{ MeV} \;\ll\; m_\tau = 1777 \text{ MeV}. \tag{20}$$

Similar to the quarks, the masses of charged leptons form a hierarchy; we do no not know why.

Weak Currents

Altogether, the fermionic fields of the electroweak theory and their couplings to the bosonic gauge and Higgs fields can be summarized by the Lagrangian

$$\mathcal{L}_F \;=\; \sum_{\substack{\text{LH quarks}\\ \text{\& leptons}}} i\psi_L^\dagger \bar\sigma^\mu D_\mu \psi_L \;+\; \sum_{\substack{\text{RH quarks}\\ \text{\& leptons}}} i\psi_R^\dagger \sigma^\mu D_\mu \psi_R \;+\; \mathcal{L}_{\text{Yukawa}}\,. \tag{21}$$

In the first section of these notes I was focused on the Yukawa couplings that give rise to the fermion masses when the Higgs field gets its VEV, but now let's turn our attention to the interactions of quarks and leptons with the electroweak $SU(2) \times U(1)$ gauge fields. In the Lagrangian (21), the gauge interactions are hidden inside the covariant derivatives D_μ, so let me spell them out in detail:

- The left-handed quarks form $SU(2)$ doublets

$$\psi_L^i \;=\; \begin{pmatrix} u \\ d \end{pmatrix}_L \quad \text{or} \quad \begin{pmatrix} c \\ s \end{pmatrix}_L \quad \text{or} \quad \begin{pmatrix} t \\ b \end{pmatrix}_L \tag{22}$$

  of hypercharge $Y = +\frac{1}{6}$, so for the LH quark fields

$$D_\mu \psi_L^i \;=\; \partial_\mu \psi_L^i \;+\; \frac{ig_2}{2}\, W_\mu^a (\tau^a)^i{}_j \psi_L^j \;+\; \frac{ig_1}{6}\, B_\mu \psi_L^i\,.$$

- The left-handed leptons also form $SU(2)$ doublets

$$\psi_L^i \;=\; \begin{pmatrix} \nu_e \\ e^- \end{pmatrix}_L \quad \text{or} \quad \begin{pmatrix} \nu_\mu \\ \mu^- \end{pmatrix}_L \quad \text{or} \quad \begin{pmatrix} \nu_\tau \\ \tau^- \end{pmatrix}_L \tag{23}$$

but of hypercharge $Y = -\frac{1}{2}$, so for the LH lepton fields

$$D_\mu \psi_L^i \;=\; \partial_\mu \psi_L^i \;+\; \frac{ig_2}{2}\, W_\mu^a (\tau^a)^i{}_j \psi_L^j \;-\; \frac{ig_1}{2}\, B_\mu \psi_L^i \,.$$

- The right handed quarks are $SU(2)$ singlets of hypercharges $Y = +\frac{2}{3}$ or $Y = -\frac{1}{3}$, thus

$$\begin{aligned} &\text{for } \psi_R \;=\; u_R \text{ or } c_R \text{ or } t_R \,, \quad D_\mu \psi_R \;=\; \partial_\mu \psi_R \;+\; \frac{2ig_1}{3}\, B_\mu \psi_R \,, \\ &\text{for } \psi_R \;=\; d_R \text{ or } s_R \text{ or } b_R \,, \quad D_\mu \psi_R \;=\; \partial_\mu \psi_R \;-\; \frac{ig_1}{3}\, B_\mu \psi_R \,. \end{aligned} \tag{24}$$

- The right-handed charged leptons are $SU(2)$ singlets of hypercharge $Y = -1$, thus

$$\text{for } \psi_R \;=\; e_R^- \text{ or } \mu_R^- \text{ or } \tau_R^- \,, \quad D_\mu \psi_R \;=\; \partial_\mu \psi_R \;-\; ig_1\, B_\mu \psi_R \,. \tag{25}$$

— Finally, if the right-handed neutrino fields exist at all, they are $SU(2)$ singlets and have zero hypercharge, thus

$$\text{for } \psi_R \;=\; \nu_R^e \text{ or } \nu_R^\mu \text{ or } \nu_R^\tau \,, \quad D_\mu \psi_R \;=\; \partial_\mu \psi_R \;+\; 0. \tag{26}$$

Now let's plug these covariant derivatives into the Lagrangian (21), extract the terms containing the gauge fields, and organize the fermionic fields interacting with those gauge fields into the currents according to

$$\mathcal{L} \;\supset\; -g_2 W_\mu^a J_{Ta}^\mu \;-\; g_1 B_\mu J_Y^\mu \,, \tag{27}$$

cf. eq. (21) from the first set of my notes on the electroweak theory. Since the right-handed

quarks and leptons are $SU(2)$ singlets, the $SU(2)$ currents turn out to be purely left-handed,

$$J^\mu_{Ta} \;=\; \sum_{(u,d),(c,s),(t,b)}^{\text{LH quarks}} \psi^\dagger_{L,i} \left(\frac{\tau^a}{2}\right)^i_j \bar\sigma^\mu \psi^j_L \;+\; \sum_{(\nu_e,e),(\nu_\mu,\mu),(\nu_\tau,\tau)}^{\text{LH leptons}} \psi^\dagger_{L,i} \left(\frac{\tau^a}{2}\right)^i_j \bar\sigma^\mu \psi^j_L \,. \tag{28}$$

However, the $U(1)$ current has both left-handed and right handed contributions,

$$\begin{aligned} J^\mu_Y \;=\; & \sum_{u,c,t \text{ quarks}} \left(\tfrac{1}{6}\psi^\dagger_L \bar\sigma^\mu \psi_L \,+\, \tfrac{2}{3}\psi^\dagger_R \sigma^\mu \psi_R\right) \,+\, \sum_{d,s,b \text{ quarks}} \left(\tfrac{1}{6}\psi^\dagger_L \bar\sigma^\mu \psi_L \,-\, \tfrac{1}{3}\psi^\dagger_R \sigma^\mu \psi_R\right) \\ & +\, \sum_{e,\mu,\tau \text{ leptons}} \left(-\tfrac{1}{2}\psi^\dagger_L \bar\sigma^\mu \psi_L \,-\, \psi^\dagger_R \sigma^\mu \psi_R\right) \,+\, \sum_{\text{neutrinos}} \left(-\tfrac{1}{2}\psi^\dagger_L \bar\sigma^\mu \psi_L \,+\, 0\right). \end{aligned} \tag{29}$$

In the first set of notes I had re-organized these 4 gauge currents into currents which couple to the specific electroweak gauge field, namely the electric current

$$J^\mu_{\text{EM}} \;=\; J^\mu_{T3} \,+\, J^\mu_Y \tag{30}$$

which couples to the EM field A_μ, the *charged weak currents*

$$J^{+\mu} \;=\; J^\mu_{T1} \,-\, i J^\mu_{T2} \quad \text{and} \quad J^{-\mu} \;=\; J^\mu_{T1} \,+\, i J^\mu_{T2} \tag{31}$$

which couple to the charged $W^\pm_\mu$ massive vector fields, and the *neutral weak current*

$$J^\mu_Z \;=\; J^\mu_{T3} \,-\, \sin^2\theta J^\mu_{\text{EM}} \tag{32}$$

which couples to the neutral massive vector field Z^0_μ.

Now let's spell out all these currents in terms of the fermionic fields. For the charged currents, reorganizing the weak isospin currents (28) into the $J^{\pm\mu}$ amounts to combining the

isospin Pauli matrices τ^a in the same way as the currents (31),

$$\tau^+ \equiv \tau^1 - i\tau^2 = \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}, \qquad \tau^- \equiv \tau^1 + i\tau^2 = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}. \tag{33}$$

Consequently, in eqs. (28) we have

$$\psi^\dagger_{L,i} \left(\frac{\tau^+}{2}\right)^i_j \bar\sigma^\mu \psi^j_L = \psi^\dagger_{L,2} \bar\sigma^\mu \psi^1_L, \qquad \psi^\dagger_{L,i} \left(\frac{\tau^-}{2}\right)^i_j \bar\sigma^\mu \psi^j_L = \psi^\dagger_{L,1} \bar\sigma^\mu \psi^2_L, \tag{34}$$

and therefore

$$\begin{aligned} J^{+\mu} &= \psi^\dagger_L(d)\bar\sigma^\mu\psi_L(u) + \psi^\dagger_L(s)\bar\sigma^\mu\psi_L(c) + \psi^\dagger_L(b)\bar\sigma^\mu\psi_L(t) \\ &\quad + \psi^\dagger_L(e)\bar\sigma^\mu\psi_L(\nu_e) + \psi^\dagger_L(\mu)\bar\sigma^\mu\psi_L(\nu_\mu) + \psi^\dagger_L(\tau)\bar\sigma^\mu\psi_L(\nu_\tau), \\ J^{-\mu} &= \psi^\dagger_L(u)\bar\sigma^\mu\psi_L(d) + \psi^\dagger_L(c)\bar\sigma^\mu\psi_L(s) + \psi^\dagger_L(t)\bar\sigma^\mu\psi_L(b) \\ &\quad + \psi^\dagger_L(\nu_e)\bar\sigma^\mu\psi_L(e) + \psi^\dagger_L(\nu_\mu)\bar\sigma^\mu\psi_L(\mu) + \psi^\dagger_L(\nu_\tau)\bar\sigma^\mu\psi_L(\tau). \end{aligned} \tag{35}$$

In terms of Dirac fermions for the quarks and leptons,

$$\psi^\dagger_L \bar\sigma^\mu \psi_L = \overline{\Psi}\gamma^\mu \frac{1-\gamma^5}{2}\Psi, \tag{36}$$

hence

$$\begin{aligned} J^{+\mu} &= \overline{\Psi}^d\gamma^\mu\frac{1-\gamma^5}{2}\Psi^u + \overline{\Psi}^s\gamma^\mu\frac{1-\gamma^5}{2}\Psi^c + \overline{\Psi}^b\gamma^\mu\frac{1-\gamma^5}{2}\Psi^t \\ &\quad + \overline{\Psi}^e\gamma^\mu\frac{1-\gamma^5}{2}\Psi^{\nu_e} + \overline{\Psi}^\mu\gamma^\mu\frac{1-\gamma^5}{2}\Psi^{\nu_\mu} + \overline{\Psi}^\tau\gamma^\mu\frac{1-\gamma^5}{2}\Psi^{\nu_\tau}, \\ J^{-\mu} &= \overline{\Psi}^u\gamma^\mu\frac{1-\gamma^5}{2}\Psi^d + \overline{\Psi}^c\gamma^\mu\frac{1-\gamma^5}{2}\Psi^s + \overline{\Psi}^t\gamma^\mu\frac{1-\gamma^5}{2}\Psi^b \\ &\quad + \overline{\Psi}^{\nu_e}\gamma^\mu\frac{1-\gamma^5}{2}\Psi^e + \overline{\Psi}^{\nu_\mu}\gamma^\mu\frac{1-\gamma^5}{2}\Psi^\mu + \overline{\Psi}^{\nu_\tau}\gamma^\mu\frac{1-\gamma^5}{2}\Psi^\tau. \end{aligned} \tag{37}$$

As promised, these charged weak currents are purely left-handed, so they completely violate the parity and the charge-conjugation symmetries. But please note that this left-handedness is in terms of chirality of the fermionic fields rather than helicities of the fermionic particles. In terms of helicities, *the quarks and the leptons participating in charged-current weak interactions are polarized left, but the antiquarks and the antileptons are polarized right;* the degree of polarization is $\beta = v/c$, which approaches 100% for the ultrarelativistic particles.

On the other hand, the electric current is left-right symmetric,

$$J^\mu_{\rm EM} \;=\; \tfrac{2}{3}\sum_{q=u,c,t}^{\rm quarks}\overline{\Psi}^q\gamma^\mu\Psi^q \;-\; \tfrac{1}{3}\sum_{q=d,s,b}^{\rm quarks}\overline{\Psi}^q\gamma^\mu\Psi^q \;-\; \sum_{\ell=e,\mu\tau}^{\rm leptons}\overline{\Psi}^\ell\gamma^\mu\Psi^\ell\,. \tag{38}$$

Finally, the neutral weak current has both left-handed and right-handed components but it is not left-right symmetric. In terms of Dirac spinor fields,

$$\begin{aligned} J^\mu_Z \;&=\; J^\mu_{T3}[\text{left-handed}] \;-\; \sin^2\theta\times J^\mu_{\rm EM}[\text{left-right symmetric}] \\ &= \sum_{q=u,c,t}^{\rm quarks}\overline{\Psi}^q\gamma^\mu\left(+\frac{1-\gamma^5}{4} - \frac{2}{3}\sin^2\theta\right)\Psi^q \;+\; \sum_{q=d,s,b}^{\rm quarks}\overline{\Psi}^q\gamma^\mu\left(-\frac{1-\gamma^5}{4} + \frac{1}{3}\sin^2\theta\right)\Psi^q \\ &\quad+\sum_{\ell=e,\mu,\tau}^{\rm leptons}\overline{\Psi}^\ell\gamma^\mu\left(-\frac{1-\gamma^5}{4} + \sin^2\theta\right)\Psi^\ell \;+\; \sum_{\nu=\nu_e,\nu_\mu,\nu_\tau}^{\rm neutrinos}\overline{\Psi}^\nu\gamma^\mu\left(+\frac{1-\gamma^5}{4} - 0\right)\Psi^\nu\,. \end{aligned} \tag{39}$$

Flavor Mixing and the Cabibbo–Kobayashi–Maskawa Matrix

Actually, the charged weak currents are more complicated then I wrote down in eq. (37). Since we have 3 quark flavors of each charge $+\frac{2}{3}$ or $-\frac{1}{3}$, we need to be careful as to how they form 3 $SU(2)$ doublets. Normally, one defines the specific flavors of quarks as eigenstates of the quark mass matrix, but this definition does not respect the doublet structure: the $SU(2)$ partner of say the u quark is not the d quark but rather some linear combination of the d, s, b quarks, and likewise for the partners of the c and t quarks. Thus, the $SU(2)$ doublets are

$$\begin{pmatrix} u \\ d' \end{pmatrix}, \quad \begin{pmatrix} c \\ s' \end{pmatrix}, \quad \begin{pmatrix} t \\ b' \end{pmatrix}, \quad \text{for} \quad \begin{pmatrix} d' \\ s' \\ b' \end{pmatrix} \;=\; V\times\begin{pmatrix} d \\ s \\ b \end{pmatrix} \tag{40}$$

where V is a unitary 3×3 matrix called the *Cabibbo–Kobayashi–Maskawa* matrix (CKM). In this section, I shall first explain where this matrix comes from, and then I'll tell you its physical consequence for the weak interactions.

In the un-broken $SU(2)\times U(1)$ theory the quarks are massless and we cannot tell which quark is u, which is c, *etc.*, *etc.*; we cannot even tell which left-handed Weyl field pairs up

with which right-handed Weyl field into a Dirac spinor. We can use the $SU(2)$ symmetry to form doublets, but we are free to choose any basis we like for the 3 doublets — let's call them Q_α for $\alpha = 1, 2, 3$ — and we are free to change this basis by a unitary field re-definition,

$$\psi^i_L(Q_\alpha) \;\to\; \psi^i_L(Q'_\alpha) \;=\; \sum_\beta \bigl(\mathcal{U}^Q\bigr)_{\alpha,\beta} \times \psi^i_L(Q_\beta), \tag{41}$$

where $\mathcal{U}^Q$ is a unitary 3×3 matrix. Similarly, we may use any basis D_α for the 3 right-handed quarks of charge $-\frac{1}{3}$, any basis U_α for the 3 right-handed quarks of charge $+\frac{2}{3}$, and we are free to change these two bases by unitary transforms

$$\psi_R(U_\alpha) \;\to\; \psi_R(U'_\alpha) \;=\; \sum_\beta \bigl(\mathcal{U}^U\bigr)_{\alpha,\beta} \times \psi_R(U_\beta), \quad \psi_R(D_\alpha) \;\to\; \psi_R(D'_\alpha) \;=\; \sum_\beta \bigl(\mathcal{U}^D\bigr)_{\alpha,\beta} \times \psi_R(D_\beta), \tag{42}$$

where $\mathcal{U}^U$ and $\mathcal{U}^D$ are two independent unitary 3×3 matrices. However, we cannot mix the U_α with the D_α because of their different $U(1)$ hypercharges.

Likewise, we are free to use any basis L_α for the 3 doublets of left-handed leptons, any basis E_α for the 3 right-handed charged leptons, and we are free to changes all these bases by unitary transforms,

$$\psi^i_L(L_\alpha) \;\to\; \psi^i_L(L'_\alpha) \;=\; \sum_\beta \bigl(\mathcal{U}^L\bigr)_{\alpha,\beta} \times \psi^i_L(L_\beta), \quad \psi_R(E_\alpha) \;\to\; \psi_R(E'_\alpha) \;=\; \sum_\beta \bigl(\mathcal{U}^E\bigr)_{\alpha,\beta} \times \psi_R(E_\beta). \tag{43}$$

(I'll take care of the neutrinos in a later section.)

The Yukawa couplings involve one Higgs field H^i or H^*_i and two fermion fields, — one left-handed, one right-handed — and for each choice of their $SU(2)\times U(1)$ quantum numbers, there are three ψ_L fields and three ψ_R fields. Consequently, there is a big lot of the Yukawa terms in the Lagrangian, namely

$$\begin{aligned}\mathcal{L}_{\text{Yukawa}} \;=\; &-\sum_{\alpha,\beta} Y^U_{\alpha\beta} \times \psi^\dagger_R(U_\alpha)\psi^i_L(Q_\beta) \times \epsilon_{ij}H^j \;-\; \sum_{\alpha,\beta} Y^D_{\alpha\beta} \times \psi^\dagger_R(D_\alpha)\psi^i_L(Q_\beta) \times H^*_i \\ &-\sum_{\alpha,\beta} Y^E_{\alpha\beta} \times \psi^\dagger_R(E_\alpha)\psi^i_L(L_\beta) \times H^*_i \;+\; \text{Hermitian conjugates},\end{aligned} \tag{44}$$

where the $Y^U_{\alpha,\beta}$, the $Y^D_{\alpha,\beta}$, and the $Y^E_{\alpha,\beta}$ comprise three 3×3 complex matrices of the Yukawa coupling constants. And when the Higgs develops symmetry-breaking VEV, these matrices

of Yukawa couplings give rise to the complex 3×3 *mass matrices*

$$M^U_{\alpha,\beta} \;= \frac{v}{\sqrt{2}} \times Y^U_{\alpha,\beta}\,, \qquad M^U_{\alpha,\beta} \;= \frac{v}{\sqrt{2}} \times Y^D_{\alpha,\beta}\,, \qquad M^E_{\alpha,\beta} \;= \frac{v}{\sqrt{2}} \times Y^E_{\alpha,\beta}\,, \tag{45}$$

$$\mathcal{L}_{\rm mass} \;=\; -\sum_{\alpha,\beta} M^U_{\alpha\beta} \times \psi^\dagger_R(U_\alpha)\psi^1_L(Q_\beta) \;- \sum_{\alpha,\beta} M^D_{\alpha\beta} \times \psi^\dagger_R(D_\alpha)\psi^2_L(Q_\beta) \tag{46}$$

$$- \sum_{\alpha,\beta} M^E_{\alpha\beta} \times \psi^\dagger_R(E_\alpha)\psi^2_L(L_\beta) \;+\; \text{Hermitian conjugates.} \tag{47}$$

To get the physical masses of quarks and leptons, we need to diagonalize these mass matrices via suitable unitary transforms (41)–(43). In matrix notations, these transforms lead to

$$\big(Y^U\big)' \;=\; \mathcal{U}^U \times Y^U \times \big(\mathcal{U}^Q\big)^\dagger, \quad \big(Y^D\big)' \;=\; \mathcal{U}^D \times Y^D \times \big(\mathcal{U}^Q\big)^\dagger, \quad \big(Y^E\big)' \;=\; \mathcal{U}^E \times Y^E \times \big(\mathcal{U}^L\big)^\dagger, \tag{48}$$

and consequently

$$\big(M^U\big)' \;=\; \mathcal{U}^U \times M^U \times \big(\mathcal{U}^Q\big)^\dagger, \quad \big(M^D\big)' \;=\; \mathcal{U}^D \times M^D \times \big(\mathcal{U}^Q\big)^\dagger, \quad \big(M^E\big)' \;=\; \mathcal{U}^E \times M^E \times \big(\mathcal{U}^L\big)^\dagger. \tag{49}$$

Now, any complex matrix M can be written as a product $M = W_1 D W_2$ where W_1 and W_2 are unitary matrices while D is diagonal, real, and non-negative.[⋆] Consequently, using appropriate unitary matrices $\mathcal{U}^E$ and $\mathcal{U}^Q$ we can make the charged lepton's mass matrix diagonal and real

$$M^E \;\to\; \big(M^E\big)' \;=\; \mathcal{U}^E \times M^E \times \big(\mathcal{U}^L\big)^\dagger \;=\; \begin{pmatrix} m_e & 0 & 0 \\ 0 & m_\mu & 0 \\ 0 & 0 & m_\tau \end{pmatrix}. \tag{50}$$

Note that it is in the transformed bases — where the $(M^E)'$ is diagonal — that the LH and

⋆ To prove, start with a polar decomposition $M = UH$ where U is unitary and $H = \sqrt{M^\dagger M}$ is hermitian and positive semi-definite. Then diagonalize the hermitian matrix H, *i.e.*, write it as $H = W^\dagger D W$ for some unitary matrix W. Consequently, $M = UW^\dagger D W = W_1 D W_2$ for $W_2 = W$ and $W_1 = UW^\dagger$.

RH Weyl fields combine into Dirac fields of the physical electron, muon, and the tau,

$$\begin{aligned}
\Psi_e &= \begin{pmatrix} \psi^2_L(L'_1) = \mathcal{U}^L_{1\beta}\psi^2_L(L_\beta) \\ \psi_R(E'_1) = \mathcal{U}^E_{1\beta}\psi_R(E_\beta) \end{pmatrix}, \\
\Psi_\mu &= \begin{pmatrix} \psi^2_L(L'_2) = \mathcal{U}^L_{2\beta}\psi^2_L(L_\beta) \\ \psi_R(E'_2) = \mathcal{U}^E_{2\beta}\psi_R(E_\beta) \end{pmatrix}, \\
\Psi_\tau &= \begin{pmatrix} \psi^2_L(L'_3) = \mathcal{U}^L_{3\beta}\psi^2_L(L_\beta) \\ \psi_R(E'_3) = \mathcal{U}^E_{3\beta}\psi_R(E_\beta) \end{pmatrix}.
\end{aligned} \tag{51}$$

Likewise, using the $\mathcal{U}^U$ and the $\mathcal{U}^Q$ unitary matrices we may diagonalize the mass matrix for the charge $+\frac{2}{3}$ quarks,

$$\begin{aligned}
M^U \;\to\; \left(M^U\right)' &= \mathcal{U}^U \times M^U \times \left(\mathcal{U}^Q\right)^\dagger = \begin{pmatrix} m_u & 0 & 0 \\ 0 & m_c & 0 \\ 0 & 0 & m_t \end{pmatrix}, \\
\Psi_u &= \begin{pmatrix} \psi^1_L(Q'_1) = \mathcal{U}^Q_{1\beta}\psi^1_L(Q_\beta) \\ \psi_R(U'_1) = \mathcal{U}^U_{1\beta}\psi_R(U_\beta) \end{pmatrix}, \\
\Psi_c &= \begin{pmatrix} \psi^1_L(Q'_2) = \mathcal{U}^Q_{2\beta}\psi^1_L(Q_\beta) \\ \psi_R(U'_2) = \mathcal{U}^U_{2\beta}\psi_R(U_\beta) \end{pmatrix}, \\
\Psi_t &= \begin{pmatrix} \psi^1_L(Q'_3) = \mathcal{U}^Q_{3\beta}\psi^1_L(Q_\beta) \\ \psi_R(U'_3) = \mathcal{U}^U_{3\beta}\psi_R(U_\beta) \end{pmatrix},
\end{aligned} \tag{52}$$

and similarly for the charge $-\frac{1}{3}$ quarks,

$$\begin{aligned}
M^D \;\to\; \left(M^D\right)' &= \mathcal{U}^D \times M^D \times \left(\widetilde{\mathcal{U}}^Q\right)^\dagger = \begin{pmatrix} m_d & 0 & 0 \\ 0 & m_s & 0 \\ 0 & 0 & m_b \end{pmatrix}, \\
\Psi_d &= \begin{pmatrix} \psi^2_L(Q'_1) = \widetilde{\mathcal{U}}^Q_{1\beta}\psi^2_L(Q_\beta) \\ \psi_R(U'_1) = \mathcal{U}^D_{1\beta}\psi_R(D_\beta) \end{pmatrix}, \\
\Psi_s &= \begin{pmatrix} \psi^2_L(Q'_2) = \widetilde{\mathcal{U}}^Q_{2\beta}\psi^1_L(Q_\beta) \\ \psi_R(U'_2) = \mathcal{U}^D_{2\beta}\psi_R(D_\beta) \end{pmatrix}, \\
\Psi_b &= \begin{pmatrix} \psi^2_L(Q'_3) = \widetilde{\mathcal{U}}^Q_{3\beta}\psi^1_L(Q_\beta) \\ \psi_R(U'_3) = \mathcal{U}^D_{3\beta}\psi_R(D_\beta) \end{pmatrix},
\end{aligned} \tag{53}$$

However, *it takes different unitary matrices* $\mathcal{U}^Q \neq \widetilde{\mathcal{U}}^Q$ *to diagonalize the up-type and down-type quark mass matrices, and that's what messes up the* $SU(2)$ *doublet structure!* Indeed, in terms of the upper components $\psi^1_L(Q_\alpha)$ of the original doublets, the left-handed u, c, t quarks of definite mass are linear combinations

$$\begin{pmatrix} u_L \\ c_L \\ t_L \end{pmatrix} = \mathcal{U}^Q \times \begin{pmatrix} \psi^1_L(Q_1) \\ \psi^1_L(Q_2) \\ \psi^1_L(Q_3) \end{pmatrix}, \tag{54}$$

so their $SU(2)$ partners are *similar* linear combinations of the lower components $\psi^2_L(Q_\alpha)$ of the original doublets,

$$\begin{pmatrix} d'_L \\ s'_L \\ b'_L \end{pmatrix} = \mathcal{U}^Q \times \begin{pmatrix} \psi^2_L(Q_1) \\ \psi^2_L(Q_2) \\ \psi^2_L(Q_3) \end{pmatrix}, \tag{55}$$

for the same $\mathcal{U}^Q$ matrix as the up-type quarks. On the other hand, the d, s, b quarks defined as mass eigenstates obtain from different linear combinations

$$\begin{pmatrix} d_L \\ s_L \\ b_L \end{pmatrix} = \widetilde{\mathcal{U}}^Q \times \begin{pmatrix} \psi^2_L(Q_1) \\ \psi^2_L(Q_2) \\ \psi^2_L(Q_3) \end{pmatrix}. \tag{56}$$

Comparing the sets of down-type quark fields, we immediately see that

$$\begin{pmatrix} d'_L \\ s'_L \\ b'_L \end{pmatrix} = \mathcal{U}^Q \times \widetilde{\mathcal{U}}^{Q\dagger} \times \begin{pmatrix} d_L \\ s_L \\ b_L \end{pmatrix}, \tag{57}$$

which gives us the Cabibbo–Kobayashi–Maskawa matrix

$$V_{\text{CKM}} = \mathcal{U}^Q \times \widetilde{\mathcal{U}}^{Q\dagger}. \tag{58}$$

Now let's go back to the charged weak currents $J^{\pm\mu}$. Since they are gauge currents of the $SU(2)_W$, they connect a fermion in some $SU(2)$ doublet into the other fermion in exactly same doublet! Thus, the J^+ current would turn the u quark into its partner d', or the c quark

into its partner s', *etc.*, and vice verse for the J^- current. In terms of the Dirac spinor fields, this means

$$\begin{aligned} J^{-\mu}(\text{quarks}) &= \overline{\Psi}^u \gamma^\mu \frac{1-\gamma^5}{2} \Psi^{d'} + \overline{\Psi}^c \gamma^\mu \frac{1-\gamma^5}{2} \Psi^{s'} + \overline{\Psi}^t \gamma^\mu \frac{1-\gamma^5}{2} \Psi^{b'} \\ &= \sum_{\alpha=u,c,t} \sum_{\beta=d,s,b} V_{\alpha,\beta} \times \overline{\Psi}^\alpha \gamma^\mu \frac{1-\gamma^5}{2} \Psi^\beta, \\ J^{+\mu}(\text{quarks}) &= \overline{\Psi}^{d'} \gamma^\mu \frac{1-\gamma^5}{2} \Psi^u + \overline{\Psi}^{s'} \gamma^\mu \frac{1-\gamma^5}{2} \Psi^c + \overline{\Psi}^{b'} \gamma^\mu \frac{1-\gamma^5}{2} \Psi^t \\ &= \sum_{\alpha=u,c,t} \sum_{\beta=d,s,b} V^*_{\alpha,\beta} \times \overline{\Psi}^\beta \gamma^\mu \frac{1-\gamma^5}{2} \Psi^\alpha, \end{aligned} \tag{59}$$

where $V_{\alpha,\beta}$ is the CKM matrix.

The CKM matrix is very important for the physics of weak interactions. For example, without this matrix the strange particles like the K-mesons or Λ-baryons would be stable because the s quark would not be able to decay. Indeed, the $SU(2)$ partner of the s quark is the c quark, so without the CKM matrix the only flavor-changing processes involving the s quark would be $s \leftrightarrow c$. However, the c quark is much heavier than s, so the decay can only go from c to s but not from s to c. But thanks to the CKM matrix — specifically, to the non-zero matrix element $V_{u,s}$ — the s quark may also decay to the u quark (which is lighter than s), albeit with a reduced amplitude $\propto V_{u,s} \approx 0.22$.

There are many other interesting flavor-changing weak processes involving the charged currents and the CKM matrix. I wish I could spend a few weeks talking about them, but alas I do not have the time for this in my QFT class. I hope professor Çan Kiliç will explain the subject in some detail in his Phenomenology class in Fall 2017. But in these notes, I have to move on to the next subject.

Eq. (59) give the charged weak currents of the quarks, but what about the leptons? Again, we need to pick the bases for the 3 charged leptons and for the neutrinos, and if the two bases disagree with the $SU(2)$ doublet structure, we would get a CKM-like matrix for the leptons. For the charged leptons, the mass is important, so people always use the basis of mass eigenstates (e, μ, τ) as in eqs. (50) and (51). But the neutrino masses are so small, they only matter in long-baseline interferometry experiments, so for all other purposes people use

the interaction basis $(\nu_e, \nu_\mu, \nu_\tau)$ of species *defined as the $SU(2)$ partners of the corresponding charged leptons* (e, μ, τ). In this basis, there are no CKM-like matrices and

$$\begin{aligned} J^{+\mu}(\text{leptons}) &= \overline{\Psi}^{e}\gamma^{\mu}\frac{1-\gamma^5}{2}\Psi^{\nu_e} + \overline{\Psi}^{\mu}\gamma^{\mu}\frac{1-\gamma^5}{2}\Psi^{\nu_\mu} + \overline{\Psi}^{\tau}\gamma^{\mu}\frac{1-\gamma^5}{2}\Psi^{\nu_\tau}, \\ J^{-\mu}(\text{leptons}) &= \overline{\Psi}^{\nu_e}\gamma^{\mu}\frac{1-\gamma^5}{2}\Psi^{e} + \overline{\Psi}^{\nu_\mu}\gamma^{\mu}\frac{1-\gamma^5}{2}\Psi^{\mu} + \overline{\Psi}^{\nu_\tau}\gamma^{\mu}\frac{1-\gamma^5}{2}\Psi^{\tau}. \end{aligned} \tag{60}$$

On the other hand, in this basis the neutrino mass matrix is non-diagonal, which makes neutrinos slowly oscillate from one species to another. I'll come back to this issue later in these notes.

Now consider the neutral weak current J^μ_Z. The unitary transforms that diagonalize the fermion's mass matrices can only mix up fields with similar chiralities (ψ_L and ψ_R) and electric charges. In the Standard Model, this limits the mixing to fermions that have both similar weak isospins T^3 and similar hypercharges Y, which makes for similar contributions to the neutral weak current

$$J^\mu_Z \supset \sum_{\alpha}^{\text{LH species}} \bigl(T^3 - \sin^2\theta Q_{\text{el}}\bigr)\psi^\dagger_L(\alpha)\bar\sigma^\mu\psi_L(\alpha) \quad \text{or} \quad \sum_{\alpha}^{\text{RH species}} \bigl(T^3 - \sin^2\theta Q_{\text{el}}\bigr)\psi^\dagger_R(\alpha)\sigma^\mu\psi_R(\alpha). \tag{61}$$

The sums here are invariant under all unitary field redefinitions that mix only fields with similar $T^3 - \sin^2 Q_{\text{el}}$, so regardless of such redefinitions the neutral weak current remains diagonal. Specifically,

$$\begin{aligned} J^\mu_Z &= J^\mu_{T3}[\text{left-handed}] - \sin^2\theta \times J^\mu_{\text{EM}}[\text{left-right symmetric}] \\ &= \sum_{q=u,c,t}^{\text{quarks}} \overline{\Psi}^{q}\gamma^{\mu}\left(+\frac{1-\gamma^5}{4} - \frac{2}{3}\sin^2\theta\right)\Psi^{q} + \sum_{q=d,s,b}^{\text{quarks}} \overline{\Psi}^{q}\gamma^{\mu}\left(-\frac{1-\gamma^5}{4} + \frac{1}{3}\sin^2\theta\right)\Psi^{q} \\ &\quad + \sum_{\ell=e,\mu,\tau}^{\text{leptons}} \overline{\Psi}^{\ell}\gamma^{\mu}\left(-\frac{1-\gamma^5}{4} + \sin^2\theta\right)\Psi^{\ell} + \sum_{\nu=\nu_e,\nu_\mu,\nu_\tau}^{\text{neutrinos}} \overline{\Psi}^{\nu}\gamma^{\mu}\left(+\frac{1-\gamma^5}{4} - 0\right)\Psi^{\nu}, \end{aligned} \tag{39}$$

and *there are no flavor-changing neutral weak currents in the Standard Model.*

Note that this is a peculiar property of the Standard Model where all fermions of the same electric charge and chirality also have the same T^3. Historically, before the Standard

Model was fully developed and confirmed experimentally, people used to consider models with different quantum numbers for different quarks. In particular, back in the 1960's when only 3 quark flavors u, d, s were known, people assumed the left-handed s quark was un-paired $SU(2)$ singlet (with $Y = -\frac{1}{3}$ to give it the right electric charge). The mass matrix somehow mixed the two charge $-\frac{1}{3}$ quarks d and s, so the $SU(2)$ doublet was $(u, d')_L$ while the singlet was s'_L, where

$$d' \;=\; d \times \cos\theta_c \;+\; s \times \sin\theta_c\,, \qquad s' = \; s \times \cos\theta_c \;-\; d \times \sin\theta_c\,, \qquad \theta_c \;\approx\; 13^\circ. \tag{62}$$

In such a model, the s'_L and the d'_L have different $T^3 - \sin^2\theta Q_{\rm el}$, so their mixing makes for off-diagonal terms in the J^μ_Z. In other words, there would be the $s \leftrightarrow d$ flavor changing neutral current, which would lead to processes like the $K^0 \to \mu^+\mu^-$ decay. But experimentally, there are no such decays, nor any other signatures of the flavor-changing neutral currents. This made Glashow, Illiopoulos, and Maiani conjecture in 1970 that the s quark (or rather the s') should be a member of a doublet just like the d' quark — which would give them the same T^3 and hence keep the neutral weak current flavor-diagonal — and consequently there must be a fourth quark flavor c to form the (c, s') doublet. And in 1974 this fourth flavor (called the 'charm') was experimentally discovered at SLAC and BNL.

Later, when the fifth flavor b was discovered in 1977, most physicists expected it to also be a part of the doublet, so everybody was looking for the sixth flavor t. This expectation turned out to be correct, and the t quark was duly discovered in 1995. The delay was due to the very large mass of the top quark, $m_t \approx 173$ GeV, much heavier that the other 5 flavors.

CP violation

Like any chiral gauge theory, the weak interactions do not have the parity symmetry **P** or the charge conjugation symmetry **C**. In particular, the charged currents involve only the left-chirality Weyl spinors, which in particle terms mean left-helicity quarks and leptons but right-helicity anti-quarks or anti-leptons.

However, the chirality is perfectly consistent with the combined **CP** symmetry, which does not mix the ψ_L and the ψ_R fields; instead it acts as

$$\mathbf{CP}: \quad \psi_{_L}(\mathbf{x}, t) \;\to \pm\sigma_2\psi^*_L(-\mathbf{x}, t), \quad \psi_{_R}(\mathbf{x}, t) \;\to \pm\sigma_2\psi^*_R(-\mathbf{x}, t). \tag{63}$$

By the CPT theorem, the **CP** symmetry is equivalent to the time-reversal (or rather motion-reversal) symmetry **T**, so it would be nice to have it as an exact symmetry of Nature. But in 1964, Cronin and Fitch have discovered that weak decays of the neutral K-mesons are only approximately CP-symmetric, but sometimes a CP-odd state of the kaon decays into a CP-even pair of pions. Later experiments found CP violations in other weak processes involving mesons containing b quarks or c quarks.

All the experimentally measured CP-violating effects can be explained by the imaginary parts $\mathrm{Im}(V_{\alpha,\beta})$ of the CKM matrix elements. In General, the relation between the CP violation and the CKM matrix is rather complicated and involves interference between different orders of perturbations theory; at the tree level, there is no CP violation. I am not going to work out such complicated issues in these notes; instead, let me simply show that complex CKM matrix violates the CP symmetry of the electroweak Lagrangian.

Since the neutral weak current does not care about the CKM matrix, let me focus on the charged currents. Under **CP**, the charged vector fields $W^{\pm}_{\mu}(x)$ transform as

$$\mathbf{CP}: \quad W^{\pm}_0(\mathbf{x},t) \;\to\; -W^{\mp}_0(-\mathbf{x},+t), \quad W^{\pm}_i(\mathbf{x},t) \;\to\; +W^{\mp}_i(-\mathbf{x},+t), \tag{64}$$

where the exchange $W^+ \leftrightarrow W^-$ is due to charge conjugation while different signs for 3-scalar and 3-vector components are due to reflection $\mathbf{x} \to -\mathbf{x}$ of the space coordinates. Consequently, in a **CP** symmetric theory we would need a similar relation for the charged currents,

$$\mathbf{CP}: \quad J^{0\pm}(\mathbf{x},t) \;\to\; -J^{0\pm}(-\mathbf{x},+t), \quad J^{i\pm}(\mathbf{x},t) \;\to\; +J^{i\mp}(-\mathbf{x},+t), \tag{65}$$

In terms of fermions, the charged weak currents are sums of left-handed currents terms of general form

$$j^{\mu}_L \;=\; \psi^{1\dagger}_L \bar{\sigma}^{\mu} \psi^{2}_L \;=\; \overline{\Psi}^1 \gamma^{\mu} \frac{1-\gamma^5}{2} \Psi^2 \,, \tag{66}$$

so let's work out how such terms transform under **CP**. Assuming the Weyl fermions ψ^1_L and

ψ_L^2 have the similar intrinsic CP signs as members of the same $SU(2)$ doublet, we have

$$\begin{aligned}
\mathbf{CP}: \ \psi_L^{1\dagger}\bar{\sigma}^\mu\psi_L^2 \ &= \ +(\psi_L^1)^\top\sigma_2\times\bar{\sigma}^\mu\times\sigma_2(\psi_L^2)^* \\
&= \ +(\psi_L^1)^\top\times\big(\sigma_2\bar{\sigma}^\mu\sigma_2 \ = \ (\sigma^\mu)^\top\big)\times(\psi_L^2)^* \\
&= \ -\psi_L^{2\dagger}\sigma^\mu\psi_L^1 \\
&= \ \psi_L^{2\dagger}\bar{\sigma}^\mu\psi_L^1\times\begin{cases}+1 & \text{for } \mu=1,2,3,\\ -1 & \text{for } \mu=0.\end{cases}
\end{aligned} \tag{67}$$

The μ dependence of the overall sign here — which comes from comparing $-\sigma^\mu$ to $+\bar{\sigma}^\mu$ — is in perfect agreement with eq. (65). In Dirac notations, eq (67) amounts to

$$\mathbf{CP}: \ \overline{\Psi}^1\gamma^\mu\frac{1-\gamma^5}{2}\Psi^2 \ \to \ \overline{\Psi}^2\gamma^\mu\frac{1-\gamma^5}{2}\Psi^1\times\begin{cases}+1 & \text{for } \mu=1,2,3,\\ -1 & \text{for } \mu=0.\end{cases} \tag{68}$$

Besides the μ–dependent sign, the **CP** exchanges the two fermionic species $\Psi^1 \leftrightarrow \Psi^2$ involved in the current j_L^μ. For the leptonic charged weak currents (60), this exchange leads to $J^{+\mu} \leftrightarrow J^{-\mu}$, exactly as in eq. (65); indeed,

$$J^{+\mu} \ \supset \ \overline{\Psi}^e\gamma^\mu\frac{1-\gamma^5}{2}\Psi^{\nu_e} \quad \text{while} \quad J^{-\mu} \ \supset \ \overline{\Psi}^{\nu_e}\gamma^\mu\frac{1-\gamma^5}{2}\Psi^e, \quad \textit{etc., etc.} \tag{69}$$

Consequently, the interactions

$$\mathcal{L} \ \supset = \ -\frac{g_2}{\sqrt{2}}\times\Big(W_\mu^+ J^{\mu-}_{\text{leptonic}} \ + \ W_\mu^- J^{\mu+}_{\text{leptonic}}\Big) \tag{70}$$

of the leptons with the vector fields $W_\mu^\pm$ are invariant under **CP**.

But for the charged currents of the quarks, we have

$$\begin{aligned}
J^{-\mu}(\text{quarks}) \ &= \ \sum_{\alpha=u,c,t}\ \sum_{\beta=d,s,b} V_{\alpha,\beta}\times\overline{\Psi}^\alpha\gamma^\mu\frac{1-\gamma^5}{2}\Psi^\beta, \\
J^{+\mu}(\text{quarks}) \ &= \ \sum_{\alpha=u,c,t}\ \sum_{\beta=d,s,b} V^*_{\alpha,\beta}\times\overline{\Psi}^\beta\gamma^\mu\frac{1-\gamma^5}{2}\Psi^\alpha,
\end{aligned} \tag{59}$$

which transform into

$$\begin{aligned}
\mathbf{CP}:\ J^{-\mu}(\text{quarks}) \ &\to\ \pm(\mu)\times\sum_{\beta=d,s,b} V_{\alpha,\beta}\times\overline{\Psi}^{\beta}\gamma^{\mu}\frac{1-\gamma^5}{2}\Psi^{\alpha},\\
&\text{which is almost like}\quad \pm(\mu)\times J^{+\mu}(\text{quarks}),\ \text{except for } V_{\alpha,\beta} \text{ instead of } V^*_{\alpha,\beta};\\
\mathbf{CP}:\ J^{+\mu}(\text{quarks}) \ &\to\ \pm(\mu)\times\sum_{\beta=d,s,b} V^*_{\alpha,\beta}\times\overline{\Psi}^{\alpha}\gamma^{\mu}\frac{1-\gamma^5}{2}\Psi^{\beta},\\
&\text{which is almost like}\quad \pm(\mu)\times J^{-\mu}(\text{quarks}),\ \text{except for } V^*_{\alpha,\beta} \text{ instead of } V_{\alpha,\beta};
\end{aligned} \tag{71}$$

Consequently, the net effect of **CP** on the interactions

$$\mathcal{L}\ \supset =\ -\frac{g_2}{\sqrt{2}}\times\left(W^+_{\mu}J^{\mu-}_{\text{quark}}\ +\ W^-_{\mu}J^{\mu+}_{\text{quark}}\right) \tag{72}$$

of the $W^{\pm}_{\mu}$ with the quarks is equivalent to complex conjugating the CKM matrix,

$$\mathbf{CP}:\quad V_{\alpha,\beta}\ \leftrightarrow\ V^*_{\alpha,\beta}\,. \tag{73}$$

Thus, the weak interactions of quarks (and hence hadrons) are **CP** symmetric if and only if the CKM matrix is real.

For three families of quarks and leptons, the CKM matrix V is a unitary 3×3 matrix. To parametrize such a matrix we need 3 real angles — as for a real orthogonal $O(3)$ matrix — and 6 phases, for the total of $6+3=3^2$ parameters. However, some of the 6 phases can be eliminated by the unitary field redefinitions which commute with the mass matrices, namely the abelian symmetries

$$\left.\begin{array}{llll}
\Psi^u\ \to\ e^{i\theta_u}\Psi^u, & \Psi^c\ \to\ e^{i\theta_c}\Psi^c, & \Psi^t\ \to\ e^{i\theta_t}\Psi^t, & \\
\Psi^d\ \to\ e^{i\theta_d}\Psi^d, & \Psi^s\ \to\ e^{i\theta_s}\Psi^s, & \Psi^b\ \to\ e^{i\theta_b}\Psi^b, &
\end{array}\right\}\quad\Longrightarrow\quad V_{\alpha,\beta}\ \to\ e^{i\theta_\alpha-i\theta_\beta}\times V_{\alpha,\beta}\,.$$

Note that the common phase change for all the quarks does not affect the CKM matrix, but the differences between phases for different quarks do make a difference. Thus, 5 out of 6 phases in V can be eliminated, but we are stuck with one remaining phase. It is this one phase that's responsible for all the CP violations by the weak interaction!

BTW, if we had only two families of quarks and leptons (the u, s, c, s quarks but not the b and t, and the e, ν_e, μ, ν_μ leptons but not the τ, ν_τ), the CKM matrix would have 1 real angle — the Cabibbo angle $\theta_c \approx 13^\circ$ — and 3 phases, but all these phases could be eliminated by the remaining abelian symmetries of the quarks. Consequently, there would be no CP violation!

Back in 1973, only two families were known — in fact, even the charm quark was predicted but not yet discovered experimentally — and the origin of the weak CP violation was a complete mystery (although there were many far-out speculations). At that time, Kobayashi and Maskawa speculated that maybe there is a third family similar to the first two; in this case, there Cabibbo mixing matrix would be 3×3 instead of 2×2, so one of its complex phases could not be eliminated by field redefinition, and that would be a source of CP violation. Their speculation turned out to be correct, and in 2008 Kobayashi and Maskawa got a Nobel prize.

Neutrino masses

Back when the Glashow–Weinberg–Salam theory was formulated, the neutrinos were though to be exactly massless. But the later discovery of neutrino oscillations between the ν_e, ν_μ, and ν_τ species calls for tiny but non-zero neutrino masses, $m_\nu < 1$ eV. Or rather, the oscullations call for the *neutrino mass matrix* $M^\nu_{\alpha,\beta}$ that is non-diagonal in the weak-interactions basis $(\nu_e, \mu_\mu, \nu_\tau)$. Indeed, consider the effective Hamiltonian for a single ultra-relativistic neutrino particle; in the momentum-species basis,

$$\hat{H} \;=\; \sqrt{p^2 + M^2} \;\approx\; p \;+\; \frac{M^2}{2p}\,. \tag{74}$$

While a free neutrino flies from the point where it is produced to the point where it is detected, the second term here causes its species state to oscillate,

$$|p,\alpha\rangle \;\to\; \sum_\beta \exp\left(\frac{iL}{2p}\times M^2\right)_{\alpha,\beta} |p,\beta\rangle \quad \text{(up to an overall phase)}. \tag{75}$$

To illustrate how this works, let me spell out the oscillation matrix for 2 neutrino species, say

ν_e and ν_μ. In this case, the 2×2 mass matrix can be written as

$$M^2 \;=\; \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} m_1^2 & 0 \\ 0 & m_2^2 \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \tag{76}$$

where m_2^1 and m_2^2 are the eigenvalues and θ is the mixing angle between the mass eigenbasis (ν_1, ν_2) and the weak-interaction basis (ν_e, μ_μ). Consequently, the oscillation matrix in eq. (75) becomes (up to an overall phase)

$$\begin{aligned} \exp\left(\frac{iL}{2p} \times M^2\right) \;=\; & \cos\left(\frac{L(m_1^2 - m_2^2)}{4p}\right) \times \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ & +\; i \sin\left(\frac{L(m_1^2 - m_2^2)}{4p}\right) \times \begin{pmatrix} \cos(2\theta) & \sin(2\theta) \\ -\sin(2\theta) & \cos(2\theta) \end{pmatrix}. \end{aligned} \tag{77}$$

For the three neutrino species, the 3×3 oscillation matrix is more complicated, so I am not writing it here. Let me simply say that it depends on the differences $m_1^2 - m_2^2$ and $m_2^2 - m_2^3$ between the mass^2 eigenvalues and the CKM-like mixing angles between the weak-interactions basis $(\nu_e, \mu_\mu, \nu_\tau)$ and the mass eigenbasis (ν_1, ν_2, ν_3).

Experimentally, the neutrino mixing angles are rather large — $\theta_{12} \approx 33^\circ$, $\theta_{23} \approx 41^\circ$, $\theta_{13} \approx 8.5^\circ$, much larger than the CKM angles for the quarks — while the Δm^2 differences are very small, $\Delta m_{12}^2 \approx 74 \cdot 10^{-6}$ eV^2 and $\Delta m_{23}^2 \approx 26 \cdot 10^{-4}$ eV^2.

Theoretically, there are two ways to add the neutrino masses to the Glashow–Weinberg–Salam theory. The first possibility is to make the neutrino fields Dirac spinors and give them masses via Yukawa couplings to the Higgs doublet, just like the other fermions of the theory. In terms of the original theory (with massless neutrinos), this means adding 3 $SU(2)$ singlet, $Y = 0$ right-handed Weyl fields $\psi_R(N_\alpha)$ and give them Yukawa couplings

$$\mathcal{L}_{\text{Yukawa}} \;\subset\; -\sum_{\alpha,\beta} Y^N_{\alpha,\beta} \times \psi^\dagger_R(N_\alpha) \psi^i_L(L_\beta) \times \epsilon_{ij} H^j \;+\; \text{Hermitian conjugates.} \tag{78}$$

When the Higgs gets its VEV, these Yukawa couplings give rise to the Dirac mass terms for the neutrinos

$$\mathcal{L}_{\text{mass}} \;\supset\; -\sum_{\alpha,\beta} M^N_{\alpha,\beta} \times \psi^\dagger_R(N_\alpha) \psi^i_L(L_\beta) \;+\; H.\,c., \qquad M^N_{\alpha,\beta} \;=\; \frac{v}{\sqrt{2}} \times Y^N_{\alpha,\beta}\,. \tag{79}$$

The only problem with this setup is that it does not explain why the neutrinos are so light

compared to the other fermions of the Standard Model — million times lighter than even the electron, never mind the heavier leptons or quarks. Al we can say is that somehow, the Yukawa couplings for the neutrinos are extremely weak $Y^N \sim 10^{-12}$, but why?!? — we do not have a clue.

The other possibility is to make the neutrinos Majorana fermions. In Dirac-spinor notations, a Majorana fermion is a neutral field $\Psi(x) = \gamma^2\Psi^*(x)$. In terms of the Weyl spinor fields,

$$\text{Majorana } \Psi(x) \;=\; \begin{pmatrix} \psi_L(x) \\ -\sigma_2\psi_L^*(x) \end{pmatrix} \quad \text{for the same } \psi_L(x), \tag{80}$$

there is no independent $\psi_R(x)$. Thus, a majorana fermion is equivalent to a single Weyl fermion $\psi_L(x)$ together with its conjugate $\psi_L^\dagger(x)$. The Lagrangian for the free Majorana field is

$$\mathcal{L} \;=\; \tfrac{1}{2}\overline{\Psi}(i\,\partial\!\!\!/ - m)\Psi \;=\; i\psi_L^\dagger\bar\sigma^\mu\partial_\mu\psi_L \;+\; \frac{m}{2}\,\psi_L^\top\sigma_2\psi_L \;+\; \frac{m}{2}\,\psi_L^\dagger\sigma_2\psi_L^*\,. \tag{81}$$

In a general theory of multiple fermions, mass terms like in this formula are called *Majorana masses*.

To give the neutrinos Majorana masses we do not need the independent right-handed neutrino fields $\psi_R(N_\alpha)$. All we need are the left-handed neutrino fields $\psi_L^1(L_\alpha)$ and their conjugates, plus some interactions that would give rise to the Majorana mass terms

$$\mathcal{L}_{\text{mass}} \;\supset\; \tfrac{1}{2}\sum_{\alpha,\beta} M^\nu_{\alpha,\beta}\big(\psi_L^1(L_\alpha)\big)^\top\sigma_2\psi_L^1(L_\beta) \;+\; \tfrac{1}{2}\sum_{\alpha,\beta} M^{\nu*}_{\alpha,\beta}\big(\psi_L^1(L_\alpha)\big)^\dagger\sigma_2\big(\psi_L^1(L_\beta)\big)^*, \tag{82}$$

Note that the mass matrix in this formula may be complex rather than real, but it should be symmetric $M^\nu_{\beta,\alpha} = M^\nu_{\alpha,\beta}$ because

$$\big(\psi_L^1(L_\beta)\big)^\top\sigma_2\psi_L^1(L_\alpha) \;=\; +\big(\psi_L^1(L_\alpha)\big)^\top\sigma_2\psi_L^1(L_\beta) \tag{83}$$

— the σ_2 matrix is antisymmetric, but the fields are anticommuting fermions.

The neutrino mass terms (82) break the $SU(2)\times U(1)$ gauge symmetry so we cannot put them directly into the Lagrangian of the high-energy theory. Instead, they obtain from the gauge-invariant couplings of the leptons and Higgs fields, which give rise to the mass terms

after the Higgs gets its vacuum expectation value. The simplest couplings that will do this job are the Yukawa-like couplings involving two left-handed lepton fields and two Higgs scalars,

$$\mathcal{L}_{LLHH} \;=\; \tfrac{1}{2}\sum_{\alpha,\beta} C_{\alpha,\beta} \times \big(H_i^* \psi_L^i(L_\alpha)\big)^\top \sigma_2 \big(H_j^* \psi_L^j(L_\beta)\big) \;+\; H.\,c. \tag{84}$$

Note that the product $H_i^* \psi_L^i$ of the Higgs doublet and the left-handed Lepton doublet is a gauge-invariant Weyl spinor, so we can combine two such products into a gauge-invariant, Lorentz-invariant Lagrangian term.

When the Higgs VEV breaks the electroweak gauge symmetry, it also makes neutrino mass terms from the couplings (84). Indeed, substituting Higgs VEV $\langle H\rangle_i^*$ into the interaction terms (84), we obtain the Majorana mass terms for the neutrinos that look exactly like in eq. (82) for

$$M^\nu_{\alpha,\beta} \;=\; \frac{v^2}{2} \times C_{\alpha,\beta}\,. \tag{85}$$

Unlike the dimensionless gauge and Yukawa couplings, the $C_{\alpha,\beta}$ couplings have dimensionality (energy)$^{-1}$. Such couplings make trouble for the perturbation theory at high energies, so they are not allowed in UV-complete quantum field theories. However, if the Standard Model is only an effective theory that's valid up to some maximal energy $E_{\max}$ but at higher energies must be superseded by a more complete theory, then it's OK for the SM to have *small* negative-dimensionality couplings $C \lesssim (1/E_{\max})$. The key word here is *small* — it explains why the neutrinos are so much lighter than the other fermions: If $C < 1/E_{\max}$, then

$$m_\nu \;\lesssim\; \frac{v^2}{E_{\max}} \;\ll\; v. \tag{86}$$

In particular, for $E_{\max} \sim (10^{15}\,\text{GeV})$ this limit tells us $m_\nu \lesssim 0.1\,\text{eV}$, which is in the right ballpark for the neutrino masses inferred from the neutrino oscillations.