Fermionic Functional Integrals

Gaussian Integrals over Fermionic Variables

As a prelude to Functional Integrals over fermionic fields, let's study Gaussian integrals over finite numbers of fermionic variables, *i.e.*, odd Grassmann numbers. For simplicity, let's assume 'complex' Grassmann numbers for which θ and its conjugate $\bar{\theta}$ are independent, thus $\theta\bar{\theta} \neq 0$. Similar to the Hermitian conjugation of linear operators, the 'bar' conjugation of Grassmann numbers reverses their order of multiplication, thus

$$\overline{\theta_1\theta_2} \;=\; \bar{\theta}_2\bar{\theta}_1 \;=\; -\bar{\theta}_1\bar{\theta}_2. \tag{1}$$

In particular,

$$\int d^N\theta \;=\; \int d\theta_N \cdots \int d\theta_1 \quad \text{but} \quad \int d^N\bar{\theta} \;=\; \int d\bar{\theta}_1 \cdots \int d\bar{\theta}_N\,. \tag{2}$$

Theorem: for any $N \times N$ bosonic matrix A_{ij}, let

$$\Theta^\dagger A \Theta \;=\; \bar{\theta}_i A_{ij} \theta_j \;=\; \sum_{i,j=1}^{N} \bar{\theta}_i A_{ij} \theta_j\,, \tag{3}$$

then

$$\int d^N\bar{\theta} \int d^N\theta\, \exp\Bigl(-\Theta^\dagger A \Theta\Bigr) \;=\; \det(A). \tag{4}$$

Before proving this theorem for general N, let's see how it works for $N = 1$ and $N = 2$. For $N = 1$, A is just a number (real or complex), $\Theta^\dagger A \Theta$ is simply $\bar{\theta} A \theta$, and

$$\exp\bigl(-\bar{\theta} A \theta\bigr) \;=\; 1 \;-\; \bar{\theta} A \theta \;+\; \text{nothing else},$$

hence

$$\int d\bar{\theta} \int d\theta\, \exp\bigl(-\bar{\theta} A \theta\bigr) \;=\; A \times \int d\bar{\theta} \int d\theta\, (-\bar{\theta}\theta \;=\; +\theta\bar{\theta}) \;=\; A. \tag{5}$$

Next, for $N = 2$

$$\exp\bigl(-\Theta^\dagger A\Theta\bigr)\ =\ 1\ -\ \bigl(\Theta^\dagger A\Theta\bigr) +\ \tfrac{1}{2}\bigl(\Theta^\dagger A\Theta\bigr)^2 \tag{6}$$

where the highest component is

$$\tfrac{1}{2}\bigl(\Theta^\dagger A\Theta\bigr)^2\ =\ \tfrac{1}{2}A_{ij}A_{k\ell}\times\bar\theta_i\theta_j\bar\theta_k\theta_\ell\ =\ -\tfrac{1}{2}A_{ij}A_{k\ell}\times\theta_j\theta_\ell\times\bar\theta_i\bar\theta_k\,. \tag{7}$$

Moreover, since there are only two independent θ's at play and they anticommute with each other,

$$\theta_j\theta_\ell\ =\ \epsilon_{j\ell}\theta_1\theta_2 \tag{8}$$

where $\epsilon_{j\ell}$ is the 2D Levi–Civita tensor, and likewise

$$\bar\theta_i\bar\theta_k\ =\ \epsilon_{ik}\bar\theta_1\bar\theta_2\ =\ -\epsilon_{ik}\bar\theta_2\bar\theta_1\,. \tag{9}$$

Consequently,

$$\Bigl[\exp\bigl(-\Theta^\dagger A\Theta\bigr)\Bigr]^{\text{highest}}_{\text{component}}\ =\ +\tfrac{1}{2}A_{ij}A_{k\ell}\times\epsilon_{j\ell}\epsilon_{ik}\times\theta_1\theta_2\bar\theta_2\bar\theta_1 \tag{10}$$

and therefore

$$\int\!d^2\bar\theta\int\!d^2\theta\ \exp\Bigl(-\Theta^\dagger A\Theta\Bigr)\ =\ +\tfrac{1}{2}A_{ij}A_{k\ell}\times\epsilon_{j\ell}\epsilon_{ik}\,. \tag{11}$$

Finally, on the RHS here

$$\begin{aligned}+\tfrac{1}{2}A_{ij}A_{k\ell}\times\epsilon_{j\ell}\epsilon_{ik}\ &=\ \tfrac{1}{2}A_{11}A_{22}\ -\ \tfrac{1}{2}A_{12}A_{21}\ -\ \tfrac{1}{2}A_{21}A_{12}\ +\ \tfrac{1}{2}A_{22}A_{11}\\ &=\ A_{11}A_{22}\ -\ A_{12}A_{21}\ =\ \det(A),\end{aligned} \tag{12}$$

which verifies the theorem for $N = 2$.

In the same way, for $N \geq 3$, the highest component of $\exp\bigl(-\Theta^\dagger A\Theta\bigr)$ is

$$\begin{aligned}
\frac{1}{N!}\bigl(-\Theta^\dagger A\Theta\bigr)^N &= \frac{(-1)^N}{N!}\,A_{i_1j_1}A_{i_2j_2}\cdots A_{i_Nj_N}\times\bar\theta_{i_1}\theta_{j_1}\bar\theta_{i_2}\theta_{j_2}\cdots\bar\theta_{i_N}\theta_{j_N}\\
&= \frac{(-1)^N}{N!}\,A_{i_1j_1}A_{i_2j_2}\cdots A_{i_Nj_N}\times(-1)^{N(N+1)/2}\,\theta_{j_1}\cdots\theta_{j_N}\times\bar\theta_{i_1}\cdots\bar\theta_{i_N}\,.
\end{aligned} \tag{13}$$

Furthermore, since there are only N θ's at play

$$\theta_{j_1}\cdots\theta_{j_N} = \epsilon_{j_1,\ldots,j_N}\times\theta_1\cdots\theta_N \tag{14}$$

where $\epsilon_{j_1,\ldots,j_N}$ is the N-dimensional Levi–Civita tensor, and likewise

$$\bar\theta_{i_1}\cdots\bar\theta_{i_N} = \epsilon_{i_1,\ldots,i_N}\times\bar\theta_1\cdots\bar\theta_N = \epsilon_{i_1,\ldots,i_N}\times(-1)^{N(N-1)/2}\,\bar\theta_N\cdots\bar\theta_1\,. \tag{15}$$

Consequently,

$$\Bigl[\exp\bigl(-\Theta^\dagger A\Theta\bigr)\Bigr]^{\text{highest}}_{\text{component}} = \frac{+1}{N!}\,A_{i_1j_1}A_{i_2j_2}\cdots A_{i_Nj_N}\times\epsilon_{j_1,j_2,\ldots,j_N}\epsilon_{i_1,i_2,\ldots,i_N}\times\bar\theta_{i_1}\cdots\bar\theta_{i_N}\theta_N\cdots\theta_1 \tag{16}$$

and therefore

$$\int\! d^N\bar\theta\int\! d^N\theta\,\exp\Bigl(-\Theta^\dagger A\Theta\Bigr) = \frac{+1}{N!}\,A_{i_1j_1}A_{i_2j_2}\cdots A_{i_Nj_N}\times\epsilon_{j_1,j_2,\ldots,j_N}\epsilon_{i_1,i_2,\ldots,i_N} = \det(A), \tag{17}$$

which proves the theorem (4).

Note: unlike the bosonic Gaussian integral

$$\int\! d^Nz^*\int\! d^Nz\,\exp\Bigl(-z_i^*A_{ij}z_j\Bigr) = \frac{(2\pi)^N}{\det(A)}\,, \tag{18}$$

the fermionic Gaussian integral (4) is directly rather than inversely proportional to the determinant $\det(A)$. However, both types of Gaussian integrals can be easily generalized to Gaussian+ integrals such as

$$\int\! d^Nz^*\int\! d^Nz\,\exp\Bigl(-z_i^*A_{ij}z_j\Bigr)\times z_kz_\ell^* = \frac{(2\pi)^N}{\det(A)}\times\bigl(A^{-1}\bigr)_{k\ell}\,, \tag{19}$$

$$\int d^N z^* \int d^N z\, \exp\Bigl(-z_i^* A_{ij} z_j\Bigr) \times z_k z_\ell z_m^* z_n^* \;=\; \frac{(2\pi)^N}{\det(A)} \times \tag{20}$$
$$\times \Bigl(\bigl(A^{-1}\bigr)_{km} \bigl(A^{-1}\bigr)_{\ell n} + \bigl(A^{-1}\bigr)_{kn} \bigl(A^{-1}\bigr)_{\ell m} \Bigr),$$

etc., for the bosonic variables — as we saw last lecture, — and similarly

$$\int d^N \bar\theta \int d^N \theta\, \exp\Bigl(-\Theta^\dagger A \Theta\Bigr) \times \theta_k \bar\theta_\ell \;=\; \det(A) \times \bigl(A^{-1}\bigr)_{k\ell}\,, \tag{21}$$

$$\int d^N \bar\theta \int d^N \theta\, \exp\Bigl(-\Theta^\dagger A \Theta\Bigr) \times \theta_k \theta_\ell \bar\theta_m \bar\theta_n \;=\; \det(A) \times \tag{22}$$
$$\times \Bigl(-\bigl(A^{-1}\bigr)_{km} \bigl(A^{-1}\bigr)_{\ell n} + \bigl(A^{-1}\bigr)_{kn} \bigl(A^{-1}\bigr)_{\ell m} \Bigr),$$

etc., for the fermionic variables. Indeed,

$$\begin{aligned} &\int d^N \bar\theta \int d^N \theta\, \exp\Bigl(-\Theta^\dagger A \Theta\Bigr) \times \theta_k \bar\theta_\ell \\ &\qquad = \frac{\partial}{\partial A_{\ell k}} \int d^N \bar\theta \int d^N \theta\, \exp\Bigl(-\Theta^\dagger A \Theta\Bigr) \\ &\qquad = \frac{\partial}{\partial A_{\ell k}} \det(A) \;=\; \det(A) \times \bigl(A^{-1}\bigr)_{k\ell}\,, \end{aligned} \tag{23}$$

likewise

$$\begin{aligned} &\int d^N \bar\theta \int d^N \theta\, \exp\Bigl(-\Theta^\dagger A \Theta\Bigr) \times \theta_k \theta_\ell \bar\theta_m \bar\theta_n \\ &\qquad = -\frac{\partial}{\partial A_{n\ell}} \frac{\partial}{\partial A_{mk}} \int d^N \bar\theta \int d^N \theta\, \exp\Bigl(-\Theta^\dagger A \Theta\Bigr) \\ &\qquad = -\frac{\partial}{\partial A_{n\ell}} \frac{\partial}{\partial A_{mk}} \det(A) \;=\; -\frac{\partial}{\partial A_{n\ell}} \Bigl(\det(A) \times \bigl(A^{-1}\bigr)_{km}\Bigr) \\ &\qquad = \det(A) \times \Bigl(-\bigl(A^{-1}\bigr)_{km} \bigl(A^{-1}\bigr)_{\ell n} + \bigl(A^{-1}\bigr)_{kn} \bigl(A^{-1}\bigr)_{\ell m} \Bigr), \end{aligned} \tag{24}$$

and similarly for more $(\theta, \bar\theta)$ pairs outside the exponential.

Functional Integrals for Free Fermionic Fields

A 'classical' Dirac field $\Psi_\alpha(x)$ is odd–Grassmann–number valued. That is, for each spacetime point x and each Dirac component α there is an independent complex Grassmann variable $\Psi_\alpha(x)$ and its conjugate $\Psi^\dagger_\alpha(x)$, and all such variables anticommute with each other. The Dirac action

$$S[\overline{\Psi}(x),\Psi(x)] \;=\; \int d^4x\,\overline{\Psi}(i\,\partial\!\!\!/ - m)\Psi \tag{25}$$

is bi-linear in Ψ and $\overline{\Psi}$, so the Functional integral over these fields

$$\iiint\mathcal{D}[\overline{\Psi}(x)]\iiint\mathcal{D}[\Psi(x)]\,\exp\bigl(iS[\overline{\Psi},\Psi]\bigr) \;=\; \mathrm{Det}(\partial\!\!\!/ + im) \tag{26}$$

simply generalizes the Gaussian fermionic integrals from the previous section to the infinite-dimensional family of independent fermionic variables. Likewise, the correlation functions of the fermionic fields obtain from the generalization of the Gaussian+ integrals (21), (22), *etc.*:

$$\begin{aligned}\langle\Omega|\,\mathbf{T}\Psi(y)\overline{\Psi}(x)\,|\Omega\rangle \;&=\; \frac{\iiint\mathcal{D}[\overline{\Psi}]\iiint\mathcal{D}[\Psi]\,\exp\bigl(iS[\overline{\Psi},\Psi]\bigr)\times\Psi(y)\overline{\Psi}(x)}{\iiint\mathcal{D}[\overline{\Psi}]\iiint\mathcal{D}[\Psi]\,\exp\bigl(iS[\overline{\Psi},\Psi]\bigr)}\\ &=\; \langle x|\,\frac{i}{i\,\partial\!\!\!/ - m}\,|y\rangle \;=\; \int\frac{d^4p}{(2\pi)^4}\,e^{ip(x-y)}\times\frac{i}{p\!\!\!/ - m}\end{aligned} \tag{27}$$

(Dirac indices suppressed), likewise

$$\begin{aligned}\langle\Omega|\,\mathbf{T}\Psi(w)\Psi(z)\overline{\Psi}(y)\overline{\Psi}(x)\,|\Omega\rangle \;&=\; \frac{\iiint\mathcal{D}[\overline{\Psi}]\iiint\mathcal{D}[\Psi]\,\exp\bigl(iS[\overline{\Psi},\Psi]\bigr)\times\Psi(w)\Psi(z)\overline{\Psi}(y)\overline{\Psi}(x)}{\iiint\mathcal{D}[\overline{\Psi}]\iiint\mathcal{D}[\Psi]\,\exp\bigl(iS[\overline{\Psi},\Psi]\bigr)}\\ &=\; -\,\langle x|\,\frac{i}{i\,\partial\!\!\!/ - m}\,|z\rangle\times\langle y|\,\frac{i}{i\,\partial\!\!\!/ - m}\,|w\rangle\\ &\quad+\;\langle x|\,\frac{i}{i\,\partial\!\!\!/ - m}\,|w\rangle\times\langle y|\,\frac{i}{i\,\partial\!\!\!/ - m}\,|z\rangle\,.\end{aligned} \tag{28}$$

etc., *etc.*

For closer similarity with functional integrals over the bosonic fields, let's analytically continue the Dirac fields to the Euclidean spacetime and introduce the sources. In Euclidean spacetime, all 4 Dirac matrices γ_E^μ are Hermitian, specifically

$$\gamma_E^4 \;=\; \gamma_M^0\,, \quad \vec{\gamma}_E \;=\; -i\vec{\gamma}_M\,, \quad \Longrightarrow \quad \{\gamma_E^\mu,\gamma_E^\nu\} \;=\; 2\delta^{\mu\nu} \tag{29}$$

and also

$$\not{\partial}_E \;=\; \gamma_M^0(-i\partial_0)_M \;+\; (-i\vec{\gamma}_M)\cdot\nabla_M \;=\; -i\,\not{\partial}_M\,. \tag{30}$$

Consequently,

$$iS_M \;=\; i\!\int\! d^4x_M\,\overline{\Psi}(i\,\not{\partial}_M - m)\Psi \;=\; \int\! d^4x_E\,\overline{\Psi}(-\,\not{\partial}_E - m)\Psi \;=\; -\int\! d^4x_E\,\mathcal{L}_E \tag{31}$$

for

$$\mathcal{L}_E \;=\; \overline{\Psi}(\not{\partial}_E + m)\Psi. \tag{32}$$

As to the sources, since $\Psi_\alpha(x)$ and $\overline{\Psi}_\alpha(x)$ are independent fermionic fields, we have independent sources for both of them, $\eta_\alpha(x)$ and $\overline{\eta}_\alpha(x)$. Altogether, the Euclidean action including the source terms is

$$S_E[\Psi,\overline{\Psi};\eta,\overline{\eta}] \;=\; \int\! d^4x_E\Big(\mathcal{L} \;-\; \overline{\eta}\Psi \;-\; \overline{\Psi}\eta\Big), \tag{33}$$

the partition function is

$$Z[\eta,\overline{\eta}] \;=\; \iiint\!\!\mathcal{D}[\overline{\Psi}] \iiint\!\!\mathcal{D}[\Psi]\;\exp\bigl(-S_E[\Psi,\overline{\Psi};\eta,\overline{\eta}]\bigr), \tag{34}$$

and its logarithm (or rather $-\log(Z)$) is the generation functional of the connected correlation functions,

$$G_2^{\rm conn}(x;y) \;=\; -\frac{\delta^2\log Z[\eta,\bar{\eta}]}{\delta\bar{\eta}(x)\,\delta\eta(y)}\,, \quad etc. \tag{35}$$

For the free fermions, this generation functional — or rather its dependence on η and $\bar{\eta}$ sources — can be completed exactly by completing the action (33) to a full square: For any given $\eta(x)$ and $\bar{\eta}(x)$, let

$$\Psi'(x) \;=\; \Psi(x) \;+\; (\not{\partial}+m)^{-1}\eta(x), \quad \overline{\Psi}'(x) \;=\; \overline{\Psi}(x) \;+\; \overline{\eta}(x)(\overleftarrow{\not{\partial}}+m)^{-1}, \tag{36}$$

then

$$S_E \;=\int\! d^4x_E\Big(\overline{\Psi}(\not{\partial}+m)\Psi \;-\; \overline{\eta}\Psi \;-\; \overline{\Psi}\eta\Big) \;=\int\! d^4x_E\Big(\overline{\Psi}'(\not{\partial}+m)\Psi' \;-\; \overline{\eta}(\not{\partial}+m)^{-1}\eta\Big) \tag{37}$$

and consequently

$$\begin{aligned}
Z[\eta,\overline{\eta}] \;&=\; \iiint\!\mathcal{D}[\overline{\Psi}] \iiint\!\mathcal{D}[\Psi]\, \exp\bigl(-S_E[\Psi,\overline{\Psi};\eta,\overline{\eta}]\bigr)\\
&=\; \iiint\!\mathcal{D}[\overline{\Psi}'] \iiint\!\mathcal{D}[\Psi']\, \exp\left(-\int\! d^4x_E\,\overline{\Psi}'(\not{\partial}_m)\Psi'\right) \times \exp\left(+\int\! d^4x_e\,\overline{\eta}(\not{\partial}+m)^{-1}\eta\right)\\
&=\; \exp\left(+\int\! d^4x_e\,\overline{\eta}(\not{\partial}+m)^{-1}\eta\right) \times Z[0,0].
\end{aligned} \tag{38}$$

Or in terms of the generating functional of the connected correlators,

$$-\log Z[\eta,\bar{\eta}] \;=\; -\log Z_0 \;-\int\! d^4x_e\,\overline{\eta}(\not{\partial}+m)^{-1}\eta \quad \text{(exactly)}. \tag{39}$$

Thus, the free Dirac fields have only one connected correlation function, namely the free propagator

$$G_2^{\rm conn}(x,y) \;=\; \frac{\delta}{\delta\overline{\eta}(y)}\frac{\delta}{\delta\eta(x)}\bigl(-\log Z\bigr) \;=\; +\langle y|\,(\not{\partial}+m)^{-1}\,|x\rangle \;=\int\frac{d^4p_E}{(2\pi)^4}\,e^{ip(x-y)}\times\frac{1}{i\,\not{p}_E+m}\,. \tag{40}$$

Note: in the Euclidean spacetime, the Dirac propagator is

$$\;=\; \frac{1}{i\,\not{p}_E+m} \;=\; i\times\frac{i}{\not{p}_M-m} \tag{41}$$

where the overall factor of i between Euclidean and Minkowski propagators is common to all field types, scalars, vectors, spinors, *etc.*, *etc.* As to the denominator here, due to different

$\gamma\mu$ matrices in Euclidean and Minkowski spaces, we have

$$\not{p}_E \;=\; \gamma_E^4 p_E^4 \;+\; \vec{\gamma}_E\cdot\vec{p} \;=\; \gamma^0(ip^0) \;+\; (-i\vec{\gamma})\cdot\vec{p} \;=\; i\gamma^\mu p_\mu \;=\; i\,\not{p}_M \tag{42}$$

and therefore

$$\left(\frac{1}{i\,\not{p}+m}\right)_E \;=\; \left(\frac{1}{-\not{p}+m}\right)_M \;=\; i\times\left(\frac{i}{\not{p}-m}\right)_M . \tag{43}$$

Fermionic Functional Integrals in QED

In the simplest version of Quantum ElectroDynamics — EM and electron fields, and nothing else — the Euclidean Lagrangian is

$$\mathcal{L}_E \;=\; +\tfrac{1}{4}F_{\mu\nu}^2 \;+\; \overline{\Psi}(\not{D}+m)\Psi \tag{44}$$

where $D_\mu = \partial_\mu - ieA_\mu$ is the covariant derivative, the Euclidean action including the source terms is

$$S_E \;=\; \int\! d^4x_E\Big(\mathcal{L} \;-\; J_\mu A_\mu \;-\; \overline{\Psi}\eta \;-\; \overline{\eta}\Psi\Big), \tag{45}$$

and the partition action is

$$\begin{aligned} Z[J_\mu,\eta,\bar{\eta}] \;=\; &\iiint\!\mathcal{D}[A_\mu]\,\exp\left(-\int\! d^4x_E\Big(\tfrac{1}{4}F_{\mu\nu}^2 \;-\; J_\mu A_\mu\Big)\right)\times\\ &\times\iiint\!\mathcal{D}[\overline{\Psi}]\iiint\!\mathcal{D}[\Psi]\,\exp\left(-\int\! d^4x_E\Big(\overline{\Psi}(\not{D}+m)\Psi \;-\; \overline{\eta}\Psi \;-\; \overline{\Psi}\eta\Big)\right). \end{aligned} \tag{46}$$

The functional integral over the EM fields $A_\mu(x)$ has its own issues, and I address it in a separate set of notes. For the moment, let's focus on the fermionic functional integral in a background of given EM fields $A^\mu(x)$. Thus, we identify the integral on the second line of eq. (46) as a fermionic partition function

$$\widehat{Z}[A_\mu,\eta,\bar{\eta}] \;=\; \iiint\mathcal{D}[\overline{\Psi}]\iiint\mathcal{D}[\Psi]\,\exp\left(-\int d^4x_E\left(\overline{\Psi}(\not{D}+m)\Psi\,-\,\overline{\eta}\Psi\,-\,\overline{\Psi}\eta\right)\right). \tag{47}$$

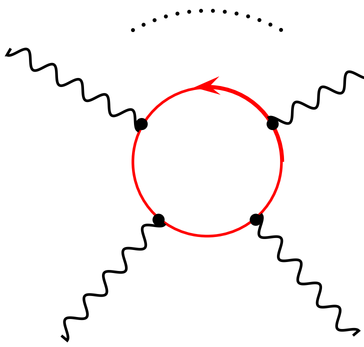
The integral here is Gaussian, so it formally evaluates to

$$\widehat{Z}[A_\mu,\eta,\bar{\eta}] \;=\; \mathrm{Det}(\not{D}+m)\times\exp\left(\int d^4x_e\,\overline{\eta}\frac{1}{\not{D}+m}\eta\right) \tag{48}$$

or in terms of the generating functional $-\log\widehat{Z}$,

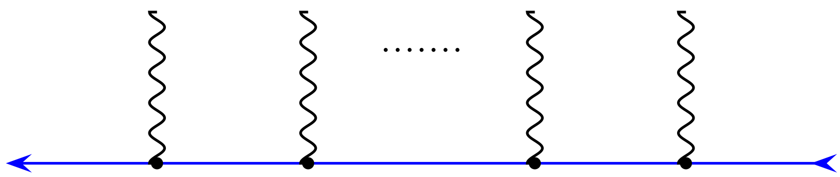
$$-\log\widehat{Z}[A_\mu,\eta,\bar{\eta}] \;=\; -\log\det(\not{D}+m)\;-\int d^4x_e\,\overline{\eta}\frac{1}{\not{D}+m}\eta. \tag{49}$$

Physically, the red term generates one loop diagrams where a bunch of external photons are connected to an electron loop



(50)

while the blue term generates tree diagrams where photons are connected to an open electron line



(51)

To see how this works, let's start with the functional determinant $\mathrm{Det}(\not{D}+m)$. To evaluate this determinant, we note that

$$\not{D} + m \; = \not{\partial} \, - \, ie\not{A} \, + \, m \;\; = \;\; (\not{\partial} + m) \times \left[1 \, - \, \frac{1}{\not{\partial} + m} (ie\not{A}) \right] \tag{52}$$

and therefore

$$\mathrm{Det}(\not{D} + m) \;\; = \;\; \mathrm{Det}(\not{\partial} + m) \times \mathrm{Det} \times \left[1 \, - \, \frac{1}{\not{\partial} + m} (ie\,\not{A}) \right] \tag{53}$$

where the first factor is badly divergent but it does not depend on the background EM field $A_\mu(x)$. Therefore, we may treat it as an overall *constant* factor of the partition function. But the second factor in eq. (53) does depend on the EM background, so when we eventually integrate over EM fields $A_\mu(x)$, this factor will appear in the context of

$$\iiint \mathcal{D}[A_\mu] \, \exp\bigl(-S_E[A_\mu] \bigr) \times \left[1 \, - \, \frac{1}{\not{\partial} + m} (ie\not{A}) \right] . \tag{54}$$

which we may interpret as

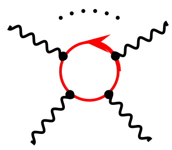
$$\iiint \mathcal{D}[A_\mu] \, \exp\bigl(-S_E^{\mathrm{eff}}[A_\mu] \; = \; -S_E^{\mathrm{tree}}[A_\mu] \, - \, \Delta S_E[A_\mu] \bigr) \tag{55}$$

where

$$\Delta S_E[A_\mu] \;\; = \;\; -\log \mathrm{Det} \left[1 \, + \, \frac{1}{\not{\partial} + m} (ie\not{A}) \right] \tag{56}$$

acts as an extra bit of effective action for the EM field due to electrons living in the EM background. In terms of Feynman rules, $-\Delta S_E[A_\mu]$ generates effective vertices for the photon fields. Physically, such effective vertices stem from the electron loops. Indeed,

$$\begin{aligned} -\Delta S_E[A_\mu] \;\; &= \;\; \log \mathrm{Det} \left[1 \, - \, \frac{1}{\not{\partial} + m} (ie\not{A}) \right] \;\; = \;\; \mathrm{Tr} \log \left[1 \, - \, \frac{1}{\not{\partial} + m} (ie\not{A}) \right] \\ &= \;\; -\sum_{n=1}^{\infty} \frac{1}{n} \, \mathrm{Tr} \left[\left(\frac{1}{\not{\partial} + m} (ie\not{A}) \right)^n \right] \\ &= \;\; \sum_{n=1}^{\infty} n\text{-photon amputated diagram} \end{aligned} \tag{57}$$



Note that each such diagram carries overall $-$ sign due to one fermion loop and a combinatorial factor $1/n$ due to cyclic symmetry of the diagram. Also, in the context of $-\Delta S_E$, each

n-photon diagram should be multiplied by the appropriate $A_\mu(x)$ factors for each external leg, hence in the coordinate space

$$n\text{-photon loop} \;=\; \frac{-1}{n}\int\! d^4x_1\cdots\int\! d^4x_n\ \mathrm{tr}\begin{pmatrix} (ie\not{A}(x_n))\times G_\psi(x_n;x_{n-1})\times \\ (ie\not{A}(x_{n-1}))\times G_\psi(x_{n-1};x_{n-2})\times \\ \cdots \\ \times(ie\not{A}(x_2))\times G_\psi(x_2;x_1)\times \\ (ie\not{A}(x_1))\times G_\psi(x_1;x_n)\end{pmatrix} \tag{58}$$

$$=\; \frac{-1}{n}\times(\text{functional trace})\ \ \mathrm{Tr}\left[\left(\frac{1}{\not\partial+m}(ie\not{A})\right)^n\right],$$

exactly as in eq. (57).

Thus we see that the red term in the fermionic free energy

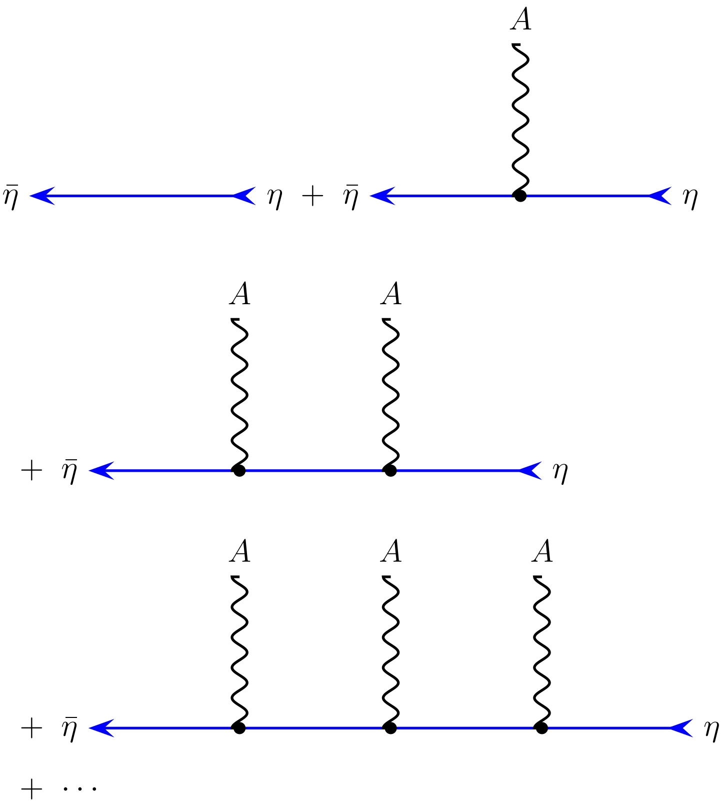
$$-\log\widehat{Z}[A_\mu,\eta,\bar\eta] \;=\; -\log\det(\not{D}+m)\; -\int\! d^4x_e\,\bar\eta\frac{1}{\not{D}+m}\eta. \tag{49}$$

indeed generates electron loops acting as effective vertices for the photon lines attached to them. As to the blue term involving the fermionic sources η and $\bar\eta$, it generates tree diagrams where a bunch of photonic lines are connected to a single open electron line. To see that, we expand

$$\begin{aligned}\frac{1}{\not{D}+m} &= \frac{1}{(\not\partial+m)-(ie\not{A})}\\ &= \frac{1}{\not\partial+m} + \frac{1}{\not\partial+m}(ie\not{A})\frac{1}{\not\partial+m}\\ &\quad + \frac{1}{\not\partial+m}(ie\not{A})\frac{1}{\not\partial+m}(ie\not{A})\frac{1}{\not\partial+m}\\ &\quad + \frac{1}{\not\partial+m}(ie\not{A})\frac{1}{\not\partial+m}(ie\not{A})\frac{1}{\not\partial+m}(ie\not{A})\frac{1}{\not\partial+m}\\ &\quad + \cdots\end{aligned} \tag{59}$$

Consequently,

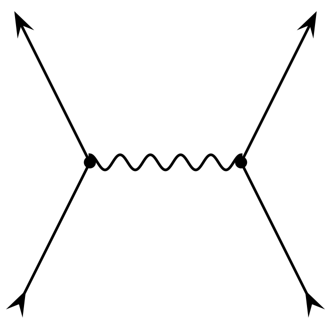
$$\int d^4x_e\,\bar\eta\frac{1}{\not{D}+m}\eta \;=\; \qquad (60)$$



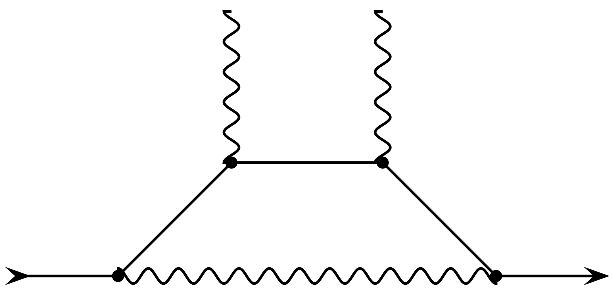
Altogether, the fermionic functional integral

$$\widehat{Z}[A_\mu,\eta,\bar\eta] \;=\; \iiint\mathcal{D}[\overline{\Psi}]\iiint\mathcal{D}[\Psi]\,\exp\left(-\int d^4x_E\Big(\overline{\Psi}(\not{D}+m)\Psi\;-\;\overline{\eta}\Psi\;-\;\overline{\Psi}\eta\Big)\right). \qquad (61)$$

takes care of all the electron lines — open or closed — in QED Feynman rules. However, at this point, all photonic lines are treated as external. To get the photon propagators — and hence diagrams like



or



we need to integrate over the $A_\mu(x)$ fields as well as the fermions. Such functional integrals over the gauge fields pose their own problems due to gauge symmetry and its fixing. These issues are discussed in detail in the next set of my notes.