

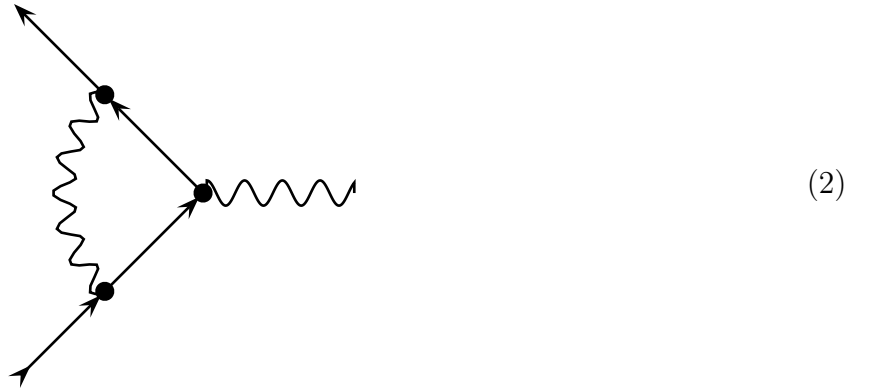
Gauge Dependence

In QED, the on-shell physical amplitudes do not depend on the gauge-fixing condition, but that's unfortunately not true for the off-shell amplitudes. Even the UV divergences — and hence the counterterms depend on the gauge-fixing conditions. In particular, in the Lorenz-invariant gauges where the photon propagator is

$$\text{wavy line} = \frac{-i}{k^2 + i0} \left[g^{\mu\nu} + (\xi - 1) \frac{k^\mu k^\nu}{k^2 + i0} \right], \quad (1)$$

the counterterms depend on the ξ parameter. In these notes, we shall focus on the ξ dependence of the δ_1 and the δ_2 counterterms.

Let's start with the one-loop δ_1 counterterm which cancels the UV divergence of the vertex correction



Evaluating this diagram for the general ξ gauge, we get

$$\begin{aligned} ie\Gamma_{1\text{loop}}^\mu(p', p) &= \int_{\text{reg}} \frac{d^4k}{(2\pi)^4} ie\gamma_\nu \times \frac{i}{\not{p}' + \not{k} - m + i0} \times ie\gamma^\mu \times \frac{i}{\not{p} + \not{k} - m + i0} \times ie\gamma_\lambda \times \\ &\quad \times \frac{-i}{k^2 + i0} \left[g^{\lambda\nu} + (\xi - 1) \frac{k^\lambda k^\nu}{k^2 + i0} \right] \\ &= e^3 \int_{\text{reg}} \frac{d^4k}{(2\pi)^4} \frac{1}{k^2 + i0} \times \gamma^\nu \frac{1}{\not{p}' + \not{k} - m + i0} \gamma^\mu \frac{1}{\not{p} + \not{k} - m + i0} \gamma_\nu \\ &\quad + (\xi - 1) e^3 \int_{\text{reg}} \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2 + i0)^2} \times \not{k} \frac{1}{\not{p}' + \not{k} - m + i0} \gamma^\mu \frac{1}{\not{p} + \not{k} - m + i0} \not{k} \\ &= ie\Gamma_F^\mu(p', p) + (\xi - 1) \times ie\Delta\Gamma^\mu(p', p) \end{aligned} \quad (3)$$

where Γ_F^μ stands for the $\Gamma_{1\text{loop}}^\mu$ which obtains in the Feynman gauge $\xi = 0$ — see [my previous set](#)

of notes for detail, — while

$$\Delta\Gamma^\mu(p', p) = -ie^2 \int_{\text{reg}} \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2 + i0)^2} \times k \frac{1}{\not{p}' + \not{k} - m + i0} \gamma^\mu \frac{1}{\not{p} + \not{k} - m + i0} k \quad (4)$$

is the gauge-dependent correction. Fortunately, this correction drastically simplifies for the on-shell electrons when $\Delta\Gamma^\mu$ appears in the context of $\bar{u}(p')\Delta\Gamma^\mu u(p)$. Indeed, in this context

$$\frac{1}{\not{p} + \not{k} - m + i0} k = 1 - \frac{1}{\not{p} + \not{k} - m + i0} (\not{p} - m) \cong 1$$

because $(\not{p} - m)u(p) = 0$, and likewise

$$k \frac{1}{\not{p}' + \not{k} - m + i0} = 1 - (\not{p}' - m) \frac{1}{\not{p}' + \not{k} - m + i0} \cong 1.$$

Consequently, eq. (4) simplifies to

$$\Delta\Gamma^\mu = e^2 \gamma^\mu \times \int_{\text{reg}} \frac{d^4k}{(2\pi)^4} \frac{-i}{(k^2 + i0)^2} \quad (5)$$

which does not depend on any momenta, p , p' , or q , but only on the UV and the IR regulators.

Finally, since the complete dressed vertex involves not only the loop diagram (2) but also the δ_1 counterterm, we see that the gauge-dependent correction (5) can be completely canceled by the gauge-dependent correction to the δ_1 , namely

$$\delta_1(\xi) = \delta_1^{\text{Feynman gauge}} - (\xi - 1) \times e^2 \int_{\text{reg}} \frac{d^4k}{(2\pi)^4} \frac{-i}{(k^2 + i0)^2}. \quad (6)$$

Now consider the δ_2 counterterm. which together with the δ_m counterterm cancels the UV divergence of the electron's self energy

$$\Sigma_{\text{net}}(\not{p}) = \Sigma_{\text{loops}}(\not{p}) + \delta_m - \Delta_2 \not{p}. \quad (7)$$

At the one loop level,

$$-i\Sigma_{1\text{ loop}}(\not{p}) = \text{---} \rightarrow \text{---} \left[\text{---} \bullet \begin{array}{c} \text{---} \text{wavy line} \text{---} \\ \text{---} \text{curved line} \text{---} \end{array} \bullet \text{---} \rightarrow \text{---} \right] \quad (8)$$

which evaluates to

$$\begin{aligned} -i\Sigma^{1\text{ loop}}(\not{p}) &= \int_{\text{reg}} \frac{d^4k}{(2\pi)^4} ie\gamma_\lambda \frac{i}{\not{k} + \not{p} - m_e + i0} \times ie\gamma_\nu \times \frac{-i}{k^2 + i0} \left[g^{\lambda\nu} + (\xi - 1) \frac{k^\lambda k^\nu}{k^2 + i0} \right] \\ &= -e^2 \int_{\text{reg}} \frac{d^4k}{(2\pi)^4} \frac{1}{k^2 + i0} \times \gamma^\nu \frac{1}{\not{k} + \not{p} - m_e + i0} \gamma_\nu \\ &\quad + (\xi - 1) \times (-e^2) \int_{\text{reg}} \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2 + i0)^2} \times \not{k} \frac{1}{\not{k} + \not{p} - m_e + i0} \not{k} \\ &= -i\Sigma_F(\not{p}) - i(\xi - 1) \times \Delta\Sigma(\not{p}), \end{aligned} \quad (9)$$

where $\Sigma_F(\not{p})$ is the $\Sigma_{1\text{ loop}}$ which obtains in the Feynman gauge — and which you should calculate in [homework set#17](#), — while

$$\Delta\Sigma(\not{p}) = -ie^2 \int_{\text{reg}} \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2 + i0)^2} \times \not{k} \frac{1}{\not{k} + \not{p} - m_e + i0} \not{k} \quad (10)$$

is the gauge-dependent correction. To compensate for this correction, we should also correct the δ_2 and δ_m counterterms to assure that Σ_{net} and $d\Sigma_{\text{net}}/d\not{p}$ both vanish at $\not{p} = m$, hence

$$\delta_2 = \delta_2^{\text{Feynman gauge}} + (\xi - 1)\Delta\delta_2, \quad \delta_m = \delta_m^{\text{Feynman gauge}} + (\xi - 1)\Delta\delta_m, \quad (11)$$

for

$$\Delta\delta_2 = \left. \frac{d\Delta\Sigma}{d\not{p}} \right|_{\not{p} \neq m} \quad \text{and} \quad \Delta\delta_m - m\Delta\delta_2 = -\Delta\Sigma(\not{p} = m). \quad (12)$$

Taking the derivative of $\Delta\Sigma$ from eq. (10), we get

$$\frac{d\Delta\Sigma}{d\not{p}} = e^2 \int_{\text{reg}} \frac{d^4k}{(2\pi)^4} \frac{-i}{(k^2 + i0)^2} \times \not{k} \frac{-1}{(\not{k} + \not{p} - m_e + i0)^2} \not{k}, \quad (13)$$

where

$$\begin{aligned} \not{k} \frac{1}{(\not{k} + \not{p} - m_e + i0)^2} \not{k} &= \left(1 - (\not{p} - m) \frac{1}{\not{p} + \not{k} - m + i0} \right) \times \left(1 - \frac{1}{\not{p} + \not{k} - m + i0} (\not{p} - m) \right) \\ &\rightarrow 1 \text{ for } \not{p} = m. \end{aligned} \quad (14)$$

Consequently,

$$\Delta\delta_2 = \left. \frac{d\Delta\Sigma}{d\not{p}} \right|_{\not{p} \neq m} = -e^2 \int_{\text{reg}} \frac{d^4k}{(2\pi)^4} \frac{-i}{(k^2 + i0)^2}. \quad (15)$$

As to the $\Delta\delta_m$ corrections to the mass counterterm, I leave its calculation to your [next homework](#).

Finally, comparing eqs. (6) and (15), we see that the gauge0dependent corrections to the δ_1 and δ_2 counterterms are exactly the same,

$$\Delta\delta_1 = \Delta\delta_2 = -e^2 \int_{\text{reg}} \frac{d^4k}{(2\pi)^4} \frac{-i}{(k^2 + i0)^2}. \quad (16)$$

Therefore, once we verify the Ward identity $\delta_1 = \delta_2$ in the Feynman gauge, it follows that

$$\delta_1(\xi) = \delta_2(\xi) \text{ in any gauge.} \quad (17)$$