Finite Multiplets of the Spin(3,1) Group.

In these notes I classify all the finite multiplets of the continuous Lorentz group $SO^+(3,1)$, or rather of its double-covering group $\mathrm{Spin}(3,1)$. The notes are insterspersed with optional exercises for the students. The solutions to the exercises will appear in a separate page separate page.

I presume you read these notes after finishing your homework#5 and homework#6, so you should be familiar with the Lorents $\hat{\mathbf{J}}$ and $\hat{\mathbf{K}}$ generators and their Dirac spinor representations. In these notes, it's convenient to re-organize the $\hat{\mathbf{J}}$ and $\hat{\mathbf{K}}$ generators into two non-hermitian 3-vectors

$$\hat{\mathbf{J}}_+ \;=\; \tfrac{1}{2}(\hat{\mathbf{J}} + i\hat{\mathbf{K}}) \quad \text{and} \quad \hat{\mathbf{J}}_- \;=\; \tfrac{1}{2}(\hat{\mathbf{J}} - i\hat{\mathbf{K}}) \;=\; \hat{\mathbf{J}}_+^\dagger. \tag{1}$$

1. Show that the two 3-vectors commute with each other, $[\hat{J}_+^k, \hat{J}_-^\ell] = 0$, while the components of each 3-vector satisfy angular momentum commutation relations, $[\hat{J}_+^k, \hat{J}_+^\ell] = i\epsilon^{k\ell m}\hat{J}_+^m$ and $[\hat{J}_-^k, \hat{J}_-^\ell] = i\epsilon^{k\ell m}\hat{J}_-^m$.

By themselves, the 3 $\hat{J}_+^k$ generate a symmetry group similar to rotations of a 3D space, but since the $\hat{J}_+^k$ are non-hermitian, the finite irreducible multiplets of this symmetry are non-unitary analytic continuations (to complex "angles") of the ordinary angular momentum multiplets (j) of spin $j = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \ldots$. Likewise, the finite irreducible multiplets of the symmetry group generated by the $\hat{J}_-^k$ are analytic continuations of the spin-j multiplets of angular momentum. Moreover, the two symmetry groups commute with each other, so the finite irreducible multiplets of the net Lorentz symmetry are tensor products $(j_+) \otimes (j_-)$ of the $\hat{\mathbf{J}}_+$ and $\hat{\mathbf{J}}_-$ multiplets. In other words, distinct finite irreducible multiplets of the Lorentz symmetry may be labeled by *two* integer or half-integer 'spins' j_+ and j_-, while the states within such a multiplet are $|j_+, j_-, m_+, m_-\rangle$ for $m_+ = -j_+, \ldots, +j_+$ and $m_- = -j_-, \ldots, +j_-$.

The simplest non-trivial Lorentz multiplets are two inequivalent doublets, the left-handed Weyl spinor $\mathbf{2}$ and the right-handed Weyl spinor $\mathbf{2}^*$. The $\mathbf{2}$ multiplet has $j_+ = \frac{1}{2}$ while $j_- = 0$, hence $\hat{\mathbf{J}}_+$ acts as $\frac{1}{2}\boldsymbol{\sigma}$ while $\hat{\mathbf{J}}_-$ does not act at all, or in terms of the $\hat{\mathbf{J}}$ and $\hat{\mathbf{K}}$ generators

$\mathbf{J} = \frac{1}{2}\boldsymbol{\sigma}$ while $\mathbf{K} = -\frac{i}{2}\boldsymbol{\sigma}$. The conjugate $\mathbf{2}^*$ multiplet has $j_- = \frac{1}{2}$ while $j_+ = 0$, hence $\hat{\mathbf{J}}$ acts as $\frac{1}{2}\boldsymbol{\sigma}$ while $\hat{\mathbf{K}}$ acts as $+\frac{i}{2}\boldsymbol{\sigma}$.

2. Check that these two doublets are indeed the LH Weyl spinors and the RH Weyl spinor from the homework set#6 (problem 2).

3. Check that for finite Lorentz symmetries, the 2×2 matrices M_L and M_R representing them in the LH and the RH Weyl spinor multiplets have determinant $= 1$.

The complex (but not necessary unitary) 2×2 matrices of unit determinant form a non-compact group called the $SL(2, \mathbf{C})$. This group is isomorphic to the $\mathrm{Spin}(3, 1)$, the double cover of the continuous Lorentz group $SO^+(3, 1)$. Just like the $SU(2)$ is isomorphic to the $\mathrm{Spin}(3)$, the double cover of the $SO(3)$ rotation group.

For the $\mathrm{Spin}(3) = SU(2)$ group, one can construct a multiplet of any spin j from a symmetric tensor product of $2j$ doublets. This procedure gives us an object $\Phi_{\alpha_1,\ldots,\alpha_{2j}}$ with $2j$ spinor indices $\alpha_1, \ldots, \alpha_{2j} = 1, 2$ that's totally symmetric under permutation of those indices and transforms under an $SU(2)$ symmetry U as

$$\Phi_{\alpha_1,\alpha_2\ldots,\alpha_{2j}} \;\to\; U_{\alpha_1}^{\beta_1} U_{\alpha_2}^{\beta_2} \cdots U_{\alpha_{2j}}^{\beta_{2j}}\, \Phi_{\beta_1,\beta_2\ldots,\beta_{2j}}. \tag{2}$$

For integer j, such objects are equivalent to tensors of the $SO(3)$; for example, for $j = 2$ $\Phi_{\alpha\beta} \equiv \Phi_{\beta\alpha}$ is equivalent to an $SO(3)$ vector $\vec{\Phi}$.

For the Lorentz group $\mathrm{Spin}(3, 1)$ we have a similar situation — any multiplet can be constructed by tensoring together a bunch of two-component spinors of the $SL(2, \mathbf{C})$. But unlike the $SU(2)$, the $SL(2, \mathbf{C})$ has two different spinors $\mathbf{2} \not\cong \mathbf{2}^*$ transforming under different rules. Notationally, we shall distinguish them by different index types: the un-dotted Greek indices belong to spinor that transform according to $M \in SL(2, \mathbf{C})$ while the dotted Greek indices belong to spinors that transform according to M^*:

$$(\psi_L)\alpha \;\to\; M_\alpha^\beta (\psi_L)_\beta \quad \not\cong \quad (\sigma_2\psi_R)_{\dot{\gamma}} \;\to\; M_{\dot{\gamma}}^{*\dot{\delta}} (\sigma_2\psi_R)_{\dot{\delta}}\,, \quad M \in SL(2, \mathbf{C}). \tag{3}$$

Combining such spinors to make a multiplet with 'spins' j_+ and j_-, we make an object $\Phi_{\alpha_1,\ldots,\alpha_{(2j_+)};\dot{\gamma}_1,\ldots,\dot{\gamma}_{(2j_-)}}$ with $2j_+$ un-dotted indices and $2j_-$ dotted indices. $\Phi_{\cdots}$ is totally symmetric under permutations of the un-dotted indices with each other or dotted indices with each other, but there is no symmetry between dotted and un-dotted indices. Under an $SL(2,\mathbf{C})$ symmetry M, the un-dotted indices transform according to M while the dotted indices transform according to the M^*, thus

$$\Phi_{\alpha_1,\ldots,\alpha_{(2j_+)};\dot\gamma_1,\ldots,\dot\gamma_{(2j_-)}} \;\to\; M_{\alpha_1}^{\beta_1}\cdots M_{\alpha_{(2j_+)}}^{\beta_{(2j_+)}} \times M_{\dot\gamma_1}^{*M\dot\delta_1}\cdots M_{\dot\gamma_{(2j_-)}}^{*M\dot\delta_{(2j_-)}}\cdots\times\Phi_{\beta_1,\ldots,\beta_{(2j_+)};\dot\delta_1,\ldots,\dot\delta_{(2j_-)}}\,. \tag{4}$$

Of particular importance among such multi-spinors is the bi-spinor $V_{\alpha\dot\gamma}$ with $j_+ = j_- = \frac{1}{2}$ — it is equivalent to the Lorentz vector V^μ. The map between bi-spinors and Lorentz vectors involves four hermitian 2×2 matrices $\sigma_\mu = (1,\boldsymbol{\sigma})$. In $SL(2,\mathbf{C})$ terms, each σ_μ matrix has one dotted and one un-dotted index, thus $(\sigma_\mu)_{\alpha\dot\gamma}$. Using the σ_μ, we may re-cast any Lorentz vector V^μ as a matrix

$$V^\mu \;\to\; V^\mu\sigma_\mu \;=\; V^0 \;+\; \mathbf{V}\cdot\boldsymbol{\sigma} \tag{5}$$

an hence as a $\left(\frac{1}{2},\frac{1}{2}\right)$ bi-spinor

$$V_{\alpha\dot\gamma} \;=\; \left(V^\mu\sigma_\mu\right)_{\alpha\dot\gamma} \;=\; V^0\delta_{\alpha\dot\gamma} \;+\; \mathbf{V}\cdot\boldsymbol{\sigma}_{\alpha\dot\gamma}\,. \tag{6}$$

Under an $SL(2,\mathbf{C})$ symmetry, the bi-spinor transforms as

$$V_{\alpha\dot\gamma} \;\to\; V'_{\alpha\dot\gamma} \;=\; M_\alpha^\beta M_{\dot\gamma}^{*\dot\delta}\, V_{\beta\dot\delta}\,, \tag{7}$$

or in matrix form,

$$V^\mu\sigma_\mu \;\to\; V'^\mu\sigma_\mu \;=\; M\,(V^\mu\sigma_\mu)\,M^\dagger. \tag{8}$$

Since the four matrices σ_μ form a complete basis of 2×2 matrices, eq. (8) defines a linear transform $V'^\mu = L^\mu_{\ \nu}(M)V^\nu$.

4. Prove that for any $SL(2,\mathbf{C})$ matrix M, the transform $L^\mu_{\ \nu}(M)$ defined by eq. (8) is real (real V'^μ for real V^μ), Lorentzian (preserves $V'_\mu V'^\mu = V_\mu V^\mu$) and orthochronous.
   Hint: prove and use $\det(V_\mu\sigma^\mu) = V_\mu V^\mu$.

   ⋆ For extra challenge, show that this transform is proper, $det(L) = +1$.

5. Verify that this $SL(2,\mathbf{C}) \to SO^+(3,1)$ map respects the group law, $L(M_2M_1) = L(M_2)L(M_1)$.

6. Show that for the $L(M)$ defined by eq. (8), the LH Weyl spinor representation of $L(M)$ is $M_L(L) = M$ while the RH Weyl spinor representation is $\overline{M} = \sigma_2 M^* \sigma_2$.

In general, any (j_+, j_-) multiplet of the $SL(2,\mathbf{C})$ with integer net spin $j_+ + j_-$ is equivalent to some kind of a Lorentz tensor. (Here, we include the scalar and the vector among the tensors.) For example, the $(1,1)$ multiplet is equivalent to a symmetric, traceless 2–index tensor $T^{\mu\nu} = +T^{\nu\mu}$, $T^\mu_\mu = 0$. For $j_+ \neq j_-$ the representation is complex, but one can make a real tensor by combining two multiplets with opposite j_+ and j_-, for example the $(1,0)$ and the $(0,1)$ multiplets are together equivalent to the antisymmetric 2–index tensor $F^{\mu\nu} = -F^{\nu\mu}$.

7. Verify the above examples.
   Hint: For any kind of angular momentum, $(j=\frac{1}{2}) \otimes (j=\frac{1}{2}) = (j=1) \oplus (j=0)$.