SPONTANEOUS SYMMETRY BREAKING

Consider QFT of a complex scalar field $\Phi(x)$ with the Lagrangian

$$\mathcal{L} = (\partial_\mu \Phi^*)(\partial^\mu \Phi) - A\Phi^*\Phi - \frac{\lambda}{2}(\Phi^*\Phi)^2 \qquad (1)$$

and a global $U(1)$ phase symmetry, $\Phi(x) \to e^{i\theta}\Phi(x)$. For $A > 0$ this theory has a unique ground state — the physical vacuum — with zero expectation value $\langle\Phi\rangle$ of the field and therefore invariant under the phase symmetry. As to the excited states, they are made from particles and antiparticles of mass $M = \sqrt{A}$ and a conserved charge #particles − #antiparticles.

For a negative $A = -\lambda v^2$, the theory behaves very differently. The scalar potential

$$V = \frac{\lambda}{2}(\Phi^*\Phi)^2 + (A = -\lambda v^2)\,\Phi^*\Phi = \frac{\lambda}{2}\Big(\Phi^*\Phi - v^2\Big)^2 + \text{const} \qquad (2)$$

has a local maximum rather than a minimum at phase-symmetric point $\Phi = 0$. Instead, it has a continuous ring of degenerate minima at $\Phi = v \times$ any phase. None of these minima is invariant under the $U(1)$ phase symmetry; instead, the symmetry relates the minima to each other. Semiclassically — and hence in perturbation theory, or even non-perturbatively for small enough λ, — this means that the theory does not have a unique physical vacuum but rather a continuous family of exactly degenerate vacua related to each other by the phase symmetry. This phenomenon is called *spontaneous breakdown of the symmetry.*

Now let's pick a vacuum state — by symmetry, it does not matter which — and find the particle spectrum of the theory. For simplicity, let's work at the semiclassical level. Take the vacuum with real $\langle\Phi\rangle = +v$, shift the field $\Phi(x)$ by its vacuum expectation value (VEV),

$$\Phi(x) = v + \varphi(x), \qquad (3)$$

and split the shifted complex field $\varphi(x)$ into its real and imaginary parts,

$$\Phi(x) = v + \frac{\sigma(x) + i\pi(x)}{\sqrt{2}}\,. \qquad (4)$$

In terms of the real σ and π fields, we have

$$\Phi^*\Phi \;-\; v^2 \;=\; v^2 \;+\; \sqrt{2}v\times\sigma \;+\; \tfrac{1}{2}\sigma^2 \;+\; \tfrac{1}{2}\pi^2 \;-\; v^2 \;=\; \sqrt{2}v\times\sigma \;+\; \tfrac{1}{2}\sigma^2 \;+\; \tfrac{1}{2}\pi^2, \qquad (5)$$

so the scalar potential becomes

$$V \;=\; \frac{\lambda}{2}\Bigl(\Phi^*\Phi - v^2\Bigr)^2 \;=\; \lambda v^2\times\sigma^2 \;+\; \frac{\lambda v}{\sqrt{2}}\times(\sigma^3+\sigma\pi^2) \;+\; \frac{\lambda}{8}(\sigma^2+\pi^2)^2. \qquad (6)$$

At the same time, the kinetic term for Φ yields the kinetic terms for the σ and π,

$$(\partial_\mu\Phi^*)(\partial^\mu\Phi) \;=\; (\partial_\mu\varphi^*)(\partial^\mu\varphi) \;=\; \tfrac{1}{2}(\partial_\mu\sigma)^2 \;+\; \tfrac{1}{2}(\partial_\mu\pi)^2, \qquad (7)$$

so altogether we have

$$\mathcal{L} \;=\; \Bigl(\tfrac{1}{2}(\partial_\mu\sigma)^2 \;-\; \lambda v^2\times\sigma^2)\Bigr) + \Bigl(\tfrac{1}{2}(\partial_\mu\pi)^2\Bigr) \;+\; \mathcal{L}^{\rm cubic} \;+\; \mathcal{L}^{\rm quartic}. \qquad (8)$$

Focusing on the free part of this Lagrangian, we see that the σ field has mass[2] $M_\sigma^2 = 2\lambda v^2$ while the π field is massless. Also, the quanta of both fields are neutral and are not related to each other by the phase symmetry.

In the fully quantum theory — to all orders of perturbation theory and even non-perturbatively — the $\pi(x)$ field remains exactly massless. This is guaranteed by the Nambu–Goldstone theorem, which I shall explain in a moment.

But before I get to the theorem, let me give you another example of *spontaneous symmetry breaking* (SSB), the so-called *linear sigma model.* Consider N real scalar fields $\Phi_i(x)$ ($i=1,\ldots,N$) with the Lagrangian

$$\mathcal{L} \;=\; \sum_i \tfrac{1}{2}(\partial_\mu\Phi_i)^2 \;-\; \frac{A}{2} \sim_i \Phi_i^2 \;-\; \frac{\lambda}{8}\left(\sum_i \Phi_i^2\right)^2 . \qquad (9)$$

This theory has an $O(N)$ global symmetry of orthogonal rotations in the N-dimensional scalar field space,

$$\Phi_i(x) \;\to\; \sum_j R_{i,j}\Phi_j(x), \quad R^\top R \;=\; \mathbf{1}_{N\times N}\,. \qquad (10)$$

For $A>0$, the scalar potential of the theory has a unique minimum at $\Phi_i = 0\ \forall i$, which is invariant under the $O(N)$ symmetry. Consequently, the quantum theory has a an $O(N)$

invariant vacuum, and the particle states form a complete multiplet of the symmetry — namely the vector multiplet $\mathbf{N}$ — of the same mass $M = \sqrt{A}$. This is the usual behavior of a theory with an *unbroken symmetry.*

But for $A < 0$, the scalar potential of the theory — which we may rewrite as

$$V \;=\; \frac{\lambda}{8}\left(\sum_i \Phi_i^2 \;-\; v^2\right)^2 \;+\; \text{const} \quad \text{for } v^2 \;=\; \frac{-A}{2\lambda} \;>\; 0, \tag{11}$$

has a local maximum rather than a minimum at the $\Phi_i = 0\ \forall i$ point. Instead, it has a continuous family of degenerate minima, namely

$$\sum_i \Phi_i^2 \;=\; v^2; \tag{12}$$

geometrically, the minima form a sphere in the N-dimensional field space. Consequently, the $O(N)$ symmetry is spontaneously broken. However, it is not completely broken, since each minimum is invariant under the $O(N-1)$ subgroup of the $O(N)$ — the rotations in the $N-1$ dimensional hyperplane perpendicular to the $\langle\Phi_i\rangle$ vector. Thus, *the $O(N)$ symmetry of the theory is spontaneously broken down to its $O(N-1$ subgroup.*

To see how this works, let's pick a particular minimum, say the "North pole" of the sphere

$$\langle\Phi\rangle \;=\; \begin{pmatrix} 0 \\ \vdots \\ 0 \\ v \end{pmatrix}, \quad \textit{i.e.},\ \langle\Phi_i\rangle \;=\; \delta_{i,N} \times v. \tag{13}$$

Similar to the complex field example, let's shift all the fields by their VEVs,

$$\Phi_N(x) \;=\; v \;+\; \sigma(x) \quad \text{while for } i = 1, \ldots, (N-1) \quad \Phi_i(x) \;=\; \pi_i(x). \tag{14}$$

In terms of the shifted fields

$$\sum_{i=1}^{N} \Phi_i^2 \;-\; v^2 \;=\; 2v \times \sigma \;+\; \sigma^2 \;+\; \sum_{i=1}^{N-1} \pi_i^2, \tag{15}$$

so the scalar potential becomes

$$V \;=\; \lambda v^2 \times \sigma^2 \;+\; \frac{\lambda v}{2} \left(\sigma^3 \,+\, \sum_i \sigma \pi_i^2 \right) \;+\; \frac{\lambda}{8} \left(\sigma^2 + \sim_i \pi_i^2 \right)^2 , \tag{16}$$

and the complete Lagrangian becomes

$$\mathcal{L} \;=\; \left(\tfrac{1}{2}(\partial_\mu \sigma)^2 \;-\; \tfrac{1}{2}\lambda v^2 \times \sigma^2 \right) \;+\; \left(\sum_{i=1}^{N-1} \tfrac{1}{2}(\partial_\mu \pi_i)^2 \right) \;+\; \mathcal{L}^{\text{cubic}} \;+\; \mathcal{L}^{\text{quartic}}. \tag{17}$$

Thus, the $\sigma(x)$ field has positive mass^2, namely $M_\sigma^2 = \lambda v^2$, while the $\pi_i(x)$ fields ($i = 1, \ldots, (N-1)$) are massless. Let me emphasize two important features of this spectrum:

- The $N-1$ species of π_i particles form a complete degenerate multiplet — the vector — of the unbroken $O(N-1)$ symmetry while the σ particle is a singlet — which is also a complete multiplet, albeit a trivial one. But we do not get complete degenerate multiplets of the spontaneously broken $O(N-1)$ symmetry.

- There is one exactly massless particle for each generator of the spontaneously broken theory $O(N)/O(N-1)$. Moreover, the massless particles have the same quantum numbers WRT the unbroken $O(N-1)$ symmetry as the spontaneously broken generators.

  Indeed, the generators of the $O(N)$ group — or rather of its continuous part $SO(N)$ — form an antisymmetric tensor multiplet of the group, $T_{ij} = -T_{ji}$. The VEVs (13) are invariant under the T_{ij} generators with $i, j < N$ but not under the $T_{i,N}$ generators, so the former remain unbroken while the latter become spontaneously broken. In other words, the T_{ij} (with $i, j = 1, \ldots, (N-1)$) generate the unbroken subgroup $SO(N-1)$ while the $T_{i,N}$ (with $i = 1, \ldots, (N-1)$ only) generate the spontaneously broken factor-group $SO(N)/SO(N-1)$. WRT to the unbroken $SO(N-1)$ subgroup, the unbroken generators form an antisymmetric tensor multiplet, while the broken $T_{i,N}$ generators form a vector multiplet. And the massless quanta of the π_i fields also form a vector multiplet — the same as the broken generators $T_{i,N}$.

Modes of Symmetries

With the above examples in mind, let me now turn to the general theory. In $d = 3+1$ dimensions, a continuous symmetry of a quantum field theory — relativistic or not — may be realized in two different modes: the Wigner mode (unbroken symmetry) or the Nambu–Goldstone mode (spontaneously broken symmetry)

Wigner Mode (unbroken symmetry)

In the Wigner mode, the ground state $|\text{ground}\rangle$ of the theory — the physical vacuum state of a relativistic theory, or the quasiparticle-vacuum of a condensed matter system — is invariant under the symmetry. Consequently, the charge operators generating the symmetry annihilate the ground state, $\hat{Q}_a\,|\text{ground}\rangle = 0$. Moreover, the currents $\hat{J}^\mu_a(x)$ also annihilate the ground state, $\hat{J}^\mu_a(x)\,|\text{ground}\rangle = 0$.

The excited states of a QFT are made by adding particles (or quasiparticles) to the ground state, $|\text{excited}\rangle = \hat{a}^\dagger \cdots \hat{a}^\dagger\,|\text{ground}\rangle$. For a symmetry realized in a Wigner mode, the particles form multiplets of the symmetry, and all particles in the same multiplet have the same mass (in a relativistic theory); in a non-relativistic theory, all particles in the same multiplet have the same dispersion relation $E(\mathbf{p})$.

Finally, the scattering amplitudes $\mathcal{M}(\text{initial particles} \to \text{final particles})$ respect the symmetries in the Wigner mode. That is, if we act with the symmetry on both the initial and the final particles — by turning each particle into a different member of the same symmetry multiplet — then the scattering amplitude should remain invariant. Consequently, the complete final state should belong to the same multiplet of the symmetry as the complete initial state, and both states should be in the same member of that multiplet.

Nambu–Goldstone Mode (spontaneously broken symmetry)

In the Nambu–Goldstone mode, the ground state $|\text{ground}\rangle$ is NOT invariant under the symmetry. Instead, there is a continuous family of exactly degenerate ground states, and the symmetry relates them to each other. Consequently, the ground states are NOT annihilated by the symmetry charges or currents, $\hat{Q}_a\,|\text{ground}\rangle \neq 0$ and $\hat{J}^\mu_a\,|\text{ground}\rangle \neq 0$.

Furthermore, the particles (or quasiparticles) do NOT form degenerate multiplets of symmetries realized in the Nambu–Goldstone mode, and the scattering amplitudes are NOT symmetric.

Instead, by the **Goldstone Theorem**, the Nambu–Goldstone-mode symmetries have other interesting consequences:

1. For every generator $\hat{Q}_a$ of a spontaneously broken symmetry there is a particle species with zero mass. Such particles are called Goldstone particles or Goldstone bosons (since in most cases they are bosons of spin $= 0$). In non-relativistic theories, the Goldstone particles (or quasiparticles) have energies $E(\mathbf{p})$ that go to zero as $|\mathbf{p}|$ for low momenta, $E(\mathbf{p}) \propto |\mathbf{p}| \to 0$ for $\mathbf{p} \to 0$.

2. The currents of broken symmetries create Goldstone particles from the vacuum,

$$\hat{J}_a^\mu(\mathbf{x})\,|\text{ground}\rangle \;\propto\; |1 \text{ Goldstone particle } a \text{ @}\mathbf{x}\rangle\,, \tag{18}$$

   or after a Fourier transform from $\mathbf{x}$ to $\mathbf{p}$,

$$\hat{J}_a^\mu(\mathbf{p}) \;=\; \int\! d^3\mathbf{x}\, e^{i\mathbf{p}\mathbf{x}}\, \hat{J}_a^\mu(\mathbf{x}), \tag{19}$$

$$\hat{J}_a^\mu(\mathbf{p})\,|\text{ground}\rangle \;\propto\; |1 \text{ Goldstone particle (species} = a,\ \text{momentum} = \mathbf{p})\rangle\,. \tag{20}$$

3. The Goldstone particles have the same quantum numbers WRT the *unbroken* symmetries — in particular, they form the same type of a multiplet — as the generators $\hat{Q}_a$ of the broken symmetries.

4. Finally, the scattering amplitudes involving low-momentum Goldstone particle vanish as $O(p)$ when the momentum p^μ of the Goldstone particle goes to zero. If multiple Goldstone particles are involved, the amplitude vanishes as $O(p)$ when *any* of the Goldstone particles momenta $\to 0$.

Please note that a continuous group G of symmetries may be *partially broken* down to a proper subgroup $H \subset G$. That is, the action and the Hamiltonian of the theory are invariant under all symmetries $\in G$, but the ground state is invariant only under the symmetries $\in H$. In this case, the symmetries in H are realized in the Wigner mode while the remaining symmetries in G/H are realized in the Nambu–Goldstone mode. Consequently, all particle species — including the Goldstone particles — form degenerate multiplets of H but not of G.

$\star \quad \star \quad \star$

Earlier in these notes I used the linear sigma model example to illustrate points (1) and (3) of the Goldstone theorem — the existence of massless Goldstone bosons and their quantum numbers WRT to the unbroken symmetry. We also saw partial symmetry breaking: the continuous symmetry group of the linear sigma model's action is $G = SO(N)$ but the ground state is invariant only WRT to a subgroup $H = SO(N-1)$ of G. Now let me turn to the remaining aspects of the Goldstone theorem: point (2) about the currents, and point (4) about the scattering amplitudes.

The conserved currents of the $SO(N)$ symmetry in the linear sigma model follow from the Noether theorem: Given the generators T_{ij} acting on the fields according to

$$T_{ij}\Phi_k(x) \;=\; \delta_{jk}\Phi_i(x) \;-\; \delta_{ik}\Phi_j(x), \tag{21}$$

the conserved currents obtain as

$$J^\mu_{ij} \;=\; \sum_k \frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi_k)} \times T_{ij}\Phi_k \;=\; (\partial^\mu\Phi_j)\Phi_i \;-\; (\partial^\mu\Phi_i)\Phi_j\,. \tag{22}$$

In terms of the shifted fields, the currents of the unbroken symmetries $(i,j = 1,\ldots,(N-1))$ become

$$J^\mu_{ij} \;=\; \pi_i\partial^\mu\pi_j \;-\; \pi_j\partial^\mu\pi_i \tag{23}$$

and hence in terms of the creation and annihilation operators for the π_i particles

$$\hat{J}^\mu_{ij}(k) \;=\int\frac{d^3\mathbf{p}}{(2\pi)^3}\,\frac{p^\mu}{2\sqrt{|\mathbf{p}|\times|\mathbf{p_k}|}}\Big(\hat{a}^\dagger(j;\mathbf{p}+\mathbf{k})\hat{a}(i;\mathbf{p}) \;-\; \hat{a}^\dagger(i;\mathbf{p}+\mathbf{k})\hat{a}(j;\mathbf{p})\Big). \tag{24}$$

Clearly, all these current operators do annihilate the vacuum state.

On the other hand, for a broken symmetry generator $T_{N,i}$ (with $i < N$), we have

$$J^\mu_{N,i} \;=\; (v+\sigma)\partial^\mu\pi_i \;-\; \pi_i\partial^\mu\sigma \;=\; v\partial^\mu\pi_i \;+\; (\sigma\partial^\mu\pi_i - \pi_i\partial^\mu\sigma). \tag{25}$$

Consequently, in terms of creations and annihilation operators,

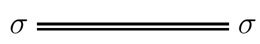
$$\hat{J}^{\mu}_{N,i}(k) \;=\; v\times\frac{k^{\mu}}{2k^{0}}\Big(\hat{a}^{\dagger}(i;\mathbf{k})+\hat{a}(i;-\mathbf{k}\Big)+\text{ a continuum of }\big(\hat{a}^{\dagger}(\sigma)\hat{a}(i)-\hat{a}^{\dagger}(i)\hat{a}(\sigma)\big)\text{ terms.} \tag{26}$$

All the $\hat{a}^{\dagger}\hat{a}$ terms here do annihilate the vacuum state, but the $\hat{a}^{\dagger}(i;k)$ term does not, thus

$$\hat{J}^{\mu}_{N,i}(k)\,|\text{vacuum}\rangle \;=\; \frac{vk^{\mu}}{2k_0}\,|1\text{ particle : species} = \pi_i;\text{momentum} = k\rangle\,. \tag{27}$$

Thus, the π_i particle is indeed the Goldstone boson created from the vacuum by the broken symmetry current $J^{\mu}_{N,i}$.

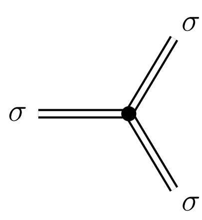
Next, consider the scattering amplitudes for the Goldstone particles in the linear sigma model. The Feynman rules of the model can be summarized as following: The propagators



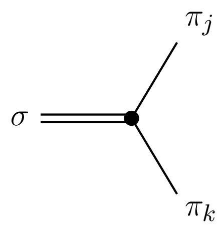
$$= \frac{i}{q^2 - M_\sigma^2 + i0} \qquad \text{and}$$

$$= \frac{i\delta_{jk}}{q^2 + i0}\,, \tag{28}$$

the 3-scalar vertices

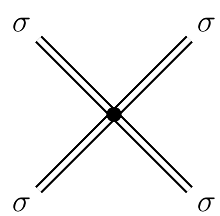


$$= -3i\lambda f,$$

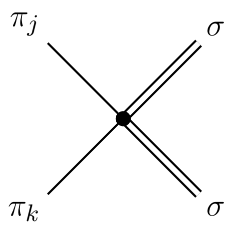


$$= -i\lambda f\delta_{jk}\,, \tag{29}$$

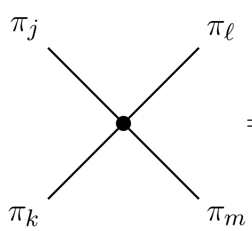
and the 4-scalar vertices



$$= -3i\lambda,$$

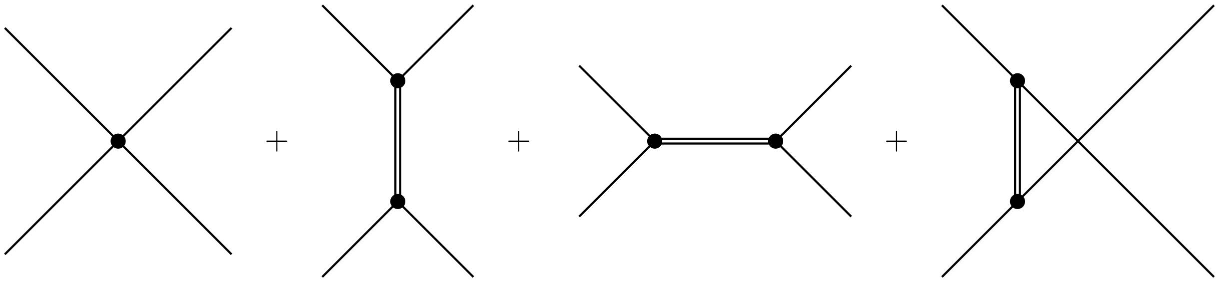


$$= -i\lambda\delta_{ik}\,, \tag{30}$$



$$= -i\lambda\big(\delta_{jk}\delta_{\ell m} + \delta_{j\ell}\delta_{km} + \delta_{jm}\delta_{k\ell}\big).$$

For simplicity, let's focus on the tree-level scattering $\pi_j + \pi_k \to \pi_\ell + \pi_m$. There are 4 tree diagram for this scattering,



hence net tree amplitude

$$\begin{aligned}\mathcal{M} &= -\lambda\Big(\big(\delta_{jk}\delta_{\ell m}+\delta_{j\ell}\delta_{km}+\delta_{jm}\delta_{k\ell}\big) \;-\; \frac{(\lambda v)^2}{s-M_\sigma^2}\times\delta_{jk}\delta_{\ell m} \\ &\qquad -\,\frac{(\lambda v)^2}{t-M_\sigma^2}\times\delta_{j\ell}\delta_{km} \;-\; \frac{(\lambda v)^2}{u-M_\sigma^2}\times\delta_{jm}\delta_{k\ell} \\ &= -\delta_{jk}\delta_{\ell m}\left(\lambda+\frac{\lambda^2v^2}{s-M_\sigma^2}\right) \;-\; \delta_{j\ell}\delta_{km}\left(\lambda+\frac{\lambda^2v^2}{t-M_\sigma^2}\right) \;-\; \delta_{jm}\delta_{k\ell}\left(\lambda+\frac{\lambda^2v^2}{u-M_\sigma^2}\right),\end{aligned} \tag{31}$$

where s,t,u are the Mandelstam's kinematic variables,

$$\begin{aligned} s &= (p_1+p_2)^2 \;=\; (p_3+p_4)^2, \\ t &= (p_1-p_3)^2 \;=\; (p_2-p_4)^2, \\ u &= (p_1-p_4)^2 \;=\; (p_2-p_3)^2. \end{aligned} \tag{32}$$

Note that all 4 Goldstone bosons here are massless, so

$$s \;=\; 2p_2p_2 \;=\; 2p_3p_4, \quad t \;=\; -2p_1p_3 \;=\; -2p_2p_4, \quad u \;=\; -2p_1p_4 \;=\; -2p_2p_3, \tag{33}$$

which means that if any one of the 4 particles' momenta becomes small, then all 3 invariants s,t,u become small proportionally to that small momentum,

$$s,t,u \;\propto\; p_{\rm small}\,. \tag{34}$$

Also note that in the linear sigma model $M_\sigma^2=\lambda v^2$, hence in the amplitude (31)

$$\lambda+\frac{\lambda^2v^2}{s-M_\sigma^2} \;=\; \lambda\left(1+\frac{M_\sigma^2}{s-M_\sigma^2}\right) \;=\; \frac{\lambda s}{s-M_\sigma^2}\,, \tag{35}$$

and likewise

$$\lambda + \frac{\lambda^2 v^2}{t - M_\sigma^2} = \frac{\lambda t}{t - M_\sigma^2}, \quad \lambda + \frac{\lambda^2 v^2}{u - M_\sigma^2} = \frac{\lambda u}{u - M_\sigma^2}. \tag{36}$$

Consequently,

$$\mathcal{M} = -\frac{\lambda s}{s - M_\sigma^2} \times \delta_{jk}\delta_{\ell m} - \frac{\lambda t}{t - M_\sigma^2} \times \delta_{j\ell}\delta_{km} - \frac{\lambda u}{u - M_\sigma^2} \times \delta_{jm}\delta_{k\ell}. \tag{37}$$

Now consider the low-energy limit of this amplitude and let all 4 particle's energies be much smaller that M_σ. In this limit $s, t, u \ll M_\sigma^2$, hence all 3 denominators in eq. (37) may be approximated as

$$\frac{1}{s - M_\sigma^2} \approx \frac{1}{t - M_\sigma^2} \approx \frac{1}{u - M_\sigma^2} \approx \frac{1}{-M_\sigma^2} = \frac{-1}{\lambda v^2}, \tag{38}$$

and the net amplitude becomes

$$\mathcal{M} \approx +\frac{1}{v^2}\big(s \times \delta_{jk}\delta_{\ell m} + t \times \delta_{j\ell}\delta_{km} + u \times \delta_{jm}\delta_{k\ell}\big). \tag{39}$$

In particular, when any one of the four particles' momenta becomes small, the s, t, u all become small with that momentum, so the entire scattering amplitude becomes small,

$$\mathcal{M} \propto p_{\text{small}}. \tag{40}$$

This completes about demonstration of point (4) of the Goldstone theorem: amplitudes involving low-energy Goldstone bosons go to zero when any of the Goldstone bosons' energy goes to zero. Although I have only demonstrated this behavior at the tree level, the Goldstone theorem assures similar behavior of the amplitudes to all order of the perturbation theory and even non-perturbatively.

Other Spacetime Dimensions

Throughout these notes, we have focused on continuous symmetries and their spontaneous breakdown in $d = 3 + 1$ dimensions. Let me briefly review the situation in other spacetime dimensions:

- In higher dimensions $d > 3 + 1$, the spontaneous symmetry breaking works exactly as in $d = 3 + 1$ dimensions, and each continuous symmetry can be realized in either the Wigner mode (unbroken) or Nambu–Goldstone mode (spontaneously broken).
- On the other hand, in $d = 1 + 1$ dimensions, the **Mermin–Wagner–Coleman Theorem** forbids any spontaneous breakdown of a continuous symmetry. This is caused by the quantum fluctuations of the massless would-be Goldstone boson: In one space dimension, they propagate to long distances undiminished, and in the process they wash out any would-be vacuum expectation values (VEVs) which are not invariant under the symmetry. Thus, no symmetry breaking VEVs, hence no SSB.
- Finally, in $d = 2 + 1$ dimensions, a continuous symmetry may be spontaneously broken at zero temperature. But at finite temperature, the thermal fluctuations have similar effect to the quantum fluctuations in $d = 1 + 1$, so there no SSB at $T > 0$.