

## Mandelstam Variables

Consider any kind of a 2 particles  $\rightarrow$  2 particles process



The 4-momenta  $p_1^\mu$ ,  $p_2^\mu$ ,  $p_1'^\mu$ , and  $p_2'^\mu$  of the 2 incoming and 2 outgoing particles satisfy 8 constraints: the on-shell conditions for each particle

$$p_1^2 = m_1^2, \quad p_2^2 = m_2^2, \quad p_1'^2 = m_1'^2, \quad p_2'^2 = m_2'^2, \quad (2)$$

and the net 4-momentum conservation

$$p_1^\mu + p_2^\mu = p_1'^\mu + p_2'^\mu. \quad (3)$$

Altogether, this gives us  $4 \times 4 - 8 = 8$  independent momentum variables, and the number of independent Lorentz-invariant combinations of these variables is only  $8 - 6 = 2$ .

However, for practical purposes it's is often convenient to use 3 Lorentz-invariant variables with a fixed sum,

$$\begin{aligned} s &= (p_1 + p_2)^2 = (p_1' + p_2')^2, \\ t &= (p_1 - p_1')^2 = (p_2' - p_2)^2, \\ u &= (p_1 - p_2')^2 = (p_1' - p_2)^2. \end{aligned} \quad (4)$$

$$s + t + u = m_1^2 + m_2^2 + m_1'^2 + m_2'^2.$$

Indeed,

$$\begin{aligned} s + t + u &= (p_1 + p_2)^2 + (p_1 - p_1')^2 + (p_1 - p_2')^2 \\ &= 3p_1^2 + p_2^2 + p_1'^2 + p_2'^2 + 2(p_1 p_2) - 2(p_1 p_1') - 2(p_1 p_2') \\ &= p_1^2 + p_2^2 + p_1'^2 + p_2'^2 + 2p_1 \times (p_1 + p_2 - p_1' - p_2') = 0 \\ &= p_1^2 + p_2^2 + p_1'^2 + p_2'^2 \\ &= m_1^2 + m_2^2 + m_1'^2 + m_2'^2. \end{aligned} \quad (5)$$

The  $s$ ,  $t$ , and  $u$  are called Mandelstam variables after Stanley Mandelstam who introduced them back in 1958.

In the center-of-mass frame where  $\mathbf{p}_1 + \mathbf{p}_2 = \mathbf{p}'_1 + \mathbf{p}'_2 = 0$ ,  $\sqrt{s}$  is the total energy of the colliding particles,  $\sqrt{s} = E_1 + E_2 = E'_1 + E'_2$ . Also, for an elastic collision in the CM frame,  $t$  parametrizes the scattering angle according to  $t = -(\mathbf{p}'_1 - \mathbf{p}_1)^2 = -\mathbf{p}^2 \times (1 - \cos \theta)$ . Hence, the Lorentz-invariant definitions (4) translate the CM-frame energy and the CM-frame scattering angle to any other frame of reference.

All Lorentz-invariant combinations of the four momenta  $p_1^\mu$ ,  $p_2^\mu$ ,  $p_1'^\mu$ , and  $p_2'^\mu$  can be expressed in terms of the Mandelstam variables. For example, the Lorentz products  $k^\mu k'_\mu$  of any two momenta are

$$\begin{aligned}
2(p_1 p_2) &= (p_1 + p_2)^2 - p_1^2 - p_2^2 = s - m_1^2 - m_2^2, \\
2(p'_1 p'_2) &= (p'_1 + p'_2)^2 - p_1'^2 - p_2'^2 = s - m_1'^2 - m_2'^2, \\
2(p_1 p'_1) &= p_1^2 + p_1'^2 - (p_1 - p'_1)^2 = m_1^2 + m_1'^2 - t, \\
2(p_2 p'_2) &= p_2^2 + p_2'^2 - (p_2 - p'_2)^2 = m_2^2 + m_2'^2 - t, \\
2(p_1 p'_2) &= p_1^2 + p_2'^2 - (p_1 - p'_2)^2 = m_1^2 + m_2'^2 - u, \\
2(p_2 p'_1) &= p_2^2 + p_1'^2 - (p_2 - p'_1)^2 = m_2^2 + m_1'^2 - u.
\end{aligned} \tag{6}$$

In particular, for an elastic scattering of 2 same-mass particles

$$\begin{aligned}
s + t + u &= 4m^2, \\
2(p_1 p_2) &= 2(p'_1 p'_2) = s - 2m^2, \\
2(p_1 p'_1) &= 2(p_2 p'_2) = 2m^2 - t, \\
2(p_1 p'_2) &= 2(p_2 p'_1) = 2m^2 - u.
\end{aligned} \tag{7}$$

For future reference, let me give you similar formulae for the  $e^-e^+ \rightarrow \mu^-\mu^+$  pair-production,

$$\begin{aligned}
s + t + u &= 2M_\mu^2 + 2m_e^2 \approx 2M_\mu^2, \\
2(p_1 p_2) &= s - 2m_e^2 \approx s, \\
2(p'_1 p'_2) &= s - 2M_\mu^2, \\
2(p_1 p'_1) &= 2(p_2 p'_2) = M_\mu^2 + m_e^2 - t \approx M_\mu^2 - t, \\
2(p_1 p'_2) &= 2(p_2 p'_1) = M_\mu^2 + m_e^2 - u \approx M_\mu^2 - u.
\end{aligned} \tag{8}$$

and for the  $e^-e^+ \rightarrow \gamma\gamma$  annihilation process  $p_- + p_+ \rightarrow k_1 + k_2$ ,

$$\begin{aligned} s + t + u &= 2m_e^2, \\ 2(p_-p_+) &= s - 2m_e^2, \\ 2(k_1k_2) &= s, \\ 2(p_-k_1) &= 2(p_+k_2) = m_e^2 - t, \\ 2(p_-k_2) &= 2(p_+k_1) = m_e^2 - u. \end{aligned} \tag{9}$$