

## Expansion of Free Relativistic Fields into Creation and Annihilation Operators

Similar to the scalar field I have discussed in class, any free relativistic field  $\widehat{\Psi}_{\aleph}(x)$  — where  $\aleph$  stands for a vector, tensor, or spinor index or multi-index — can be expanded into creation and annihilation operators:

$$\widehat{\Psi}_{\aleph}(x) = \int \frac{d^3\mathbf{k}}{(2\pi)^3} \frac{1}{2\omega_{\mathbf{k}}} \sum_{\lambda} \left( e^{-ikx} U_{\aleph}(\mathbf{k}, \lambda) \hat{a}_{\mathbf{k},\lambda} + e^{+ikx} V_{\aleph}(\mathbf{k}, \lambda) \hat{a}_{\mathbf{k},\lambda}^{\dagger} \right)^{k^0=+\omega_{\mathbf{k}}} \quad (1)$$

for a real (hermitian) quantum field, or

$$\begin{aligned} \widehat{\Psi}_{\aleph}(x) &= \int \frac{d^3\mathbf{k}}{(2\pi)^3} \frac{1}{2\omega_{\mathbf{k}}} \sum_{\lambda} \left( e^{-ikx} U_{\aleph}(\mathbf{k}, \lambda) \hat{a}_{\mathbf{k},\lambda} + e^{+ikx} V_{\aleph}(\mathbf{k}, \lambda) \hat{b}_{\mathbf{k},\lambda}^{\dagger} \right)^{k^0=+\omega_{\mathbf{k}}}, \\ \widehat{\Psi}_{\aleph}^{\dagger}(x) &= \int \frac{d^3\mathbf{k}}{(2\pi)^3} \frac{1}{2\omega_{\mathbf{k}}} \sum_{\lambda} \left( e^{-ikx} V_{\aleph}^*(\mathbf{k}, \lambda) \hat{b}_{\mathbf{k},\lambda} + e^{+ikx} U_{\aleph}^*(\mathbf{k}, \lambda) \hat{a}_{\mathbf{k},\lambda}^{\dagger} \right)^{k^0=+\omega_{\mathbf{k}}} \end{aligned} \quad (2)$$

for a complex field and its hermitian conjugate. In all these formulae:

- The quantum fields in the Heisenberg picture of QM  $\implies$  time-dependent, but the creation/annihilation operators  $\hat{a}_{\mathbf{k},\lambda}$ ,  $\hat{a}_{\mathbf{k},\lambda}^{\dagger}$ , *etc.*, are in the Schrödinger picture.
- $kx \equiv k_{\mu}x^{\mu} = \omega_k t - \mathbf{k} \cdot \mathbf{x}$  for  $\omega_k = +\sqrt{\mathbf{k}^2 + m^2}$ .
- The  $U_{\aleph}(\mathbf{k}, \lambda)$  and  $V_{\aleph}(\mathbf{k}, \lambda)$  are the coefficients of the plane-wave plane-wave solutions of the classical field equations,

$$\Psi_{\aleph}(x) = e^{-ikx} \times U_{\aleph}(\mathbf{k}, \lambda) \quad \text{and} \quad \Psi_{\aleph}(x) = e^{+ikx} \times V_{\aleph}(\mathbf{k}, \lambda) \quad \text{for } k^0 = +\omega_{\mathbf{k}}, \quad (3)$$

where  $\lambda$  labels the *polarizations* — *i.e.*, independent solutions for the same  $k^{\mu}$ .

- For the bosonic fields

$$[\hat{a}_{\mathbf{k},\lambda}, \hat{a}_{\mathbf{k}',\lambda'}^{\dagger}] = [\hat{b}_{\mathbf{k},\lambda}, \hat{b}_{\mathbf{k}',\lambda'}^{\dagger}] = \delta_{\lambda\lambda'} \times 2\omega_{\mathbf{k}} (2\pi)^3 \delta^{(3)}(\mathbf{k} - \mathbf{k}') \quad (4)$$

while all other pairs of creation or annihilation operators commute with each other.

For the fermionic fields

$$\{\hat{a}_{\mathbf{k},\lambda}, \hat{a}_{\mathbf{k}',\lambda'}^{\dagger}\} = \{\hat{b}_{\mathbf{k},\lambda}, \hat{b}_{\mathbf{k}',\lambda'}^{\dagger}\} = \delta_{\lambda\lambda'} \times 2\omega_{\mathbf{k}} (2\pi)^3 \delta^{(3)}(\mathbf{k} - \mathbf{k}') \quad (5)$$

while all other pairs of creation or annihilation operators anti-commute with each other.