GAUGE THEORIES

Gauge theories — abelian or non-abelian — are quantum theories of vector field $A^a_\mu(x)$ whose interactions with each other and with other fields follows from a *local symmetry*. So let me start these notes by explaining the difference between local and global symmetries:

⋆ A *global symmetry* — also called a *rigid symmetry* — has similar transformation of the fields at all spacetime points x. For example, a global phase symmetry of a fermion field $\Psi(x)$ acts as

$$\Psi(x) \;\to\; \Psi'(x) \;=\; e^{i\theta}\Psi(x), \quad \text{same } \theta \text{ for all } x. \tag{1}$$

⋆ In a *local symmetry* — also called a *gauge symmetry* — the field transformations at different points x have independent parameters. For example, a local phase symmetry of a fermion field $\Psi(x)$ acts as

$$\Psi(x) \;\to\; \Psi'(x) \;=\; e^{i\theta(x)}\Psi(x), \quad \text{independent } \theta(x) \text{ at each } x. \tag{2}$$

• A point of terminology: What a physicist calls a global symmetry, a mathematician would call a local symmetry and vice verse — a local symmetry to a physicist is a global symmetry to a mathematician. The terms *rigid symmetry* and *gauge symmetry* help avoid the confusion — both physicists and mathematicians agree to their meaning.

Abelian Example: Local Phase Symmetry.

Before we delve into non-abelian gauge theory, let me start with an abelian example. Consider a complex scalar field $\Phi(x)$ with a classical Lagrangian

$$\mathcal{L} \;=\; \partial^\mu\Phi^*\partial_\mu\Phi \;-\; m^2\Phi^*\Phi \;-\; \frac{\lambda}{2}\,(\Phi^*\Phi)^2, \tag{3}$$

which has a global phase symmetry $\Phi'(x) = e^{i\theta}\Phi(x)$. In fact, the potential terms here $\Phi^*\Phi$ and $(\Phi^*\Phi)^2$ have a local phase symmetry $\Phi'(x) \;=\; e^{i\theta(x)}\Phi(x)$, but the kinetic term does not

have this local symmetry. Indeed, under this would-be local symmetry

$$\partial_\mu \Phi'(x) \;=\; e^{i\theta(x)}\big(\partial_\mu \Phi(x) \,+\, i\Phi(x)\partial_\mu\theta(x)\big), \tag{4}$$

hence

$$\big|\partial_\mu \Phi'\big|^2 \;=\; \big|\partial_\mu \Phi \,+\, i\Phi\partial_\mu\theta\big|^2 \;\neq\; \big|\partial_\mu \Phi\big|^2. \tag{5}$$

However, we may repair this problem by replacing the ordinary field derivatives $\partial_\mu\Phi$ and $\partial_\mu\Phi^*$ with the *covariant derivatives* $D_\mu\Phi$ and $D_\mu\Phi^*$ which transform under the local symmetry just line the field Φ and Φ^* themselves:

$$\begin{aligned} \Phi(x) \;&\to\; e^{i\theta(x)}\Phi(x), \quad D_\mu\Phi(x) \;\to\; e^{i\theta(x)}D_\mu\Phi(x), \\ \Phi^*(x) \;&\to\; e^{-i\theta(x)}\Phi^*(x), \quad D_\mu\Phi^*(x) \;\to\; e^{-i\theta(x)}D_\mu\Phi^*(x). \end{aligned} \tag{6}$$

Given such covariant derivatives, the Lagrangian

$$\mathcal{L} \;=\; D^\mu\Phi^* D_\mu\Phi \;-\; V(\Phi^*\Phi) \tag{7}$$

would be invariant under the local rather than global phase symmetry.

Likewise, a free Dirac fermion field with the Lagrangian

$$\mathcal{L} \;=\; i\overline{\Psi}\gamma^\mu\partial_\mu\Psi \;-\; m\overline{\Psi}\Psi \tag{8}$$

has a global phase symmetry $\Psi(x) \to e^{i\theta}\Psi(x)$, but it can be promoted to a local phase symmetry $\Psi(x) \to e^{i\theta(x)}\Psi(x)$ if we replace the ordinary derivative $\partial_\mu\Psi$ with the covariant derivative $D_\mu\Psi$, thus

$$\mathcal{L} \;=\; i\overline{\Psi}\gamma^\mu D_\mu\Psi \;-\; m\overline{\Psi}\Psi. \tag{9}$$

To make the covariant derivatives, we need a *connection* — a 4-vector field $\mathcal{A}^\mu(x)$ undergoing a gauge transform parametrized by the same $\theta(x)$ as the local phase symmetry,

thus

$$\left.\begin{aligned} \Phi'(x) &= \exp(+i\theta(x)) \times \Phi(x), \\ \Phi^{*\prime}(x) &= \exp(-i\theta(x)) \times \Phi^*(x), \\ \mathcal{A}'_\mu(x) &= \mathcal{A}_\mu(x) - \partial_\mu\theta(x) \end{aligned}\right\} \quad \text{for the same } \theta(x). \tag{10}$$

Given such combined phase/gauge transformations of the fields, the covariant derivatives

$$\begin{aligned} D_\mu\Phi(x) &= \partial_\mu\Phi(x) + i\mathcal{A}_\mu(x)\Phi(x), \\ D_\mu\Phi^*(x) &= \partial_\mu\Phi^*(x) - i\mathcal{A}_\mu(x)\Phi^*(x), \end{aligned} \tag{11}$$

transform covariantly according to eq. (6). Indeed,

$$\begin{aligned} (D_\mu\Phi)' &= \partial_\mu\Phi' + i\mathcal{A}' \times \Phi = \partial_\mu\big(e^{i\theta}\Phi\big) + i\big(\mathcal{A} - \partial_\mu\theta\big) \times e^{i\theta}\Phi \\ &= e^{i\theta}\left(\partial_\mu\phi + \cancel{i\Phi\partial_\mu\theta} + i\mathcal{A}_\mu \times \Phi - \cancel{i\partial_\mu\theta \times \Phi}\right) \\ &= e^{i\theta}\big(\partial_\mu\phi + i\mathcal{A}_\mu \times \Phi\big) = e^{i\theta} \times D_\mu\Phi, \end{aligned} \tag{12}$$

and likewise for the $D_\mu\Phi^*$.

More generally, consider a theory with multiple complex fields $\varphi_a(x)$; these fields may be scalar, fermionic, vector, whatever, as long as they have definite charges q_a WRT to the phase symmetry. Under the local phase symmetry, all these fields and the connection $\mathcal{A}_\mu(x)$ transform according to

$$\left.\begin{aligned} \varphi'_a(x) &= \exp(+iq_a\theta(x)) \times \varphi_a(x), \\ \varphi^{*\prime}_a(x) &= \exp(-iq_a\theta(x)) \times \varphi^*_a(x) \\ &\qquad \langle\!\langle\, \varphi^*_a \text{ has charge } -q_a \,\rangle\!\rangle, \\ \mathcal{A}'_\mu(x) &= \mathcal{A}_\mu(x) - \partial_\mu\theta(x), \end{aligned}\right\} \quad \text{all for the same } \theta(x). \tag{13}$$

Under these transformation laws, the derivatives

$$D_\mu\varphi_a = \partial_\mu\varphi_a + iq_a\mathcal{A}_\mu \times \varphi_a\,, \quad D_\mu\varphi^*_a = \partial_\mu\varphi^*_a - iq_a\mathcal{A}_\mu \times \varphi^*_a\,, \tag{14}$$

are covariant:

$$\big(D_\mu\varphi_a(x)\big)' = \exp(iq_q\theta(x)) \times D_\mu\varphi_a(x), \quad \big(D_\mu\varphi^*_a(x)\big)' = \exp(-iq_q\theta(x)) \times D_\mu\varphi^*_a(x). \tag{15}$$

For example, let's identify the connection $\mathcal{A}_\mu(x)$ with the electromagnetic field $A_\mu(x)$ and let's couple it to a bunch of scalar and fermionic fields of electric charges q_a governed by the

net Lagrangian

$$\mathcal{L} \;=\; -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + \sum_a^{\text{scalars}} D_\mu\Phi_a^* D^\mu\Phi_a + \sum_a^{\text{fermions}} \overline{\Psi}_a(i\gamma^\mu D_\mu - m_a)\Psi_a - V(\text{scalars}) + \mathcal{L}_{\text{Yukawa}}\,. \tag{16}$$

As long as the scalar potential and the Yukawa couplings in this Lagrangian are invariant under the global phase symmetry, the net Lagrangian would be invariant under the local phase symmetry thanks to the covariance of the derivatives D_μ.

Algebra of Covariant Derivatives

- Multiple covariant derivatives of charged fields are all covariant:

$$\begin{aligned} \big(D_\mu D_\nu\varphi_a(x)\big)' \;&=\; \exp(iq_a\theta(x)) \times D_\mu D_\nu\varphi_a(x), \\ \big(D_\lambda D_\mu D_\nu\varphi_a(x)\big)' \;&=\; \exp(iq_a\theta(x)) \times D_\lambda D_\mu D_\nu\varphi_a(x), \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned} \tag{17}$$

- Leibniz rule

$$D_\mu(\varphi_a \times \varphi_b) \;=\; (D_\mu\varphi_a) \times \varphi_b \;+\; \varphi_a \times (D_\mu\varphi_b) \quad \text{for } q(\varphi_a \times \varphi_b) \;=\; q_a \;+\; q_b\,. \tag{18}$$

  Indeed,

$$\begin{aligned} D_\mu(\varphi_a \times \varphi_b) \;&=\; \partial_\mu(\varphi_a \times \varphi_b) \;+\; i(q_a + q_b)\mathcal{A}_\mu \times \varphi_a \times \varphi_b \\ &=\; (\partial_\mu\varphi_a) \times \varphi_b \;+\; \varphi_a \times (\partial_\mu\varphi_b) \;+\; iq_a\mathcal{A}_\mu\varphi_a \times \varphi_b \;+\; \varphi_a \times iq_b\mathcal{A}_\mu\varphi_b \\ &=\; (D_\mu\varphi_a) \times \varphi_b \;+\; \varphi_a \times (D_\mu\varphi_b). \end{aligned} \tag{19}$$

  - In particular, for $q_a + q_b = 0$ the product $\varphi_a \times \varphi_b$ is neutral, thus

$$(D_\mu\varphi_a) \times \varphi_b \;+\; \varphi_a \times (D_\mu\varphi_b) \;=\; \text{ordinary } \partial_\mu(\varphi_a \times \varphi_b), \tag{20}$$

    which allows us to integrate by parts:

$$\begin{aligned} \int d^4x\,(D_\mu\varphi_a) \times \varphi_b \;+ \int d^4x\,\varphi_a \times (D_\mu\varphi_b) \;&= \int d^4x\,\partial_\mu(\varphi_a \times \varphi_b) \\ &= \int\limits_{\text{boundary}} d^3x\,n_\mu(\varphi_a \times \varphi_b) \\ \text{usually} \;&=\; 0. \end{aligned} \tag{21}$$

    For example, the kinetic term for a charged scalar field Φ can be integrated by parts

as

$$\int d^4x\,(D_\mu\Phi^*)(D^\mu\Phi) \;=\; -\int d^4x\,\Phi^*(D^2\Phi) \;=\; -\int d^4x\,(D^2\Phi^*)\Phi. \tag{22}$$

But the covariance of derivatives D_μ has its price: unlike the ordinary derivatives ∂_μ, *the covariant derivatives D_μ do not commute with each other*, $D_\mu D_\nu \neq D_\nu D_\mu$. Indeed,

$$\begin{aligned} D_\mu D_\nu\varphi \;&=\; (\partial_\mu + iq\mathcal{A}_\mu)(\partial_\nu + iq\mathcal{A}_\nu)\varphi \\ &=\; \partial_\mu\partial_\nu\varphi \;+\; iq\mathcal{A}_\mu\times\partial_\nu\varphi \;+\; iq\mathcal{A}_\nu\times\partial_\mu\varphi \;+\; iq(\partial_\mu\mathcal{A}_\nu)\times\varphi \;-\; q^2\mathcal{A}_\mu\mathcal{A}_\nu\times\varphi \end{aligned} \tag{23}$$

where the blue terms on the RHS are symmetric WRT $\mu \leftrightarrow \nu$ but the red term is not symmetric. Consequently,

$$D_\mu D_\nu\varphi \;-\; D_\nu D_\mu\varphi \;=\; iq(\partial_\mu\mathcal{A}_\nu - \partial_\nu\mathcal{A}_\mu)\times\varphi \;=\; iq\mathcal{F}_{\mu\nu}\times\varphi, \tag{24}$$

or in the operator language

$$[D_\mu, D_\nu] \;=\; i\mathcal{F}_{\mu\nu}\times\hat{Q} \tag{25}$$

where $\hat{Q}$ is the electric charge operator, $\hat{Q}\varphi = q\varphi$.

Non Abelian Example: Local SU(N) Symmetry

Take N free Dirac fermions fields $\Psi^1, \ldots, \Psi^N$ of the same mass. The Lagrangian

$$\mathcal{L} \;=\; \overline{\Psi}_j(i\gamma^\mu\partial_\mu - m)\Psi^j \qquad \langle\!\langle\, \text{implicit } \textstyle\sum_j \,\rangle\!\rangle \tag{26}$$

is invariant under global symmetries which mix the $\Psi^j(x)$ fields with each other,

$$\Psi^{j\prime}(x) \;=\; U^j_{\ k}\Psi^k(x) \tag{27}$$

for a *unitary* $N\times N$ matrix $\|U^j_{\ k}\|$. Such matrices form a non-abelian group called $U(N)$, hence the $U(N)$ group of global symmetries (27) of the N fermionic fields. To make our

notations more compact, let's assemble the Ψ^j into a column vector of length N while their conjugates $\overline{\psi}_j$ form a row vector of the same length,

$$\Psi(x) \stackrel{\text{def}}{=} \begin{pmatrix} \Psi^1(x) \\ \vdots \\ \Psi^N(x) \end{pmatrix}, \qquad \overline{\Psi}(x) \stackrel{\text{def}}{=} \begin{pmatrix} \overline{\Psi}_1(x) & \cdots & \overline{\Psi}_N(x) \end{pmatrix}, \tag{28}$$

then the global symmetries (27) become simply

$$\Psi'(x) \;=\; U\Psi(x), \quad \overline{\Psi}'(x) \;=\; \overline{\Psi}(x)U^\dagger, \quad \text{same } U \in U(N) \text{ for all } x. \tag{29}$$

To promote the global unitary symmetries (29) to local symmetries

$$\Psi'(x) \;=\; U(x)\Psi(x), \quad \overline{\Psi}'(x) \;=\; \overline{\Psi}(x)U^\dagger(x) \quad \text{for independent } U(x) \in U(N) \text{ at each } x, \tag{30}$$

we turn the ordinary derivatives ∂_μ in the Lagrangian (26) into covariant derivatives D_μ such that

$$\big(D_\mu\Psi(x)\big)' \;=\; U(x)D_\mu\Psi(x), \tag{31}$$

then the new Lagrangian

$$\mathcal{L} \;=\; \overline{\Psi}(i\gamma^\mu D_\mu \,-\, m)\Psi \tag{32}$$

would be invariant under the local symmetries (30).

To construct the derivatives

$$D_\mu\Psi(x) \;=\; \partial_\mu\Psi(x) \;+\; i\mathcal{A}_\mu(x)\Psi(x) \tag{33}$$

covariant WRT to $U(N)$ symmetries, we need the *matrix-valued connection* $\mathcal{A}_\mu(x)$, or in other words, an $N \times N$ matrix $\|\mathcal{A}^j_{\mu k}(x)\|$ of vector fields. In components, the covariant derivatives (33) act as

$$D_\mu\Psi^j(x) \;=\; \partial_\mu\Psi^j(x) \;+\; i\mathcal{A}^j_{\mu k}(x)\Psi^k(x). \tag{34}$$

Similar to the abelian case, the local unitary symmetry of the $\Psi^j(x)$ fields should be accompanied by the gauge transform of the vector fields $\mathcal{A}^j_{\mu k}(x)$, but the specific form of this

gauge transform is more complicated than its abelian counterpart. Indeed, to achieve the covariance of the derivatives (33), we need

$$\begin{aligned} \big(D_\mu\Psi\big)' \;&=\; \partial_\mu(\Psi'=U\Psi) \;+\; i\mathcal{A}'_\mu(\Psi'=U\Psi) \;=\; U\partial_\mu\Psi \;+\; (\partial_\mu U)\Psi \;+\; i\mathcal{A}'_\mu U\Psi \\ &\quad\| \\ U\,D_\mu\Psi \;&=\; U\partial_\mu\Psi \;+\; iU\mathcal{A}_\mu\Psi, \end{aligned}$$

and hence

$$i\mathcal{A}'_\mu U\Psi \;=\; iU\mathcal{A}_\mu\Psi \;-\; (\partial_\mu U)\Psi. \tag{35}$$

To make sure this relation works for any $\Psi(x)$, we need

$$i\mathcal{A}'_\mu(x)U(x) \;=\; iU(x)\mathcal{A}_\mu(x) \;-\; \partial_\mu U(x), \tag{36}$$

so *the non-abelian gauge transform of the matrix-valued connection $\mathcal{A}_\mu(x)$ works according to*

$$\mathcal{A}'_\mu(x) \;=\; U(x)\mathcal{A}_\mu(x)U^{-1}(x) \;+\; i(\partial_\mu U(x))U^{-1}(x). \tag{37}$$

Note: the first term on the RHS is peculiar to the non-abelian gauge transforms — in the abelian case, it would be simply $\mathcal{A}_\mu(x)$ — while the second term generalizes the $-\partial_\mu\theta(x)$. Indeed, for $N=1$ a unitary 1×1 matrix is simply a unimodular complex number $u=e^{i\theta}$. Consequently, the $U(1)$ symmetry group is the abelian group of phase symmetries, while

$$i(\partial_\mu u)\times u^{-1} \;=\; i(\partial_\mu e^{i\theta})\times e^{-i\theta} \;=\; -\partial_\mu\theta, \tag{38}$$

hence

$$\mathcal{A}'_\mu(x) \;=\; \mathcal{A}_\mu(x) \;-\; \partial_\mu\theta(x). \tag{39}$$

Next, let's take a closer look at the non-abelian vector fields. A priori, the connection $\mathcal{A}_\mu(x)$ is a *complex* $N\times N$ matrix of vector fields, which is equivalent to $2N^2$ real vector fields.

However, we only need the Hermitian part of that matrix, $\mathcal{A}_\mu^\dagger = \mathcal{A}_\mu$, which is equivalent to N^2 real vector fields. Indeed, the second term in eq. (37) is always Hermitian,

$$\begin{aligned}\big[i(\partial_\mu U)U^{-1}\big]^\dagger \;&=\; -i\big(U^{-1}\big)^\dagger\big(\partial_\mu U^\dagger\big) \\ &\qquad \langle\!\langle\, \text{by unitarity of } U,\, U^\dagger = U^{-1} \,\rangle\!\rangle \\ &=\; -iU\big(\partial_\mu U^{-1}\big) \;=\; -iU\big(-U^{-1}(\partial_\mu U)U^{-1}\big) \\ &=\; +i(\partial_\mu U)U^{-1},\end{aligned} \tag{40}$$

hence IF $\mathcal{A}_\mu$ is Hermitian THEN so is $\mathcal{A}'_\mu$:

$$\begin{aligned}\big[U\mathcal{A}_\mu U^{-1}\big]^\dagger \;&=\; \big(U^{-1}\big)^\dagger \mathcal{A}_\mu^\dagger U^\dagger \;=\; U\mathcal{A}_\mu U^{-1} \\ &\qquad\qquad \Downarrow \\ \Big[\mathcal{A}'_\mu \;=\; U\mathcal{A}_\mu U^{-1} \,+\, i(\partial_\mu U)U^{-1}\Big]^\dagger \;&=\; U\mathcal{A}_\mu U^{-1} \,+\, i(\partial_\mu U)U^{-1} \;=\; \mathcal{A}'_\mu\,.\end{aligned} \tag{41}$$

Moreover, the unitary symmetry group $U(N)$ is a direct product of $SU(N)$ — the group of unitary matrices with unit determinants — and the $U(1)$ group of overall phases,

$$\text{any } U \in U(N) \text{ is } U \;=\; e^{i\theta} \times \tilde{U} \quad \text{where} \quad \det(\tilde{U}) = 1 \quad \text{and} \quad \theta = \frac{\arg(\det(U))}{N}\,. \tag{42}$$

In terms of the fermion fields $\Psi^j(x)$, the $U(1)$ is the common phase symmetry — with the same phase $e^{i\theta}$ for all the Ψ^j, — while the $SU(N)$ symmetries mix the fields with each other. Consequently, the $SU(N)$ and the $U(1)$ connections are completely independent from each other. Specifically, the $U(1)$ connection $\mathcal{A}_\mu^{U(1)}$ is proportional to the unit matrix, while the $SU(N)$ connection is a traceless matrix. Indeed,

$$\text{as long as } \det U(x) \equiv 1 \text{ and } \mathrm{tr}(\mathcal{A}_\mu(x)) \equiv 0, \tag{43}$$

$$\mathrm{tr}\big(-i(\partial_\mu U)U^{-1}\big) \;=\; -i\partial_\mu\,\mathrm{tr}\big(\log(U)\big) \;=\; -i\partial_\mu \log\big(\det(U) = 1\big) \;=\; 0, \tag{44}$$

$$\mathrm{tr}\big(U\mathcal{A}_\mu U^{-1}\big) \;=\; \mathrm{tr}\big(\mathcal{A}_\mu\big) \;=\; 0, \tag{45}$$

$$\text{hence } \mathrm{tr}\big(\mathcal{A}'_\mu(x)\big) \;=\; 0. \tag{46}$$

In light of complete independence of the $SU(N)$ and $U(1)$ factors of the unitary group $U(N)$, I am going to restrict the local symmetries to the $SU(N)$ factor while the $U(1)$ factor

remains a global phase symmetry. Consequently, there is no $U(1)$ connection, while *the $SU(N)$ connection $\mathcal{A}_\mu$ is a traceless Hermitian matrix* equivalent to N^2-1 real vector fields $\mathcal{A}^a_\mu(x)$, $a=1,\ldots,(N^2-1)$.

For example, for $N=2$ there are 3 independent traceless Hermitian matrices, namely the Pauli matrices

$$\tau^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tau^2 = \begin{pmatrix} 0 & -i \\ +i & 0 \end{pmatrix}, \quad \tau^3 = \begin{pmatrix} +1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{47}$$

Consequently, the $SU(2)$ connection $\mathcal{A}_\mu(x)$ can be written as

$$\big[\mathcal{A}_\mu(x)\big]^j_{\ k} = \sum_{a=1,2,3} \mathcal{A}^a_\mu(x) \times \left(\frac{\tau^a}{2}\right)^j_{\ k} \tag{48}$$

in terms of 3 ordinary real vector fields $\mathcal{A}^a_\mu(x)$.

For $N \geq 3$, there are N^2-1 independent traceless matrices, for example the Gell-Mann matrices λ^a. Here is their explicit forms for $N=3$:

$$\lambda^1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda^2 = \begin{pmatrix} 0 & -i & 0 \\ +i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda^3 = \begin{pmatrix} +1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\lambda^4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \lambda^5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ +i & 0 & 0 \end{pmatrix}, \quad \lambda^6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \lambda^7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & +i & 0 \end{pmatrix},$$

$$\lambda^8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}. \tag{49}$$

Consequently, the $SU(N)$ connection expands into N^2-1 ordinary real vector fields as

$$\big[\mathcal{A}_\mu(x)\big]^j_{\ k} = \sum_{a=1}^{N^2-1} \mathcal{A}^a_\mu(x) \times \left(\frac{\lambda^a}{2}\right)^j_{\ k} \tag{50}$$

For future reference, here are some properties of the Gell-Mann matrices:

- Similar to the Pauli matrices τ^a, the Gell-Mann matrices λ^a are Hermitian, traceless, and normalized to $\mathrm{tr}(\lambda^a\lambda^b) = 2\delta^{ab}$.
- $[\lambda^a, \lambda^b] = 2i\sum_c f^{abc}\lambda^c$ for some totally antisymmetric *structure constants* $f^{[abc]}$ of the $SU(N)$ Lie algebra. This commutation relation generalizes the isospin commutation relation $[\tau^a, \tau^b] = 2i\sum_c \epsilon^{abc}\tau^c$ for the Pauli matrices.
  - Unlike the Pauli matrices, the Gell-Mann matrices do not anticommute with each other and do not square to unit matrices, $\{\lambda^a, \lambda^b\} \neq 2\delta^{ab}1_{N\times N}$. Instead, for $N \geq 3$ we have

$$\{\lambda^a, \lambda^b\} \;=\; \frac{4\delta^{ab}}{N}\,\mathbf{1}_{N\times N} \;+\; \sum_c 2d^{abc}\lambda^c \tag{51}$$

  for some totally symmetric coefficients $d^{(abc)}$.

Now let's go back to the component vector fields $\mathcal{A}^a_\mu(x)$. Earlier in this section I wrote down the non-abelian gauge transform of the vector fields in the matrix language, but translating it in terms of the component fields is rather painful. Or rather, it is quite painful for finite local symmetries $U(x)$, but it becomes much easier for *infinitesimal symmetries*: In matrix language,

$$U(x) \;=\; \exp(i\Lambda(x)) \;=\; 1 \;+\; i\Lambda(x) \;+\; O(\Lambda^2) \tag{52}$$

for some infinitesimal matrix-valued $\Lambda(x)$. To keep the $U(x)$ unitary and $\det(U) = 1$, the $\Lambda(x)$ matrix should be Hermitian and traceless, hence

$$\Lambda(x) \;=\; \Lambda^a(x) \times \frac{\lambda^a}{2} \quad \langle\!\langle\, \text{implicit } \textstyle\sum_a \,\rangle\!\rangle \tag{53}$$

for some infinitesimal real numbers $\Lambda^a(x)$. Under such infinitesimal local symmetries, the fermionic fields $\Psi^j(x)$ transform into

$$\Psi^{j\prime}(x) \;=\; \Psi^j(x) \;+\; i\Lambda^a(x)\left(\frac{\lambda^a}{2}\right)^j_k \Psi^k(x) \;+\; O(\Lambda^2\Psi). \tag{54}$$

At the same time, for the vector fields we have

$$-i(\partial_\mu U)\,U^{-1} \;=\; -\partial_\mu\Lambda(x) \;+\; O(\Lambda^2), \tag{55}$$

$$U\,\mathcal{A}_\mu\,U^{-1} \;=\; \mathcal{A}_\mu \;+\; i[\Lambda,\mathcal{A}_\mu] \;+\; O(\mathcal{A}\Lambda^2), \tag{56}$$

and hence to first order in Λ,

$$\mathcal{A}'_\mu(x) \;=\; \mathcal{A}_\mu(x) \;+\; i[\Lambda(x),\mathcal{A}_\mu(x)] \;-\; \partial_\mu\Lambda(x). \tag{57}$$

In components,

$$\begin{aligned} i[\Lambda(x),\mathcal{A}_\mu(x)] \;&=\; \Lambda^b(x)\times\mathcal{A}^c_\mu(x)\times i\left[\frac{\lambda^b}{2},\frac{\lambda^c}{2}\right] \\ &=\; \Lambda^b(x)\times\mathcal{A}^c_\mu(x)\times\left(-f^{bca}\,\frac{\lambda^a}{2} \;=\; -f^{abc}\,\frac{\lambda^a}{2}\right) \\ &=\; -\left(f^{abc}\Lambda^b(x)\mathcal{A}^c_\mu(x)\right)\times\frac{\lambda^a}{2}\,, \end{aligned} \tag{58}$$

hence

$$\mathcal{A}'_\mu(x) \;=\; \frac{\lambda^a}{2}\times\left(\mathcal{A}^a_\mu(x) \;-\; f^{abc}\Lambda^b(x)\mathcal{A}^c_\mu(x) \;-\; \partial_\mu\Lambda^a(x)\right) \tag{59}$$

and therefore

$$\mathcal{A}^{a\prime}_\mu(x) \;=\; \mathcal{A}^a_\mu(x) \;-\; f^{abc}\Lambda^b(x)\mathcal{A}^c_\mu(x) \;-\; \partial_\mu\Lambda^a(x). \tag{60}$$

Non Abelian Tension Fields

In an abelian $U(1)$ gauge theory such as QED, the covariant derivatives D_μ do not commute with each other, and their commutators are related to the EM tensions fields as $[D_\mu, D_\nu]\Psi(x) = iqF_{\mu\nu}(x)\Psi(x)$. In non-abelian gauge theories, there is a similar relation in the matrix language,

$$[D_\mu, D_\nu]\Psi(x) \;=\; i\mathcal{F}_{\mu\nu}(x)\Psi(x) \tag{61}$$

where $\mathcal{F}_{\mu\nu}(x)$ is the matrix-valued tension field. But the relation of this tension field to the connection $\mathcal{A}_\mu(x)$ is more complicated than in the abelian case. To see how it works, let's

spell out the double covariant derivative

$$\begin{aligned} D_\mu D_\nu \Psi &= (\partial_\mu + i\mathcal{A}_\mu)(\partial_\nu + i\mathcal{A}_\nu)\Psi \\ &= \partial_\mu\partial_\nu\Psi \ + \ i\mathcal{A}_\mu \times \partial_\nu\Psi \ + \ i\mathcal{A}_\nu \times \partial_\mu\Psi \ + \ i(\partial_\mu\mathcal{A}_\nu) \times \Psi \ - \ \mathcal{A}_\mu\mathcal{A}_\nu \times \Psi. \end{aligned} \tag{62}$$

On the second line here I have color-coded in blue the terms which are symmetric WRT to the $\mu \leftrightarrow \nu$ interchange, and in red the terms which are not symmetric. Note that the last term is not symmetric because the matrices $\mathcal{A}_\mu$ and $\mathcal{A}_\nu$ generally do not commute with each other. Consequently,

$$D_\mu D_\nu \Psi \ - \ D_\nu D_\mu \Psi \ = \ i(\partial_\mu\mathcal{A}_\nu) \times \Psi \ - \ i(\partial_\nu\mathcal{A}_\mu) \times \Psi \ - \ \mathcal{A}_\mu\mathcal{A}_\nu \times \Psi \ + \ \mathcal{A}_\nu\mathcal{A}_\mu \times \Psi, \tag{63}$$

or in other words,

$$[D_\mu, D_\nu]\Psi(x) \ = \ i\mathcal{F}_{\mu\nu}(x) \times \Psi(x) \tag{64}$$

where

$$\mathcal{F}_{\mu\nu}(x) \ = \ \partial_\mu\mathcal{A}_\nu(x) \ - \ \partial_\nu\mathcal{A}_\mu(x) \ + \ i[\mathcal{A}_\mu(x), \mathcal{A}_\nu(x)]. \tag{65}$$

Or in components,

$$\mathcal{F}_{\mu\nu}(x) \ = \ \mathcal{F}^a_{\mu\nu}(x) \times \frac{\lambda^a}{2} \quad \text{for} \quad \mathcal{F}^a_{\mu\nu}(x) \ = \ \partial_\mu\mathcal{A}^a_\nu(x) \ - \ \partial_\nu\mathcal{A}^a_\mu(x) \ - \ f^{abc}\mathcal{A}^b_\mu(x)\mathcal{A}^c_\nu(x). \tag{66}$$

Unlike their abelian counterparts, the non-abelian tensions (65) are not gauge invariant. Instead, they transform covariantly under the local $SU(N)$ symmetries: In matrix language,

$$\mathcal{F}'_{\mu\nu}(x) \ = \ U(x)\mathcal{F}_{\mu\nu}(x)U^{-1}(x). \tag{67}$$

. This formula may be derived directly from eq. (65) and the non-abelian gauge transform (37) of the vector field $\mathcal{A}_\mu(x)$ — and perhaps I should make this derivation a part of a future homework, — but it is much easier to obtain eq. (67) from the commutator (64) and

the covariance of the derivative D_μ. Indeed, multiple derivatives like $D_\mu D_\nu \Psi(x)$ are just as covariant as single derivatives,

$$\text{for } \Psi'(x) \;=\; U(x)\Psi(x), \quad \big[D'_\mu, D'_\nu\big]\Psi'(x) \;=\; U(x)\big[D_\mu, D_\nu\big]\Psi(x), \tag{68}$$

hence in light of eq. (64),

$$i\mathcal{F}'_{\mu\nu}(x) \times U(x)\Psi(x) \;=\; U(x) \times i\mathcal{F}_{\mu\nu}(x) \times \Psi(x), \tag{69}$$

and therefore

$$\mathcal{F}'_{\mu\nu}(x) \;=\; U(x) \times \mathcal{F}_{\mu\nu}(x) \times U^{-1}(x). \tag{67}$$

In components, the $\mathcal{F}^a_{\mu\nu}(x)$ tension fields transform into each other according to the *adjoint representation* of the local symmetry $U(x)$,

$$\mathcal{F}^{a\prime}_{\mu\nu}(x) \;=\; R^{ab}_{\text{adj}}(U(x)) \times \mathcal{F}^b_{\mu\nu}(x), \tag{70}$$

where

$$R^{ab}_{\text{adj}}(U) \;=\; \tfrac{1}{2}\,\text{tr}\big(\lambda^a U \lambda^b U^{-1}\big) \tag{71}$$

is an orthogonal $(N^2-1)\times(N^2-1)$ matrix. For example, for the $SU(2)$ isospin symmetry, U is the iso-doublet representation of some iso-space rotation while $R^{ab}(U)$ is the iso-vector representation of the same rotation.

While the tension fields themselves are not gauge invariant, there is an invariant quadratic combination $\text{tr}\big(\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu}\big)$. Indeed,

$$\text{tr}\big(\mathcal{F}'_{\mu\nu}\mathcal{F}^{\mu\nu\prime}\big) \;=\; \text{tr}\big(U\mathcal{F}_{\mu\nu}U^{-1} \times U\mathcal{F}^{\mu\nu}U^{-1}\big) \;=\; \text{tr}\big(\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu}\big). \tag{72}$$

In components,

$$\text{tr}\big(\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu}\big) \;=\; \mathcal{F}^a_{\mu\nu}\mathcal{F}^{b\mu\nu} \times \left(\text{tr}\left(\frac{\lambda^a}{2}\,\frac{\lambda^b}{2}\right) \;=\; \frac{\delta^{ab}}{2}\right) \;=\; \frac{1}{2}\mathcal{F}^a_{\mu\nu}\mathcal{F}^{a\mu\nu}, \tag{73}$$

and that's why the adjoint representation matrix $R^{ab}_{\text{adj}}(U)$ in eq. (70) must be an orthogonal matrix.

Yang–Mills Theory

Yang–Mills theory is the theory of non-abelian gauge fields $\mathcal{A}^a_\mu(x)$ interacting with each other; there are no other fields. The physical Lagrangian of the theory is simply

$$\mathcal{L} \;=\; -\frac{1}{2g^2}\,\mathrm{tr}\big(\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu}\big) \;=\; -\frac{1}{4g^2}\,\mathcal{F}^a_{\mu\nu}\mathcal{F}^{a\mu\nu} \tag{74}$$

for

$$\mathcal{F}^a_{\mu\nu} \;\stackrel{\text{def}}{=}\; \partial_\mu\mathcal{A}^a_\nu \;-\; \partial_\nu\mathcal{A}^a_\mu \;-\; f^{abc}\mathcal{A}^b_\mu\mathcal{A}^c_\nu. \tag{75}$$

The $1/g^2$ factor in the Yang–Mills Lagrangian (74) makes for non-canonical normalization of the gauge fields $\mathcal{A}^a_\mu$. To get the canonically normalized vector fields, we rescale

$$A^a_\mu(x) \;=\; \frac{1}{g}\,\mathcal{A}^a_\mu(x) \quad\text{and}\quad F^a_{\mu\nu}(x) \;=\; \frac{1}{g}\,\mathcal{F}^a_{\mu\nu}(x), \tag{76}$$

hence

$$\mathcal{L} \;=\; -\tfrac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} \tag{77}$$

for

$$F^a_{\mu\nu} \;=\; \partial_\mu A^a_\nu \;-\; \partial_\nu A^a_\mu \;-\; g f^{abc}A^b_\mu A^c_\nu. \tag{78}$$

For small $g \ll 1$, we may treat the non-abelian parts of $F^a_{\mu\nu}$ as small perturbation, hence

$$\mathcal{L} \;=\; -\tfrac{1}{4}\big(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu\big)^2 + \frac{g}{2}\big(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu\big)\times f^{abc}A^{b\mu}A^{c\nu} - \frac{g^2}{4}\,f^{abc}f^{ade}A^b_\mu A^c_\nu A^{d\mu}A^{e\nu} \tag{79}$$

where the quadratic term on the RHS describes N^2-1 species of free photon-like gluons, while the cubic and the quartic terms describe the interactions between the gluon fields.

Quantum Chromodynamics

Quantum ChromoDynamics or QCD is the theory of quarks and gluons — from which all the strongly interacting particles are made. The quarks are Dirac fermions $\Psi_f^j(x)$ which come in three colors ($j = 1, 2, 3$) and six flavors $f = u, d, s, c, b, t$. Quark masses depend on the flavor but not on the color, and there is exact $SU(3)$ symmetry mixing the colors. This color symmetry is local rather than global, which gives rise to $N_c^2 - 1 = 8$ species of massless vector fields $A_\mu^a(x)$ called the gluons. The physical Lagrangian of QCD is quite simple: in matrix notations for the color,

$$\mathcal{L} \;=\; -\tfrac{1}{2}\,\mathrm{tr}\big(F_{\mu\nu}F^{\mu\nu}\big) \;+\; \sum_f \overline{\Psi}_f\big(i\gamma^\mu D_\mu \;-\; m_f\big)\Psi_f \tag{80}$$

$$\text{where } F_{\mu\nu} \;=\; \partial_\mu A_\nu \;-\; \partial_\nu A_\mu \;+\; ig\big[A_\mu, A_\nu\big] \tag{81}$$

$$\text{and } D_\mu\Psi_f^j \;=\; \partial_\mu\Psi_f \;+\; igA_\mu\Psi, \tag{82}$$

while in explicit color-index notations

$$\mathcal{L} \;=\; -\tfrac{1}{4}\,F_{\mu\nu}^a F^{a\mu\nu} \;+\; \sum\nolimits_f \overline{\Psi}_{f,j}\big(i\gamma^\mu D_\mu \;-\; m_f\big)\Psi_f^j\,, \tag{83}$$

$$D_\mu\Psi_f^j \;=\; \partial_\mu\Psi_f^j \;+\; \frac{ig}{2}\,A_\mu^a\big(\lambda^a\big)^j{}_k\Psi_f^k\,, \tag{84}$$

$$F_{\mu\nu}^a \;=\; \partial_\mu A_\nu^a \;-\; \partial_\nu A_\mu^a \;-\; gf^{abc}A_\mu^b A_\nu^c\,. \tag{85}$$

To set up the perturbation theory, we should expand the Lagrangian (83) in powers of g. We should also add a gauge-fixing term for each gluon field A_μ^a to give them useful photon-like propagators, for example

$$\mathcal{L}_{\text{g.f.}} \;=\; -\frac{1}{2\xi}\,\big(\partial^\mu A_\mu^a\big)^2. \tag{86}$$

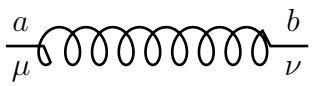
Thus altogether

$$\begin{aligned}\mathcal{L}_{\text{phys}}^{\text{QCD}} \;+\; \mathcal{L}_{\text{g.f.}}^{\text{QCD}} \;=\; &-\frac{1}{2}\,\big(\partial_\mu A_\nu^a\big)^2 \;+\; \frac{1-\xi^{-1}}{2}\,\big(\partial^\mu A_\mu^a\big)^2 \;+\; \sum_f \overline{\Psi}_{f,j}(i\gamma^\mu\partial_\mu - m_f)\Psi_f^j \\ &+\; gf^{abc}(\partial_\mu A_\nu^a)A^{b\mu}A^{c\nu} \;-\; \frac{g^2}{4}\,f^{abc}f^{ade}\,A_\mu^b A_\nu^c A^{d\mu}A^{e\nu} \\ &+\; \frac{ig}{2}\,A_\mu^a \times \sum\nolimits_f \overline{\Psi}_{f,i}\gamma^\mu(\lambda^a)^i{}_j\Psi_f^j\,.\end{aligned} \tag{87}$$

The first line of this expansion describes the free gluon and quark fields, while the second

and third lines describe their interactions. Consequently, the *tree-level* Feynman rules for QCD are as follows:

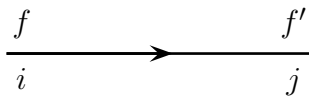
- The gluon propagator



$$= \frac{-i\delta^{ab}}{k^2 + i0}\left(g^{\mu\nu} + (\xi - 1)\frac{k^\mu k^\nu}{k^2 + i0}\right) \tag{88}$$

  where ξ is the gauge-fixing parameter. For $\xi = 0$ we have the Landau gauge while for $\xi = 1$ — the Feynman gauge.

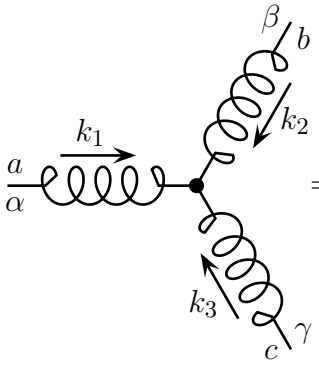
- The quark propagator



$$= \frac{i\delta^i_j \delta_{ff'}}{\not{p} - m_f + i0}\,. \tag{89}$$

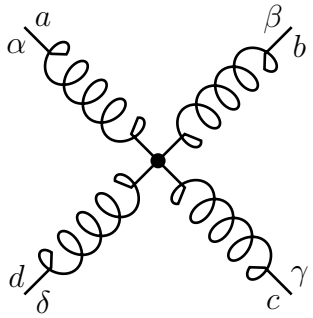
  Note: the colors and the flavors must be the same at both ends of the propagator.

- The three-gluon vertex



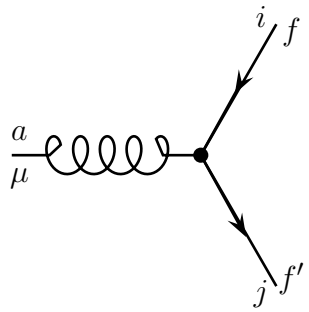
$$= -gf^{abc}\left[g^{\alpha\beta}(k_1 - k_2)^\gamma + g^{\beta\gamma}(k_2 - k_3)^\alpha + g^{\gamma\alpha}(k_3 - k_1)^\beta\right]. \tag{90}$$

- The four-gluon vertex



$$= -ig^2\begin{bmatrix} f^{abe}f^{cde}(g^{\alpha\gamma}g^{\beta\delta} - g^{\alpha\delta}g^{\beta\gamma}) \\ + f^{ace}f^{bde}(g^{\alpha\beta}g^{\gamma\delta} - g^{\alpha\delta}g^{\gamma\beta}) \\ + f^{ade}f^{bce}(g^{\alpha\beta}g^{\delta\gamma} - g^{\alpha\gamma}g^{\delta\beta}) \end{bmatrix}. \tag{91}$$

- The quark-antiquark-gluon vertex



$$= -ig\gamma^\mu \times \delta_{ff'} \times \left(\frac{\lambda^a}{2}\right)^j_i . \qquad (92)$$

  Note: the quark lines connected to the vertex must have the same flavors $f' = f$ but they may have different colors $j \neq i$.

∗ The external line factors and the sign rules of QCD are exactly the same as in QED.

The above Feynman rules are OK at the tree level but insufficient for the loop calculations. For one thing, beyond the tree level we would need a bunch of counterterm vertices to cancel the UV divergences of the loop graphs. But there is also a deeper problem stemming from the gauge-fixing the gluon fields. In abelian gauge theories like QED, linear gauge-fixing constraints like $\partial_\mu A^\mu = 0$ are harmless, but in non-abelian gauge theories such constraints screw up the analogues of Ward–Takahashi identities which we need to properly handle the loop graphs. In the path integral formalism, the gauge-fixing constraints screw up the measure of the path integral, but we can un-screw it by introducing additional un-physical fields called the *ghosts*. In Feynman rules, there are ghost propagators and ghost-ghost-gluon vertices, but no external ghost lines — the ghosts may run in loops but never as external particles.

I shall explain this issue in detail later this semester, once you have learned the basics of the path-integral formalism.

General Gauge Symmetries

There are more types of non-abelian gauge symmetry groups than $SU(N)$. In general, we may have any compact Lie group G whose generators $\hat{T}^a$ form the correspondent Lie algebra $\mathbf{G}$; that is, they obey the commutation relations

$$\left[\hat{T}^a, \hat{T}^b\right] = if^{abc}\hat{T}^c \qquad (93)$$

for the appropriate structure constants $f^{[abc]}$. For each generator $\hat{T}^a$ there is a vector field

$A^a_\mu(x)$, which acts as a component of the Lie-algebra-valued connection

$$\mathcal{A}_\mu(x) \;=\; g A^a_\mu(x) \times \hat{T}^a. \tag{94}$$

The curvature for this connection is the Lie-algebra-valued antisymmetric tensor field

$$\mathcal{F}_{\mu\nu}(x) \;=\; \partial_\mu \mathcal{A}_\nu(x) \;-\; \partial_\nu \mathcal{A}_\mu(x) \;+\; i\big[\mathcal{A}_\mu(x), \mathcal{A}_\nu(x)\big], \tag{95}$$

or in components

$$\mathcal{F}_{\mu\nu}(x) \;=\; g F^a_{\mu\nu}(x) \times \hat{T}^a \quad \text{for} \quad F^a_{\mu\nu}(x) \;=\; \partial_\mu A^a_\nu(x) \;-\; \partial_\nu A^a_\mu(x) \;-\; i g f^{abc} A^b_\mu(x) A^c_\nu(x). \tag{96}$$

The local symmetries are parametrized by $u(x) \in G$ — for each x there is an element of the gauge group G. For infinitesimal symmetries

$$u(x) \;=\; \exp\big(i\Lambda^a(x)\hat{T}^a\big) \;=\; 1 \;+\; i\Lambda^a(x) \times \hat{T}^a \;+\; O(\Lambda^2) \tag{97}$$

for some infinitesimal real parameters $\Lambda^a(x)$. Under such infinitesimal symmetries, the gauge fields $A^a_\mu(x)$ transform inhomogeneously as

$$\delta A^a_\mu(x) \;=\; -\frac{1}{g}\,\partial_\mu \Lambda^a(x) \;-\; f^{abc} \Lambda^b(x) A^c_\mu(x) \tag{98}$$

which the tension fields $F^a_{\mu\nu}(x)$ transform homogeneously as

$$\delta F^a_{\mu\nu}(x) \;=\; -f^{abc} \Lambda^b(x) F^c_{\mu\nu}(x). \tag{99}$$

The matter fields — scalars and fermions — form complete multiplets of the gauge symmetry group G. In each such multiplet (m), the generators $\hat{T}^a$ of G are represented by

$|m|\times|m|$ matrices $T^a_{(m)}$ obeying the same commutation relations as the generators themselves,

$$\left[T^a_{(m)}, T^b_{(m)}\right] \;=\; if^{abc} \times T^c_{(m)}\,. \tag{100}$$

Under infinitesimal gauge symmetries, a field Ψ^α belonging to some multiplet (m) is mixed with other fields Ψ^β belonging to the same multiplet according to

$$\delta\Psi^\alpha(x) \;=\; i\Lambda^a(x)\left[T^a_{(m)}\right]^\alpha_{\ \beta}\Psi^\beta(x). \tag{101}$$

The covariant derivatives $D_\mu\Psi^\alpha$ also mix up fields belonging to the same multiplet (m); specifically,

$$D_\mu\Psi^\alpha(x) \;=\; \partial_\mu\Psi^\alpha(x) \;+\; igA^a_\mu(x)\left[T^a_{(m)}\right]^\alpha_{\ \beta}\Psi^\beta(x). \tag{102}$$

Note different matrices $T^a_{(m)}$ for covariant derivatives of fields belonging to different multiplet types; this is similar to different fields having different electric charges in QED.

Let's verify the covariance of the derivatives (102) WRT infinitesimal gauge symmetries. In matrix language — where we treat the whole multiplet of fields Ψ^α as a column vector Ψ, we have

$$\begin{aligned}
\delta D_\mu\Psi \;&=\; \partial_\mu\delta\Psi \;+\; igA^a_\mu T^a_{(m)} \times \delta\Psi \;+\; ig\delta A^a_\mu \times T^a_{(m)}\Psi \\
&=\; i\Lambda^a T^a_{(m)} \times \partial_\mu\Psi \;+\; \cancel{i(\partial_\mu\Lambda^a) \times T^a_{(m)}\Psi} \;-\; gA^a_\mu T^a_{(m)} \times \Lambda^b T^b_{(m)}\Psi \\
&\quad -\; \cancel{i(\partial_\mu\Lambda^a) \times T^a_{(m)}\Psi} \;-\; igf^{abc}\Lambda^b A^c_\mu \times T^a_{(m)}\Psi \\
&\quad \langle\!\langle\, \text{relabeling indices} \,\rangle\!\rangle \\
&=\; i\Lambda^a T^a_{(m)} \times \partial_\mu\Psi \;-\; gA^c_\mu T^c_{(m)} \times \Lambda^a T^a_{(m)}\Psi \;-\; igf^{bac}\Lambda^a A^c_\mu \times T^b_{(m)}\Psi \\
&=\; i\Lambda^a \times \left(T^a_{(m)}\partial_\mu\Psi \;+\; igA^c_\mu \times \left(T^c_{(m)}T^a_{(m)}\Psi \;-\; if^{bac}T^b_{(m)}\Psi\right)\right)
\end{aligned} \tag{103}$$

where

$$if^{bac}T^b_{(m)} \;=\; if^{acb}T^b_{(m)} \;=\; \left[T^a_{(m)}, T^c_{(m)}\right] \quad\Longrightarrow\quad T^c_{(m)}T^a_{(m)}\Psi \;-\; if^{bac}T^b_{(m)}\Psi \;=\; T^a_{(m)}T^b_{(m)}\Psi, \tag{104}$$

hence

$$\begin{aligned}
\delta D_\mu\Psi \;&=\; i\Lambda^a \times \left(T^a_{(m)}\partial_\mu\Psi \;+\; igA^c_\mu T^a_{(m)}T^c_{(m)}\Psi\right) \;=\; i\Lambda^a T^a_{(m)} \times \left(\partial_\mu\Psi \;+\; igA^c_\mu T^c_{(m)}\Psi\right) \\
&=\; i\Lambda^a T^a_{(m)} \times D_\mu\Psi,
\end{aligned} \tag{105}$$

quod erat demonstrandum.

To save time, I am not going to prove the covariance of D_μ under finite gauge transforms $u(x)$. Instead, let me simply summarize how such finite gauge transforms act on various fields. In general,

$$\text{any finite } u \in G \text{ is } u \;=\; \exp(i\Lambda^a \hat{T}^a) \text{ for some finite } \Lambda^a, \tag{106}$$

and the representation of this finite group element in a multiplet type (m) is a finite matrix

$$R_{(m)}(u) \;=\; \exp\bigl(i\Lambda^a T^a_{(m)}\bigr). \tag{107}$$

Consequently, under a finite gauge transform $u(x) = \exp(i\Lambda^a(x)\hat{T}^a)$, matter fields $\Psi^\alpha(x)$ belonging to a multiplet (m) mix with each other — but only with the members of the same multiplet — as

$$\Psi^{\alpha\prime}(x) \;=\; \bigl[\exp\bigl(i\Lambda^a T^a_{(m)}\bigr)\bigr]^\alpha_{\ \beta} \Psi^\beta(x). \tag{108}$$

As to the gauge fields, it is best to write their transformation laws in terms of the Lie-algebra-valued connection $\mathcal{A}_\mu(x)$ and curvature $\mathcal{F}_{\mu\nu}(x)$:

$$\mathcal{A}'_\mu(x) \;=\; i(\partial_\mu u(x))u^{-1}(x) \;+\; u(x)\mathcal{A}_\mu(x)u^{-1}(x), \tag{109}$$

$$\mathcal{F}'_{\mu\nu}(x) \;=\; u(x)\mathcal{F}_{\mu\nu}(x)u^{-1}(x). \tag{110}$$

In components, eq. (109) becomes rather unwieldy, but eq. (110) amounts to $F^a_{\mu\nu}(x)$ forming an *adjoint multiplet* of G, thus

$$F^{a\prime}_{\mu\nu}(x) \;=\; R^{ab}_{\text{adj}}(u(x)) \times F^b_{\mu\nu}(x). \tag{111}$$

Note: any simple Lie group has an adjoint representation where the generators $\hat{T}^a$ are represented by

$$\bigl[T^a_{\text{adj}}\bigr]^{bc} \;=\; if^{abc}; \tag{112}$$

the commutation relations between these $\dim(G) \times \dim(G)$ matrices follow from the Jacobi identity for the Lie algebra $\mathbf{G}$. Example: for the isospin symmetry $SU(2)$, the adjoint multiplet is the iso-vector $\mathbf{3}$.

Fields $\Phi^a(x)$ in an adjoint multiplet transform under infinitesimal gauge symmetries as

$$\delta\Phi^a(x) \;=\; -f^{abc}\Lambda^b(x)\Phi^c(x) \tag{113}$$

and the covariant derivatives D_μ act on them as

$$D_\mu\Phi^a(x) \;=\; \partial_\mu\Phi^a(x) \;-\; gf^{abc}A^b_\mu(x)\Phi^c(x). \tag{114}$$

Or in matrix form — or rather Lie algebra form — $\hat\Phi(x) = \Phi^a(x)\hat T^a$,

$$\delta\hat\Phi(x) \;=\; i[\hat\Lambda(x),\hat\Phi(x)], \quad D_\mu\hat\Phi \;=\; \partial_\mu\hat\Phi(x) \;+\; i[\mathcal{A}_\mu(x),\hat\Phi(x)]. \tag{115}$$

The Lie algebra form also makes it easy to write down the finite gauge transform of an adjoint multiplet,

$$\hat\Phi'(x) \;=\; u(x)\hat\Phi(x)u^{-i}(x). \tag{116}$$

Thus, the tension fields $F^a_{\mu\nu}(x)$ indeed form an adjoint multiplet of the gauge symmetry.

Combined Gauge Symmetries

A gauge symmetry group G does not have to be simple. It may also be a direct product of several simple or abelian factors,

$$G \;=\; G_1\times G_2\times G_3\times\cdots, \tag{117}$$

where each factor G_i comes with its own gauge fields — one for each generator of G_i — and its own gauge coupling g_i, thus

$$\mathcal{L} \;=\; \sum_i \frac{-1}{2g_i^2}\,\mathrm{tr}\big(\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu}\big)_{G_i} \;+\; \mathcal{L}[\text{matter}]. \tag{118}$$

For example, the Standard Model has $G = SU(3)\times SU(2)\times U(1)$; the $SU(3)$ — which acts on quark colors — comes with 8 gluon fields $\mathcal{G}^a_\mu$ which are responsible for the strong interactions; while the 3 gauge fields $\mathcal{W}^a_\mu$ of the $SU(1)$ and 1 gauge field $\mathcal{B}_\mu$ of the $U(1)$ are

responsible for the weak and the electromagnetic interactions. The three factors of the gauge group have rather different couplings,

$$\mathcal{L}_{\rm SM} \;=\; -\frac{1}{2g_3^2}\,\mathrm{tr}\big(\mathcal{G}_{\mu\nu}\mathcal{G}^{\mu\nu}\big) \;-\; \frac{1}{2g_2^2}\,\mathrm{tr}\big(\mathcal{W}_{\mu\nu}\mathcal{W}^{\mu\nu}\big) \;-\; \frac{1}{4g_1^2}\,\mathcal{B}_{\mu\nu}\mathcal{B}^{\mu\nu} \;+\; \mathcal{L}[\text{matter}], \qquad (119)$$

for

$$\frac{4\pi}{g_3^2} \;\approx\; 9.23, \quad \frac{4\pi}{g_2^2} \;\approx\; 29.97, \quad \frac{4\pi}{g_1^2} \;\approx\; 97.76. \qquad (120)$$

(Running couplings in the $\overline{\rm MS}$ renormalization scheme at $E = M_t = 173.3$ GeV.)

The matter multiplets of product gauge groups (117) are products of multiplets of the individual factors,

$$(m) \;=\; (m_1)\otimes(m_2)\otimes(m_3)\otimes\cdots, \quad (m_1)\text{ of }G_1, \quad (m_2)\text{ of }G_2, \quad (m_3)\text{ of }G_3,\ \ldots. \qquad (121)$$

For the abelian factors of G (if any), all multiplets are singlets but they may have different $U(1)$ charges (which we need to specify). For example, the fermionic fields of the Standard Model form 5 kinds of $SU(3)\times SU(2)\times U(1)$:

- Left-handed quarks form color triplets, $SU(2)$ doublets — (u,d), (c,s), and (t,b), — and have $U(1)$ *hypercharge* $y = +\frac{1}{6}$. Consequently, for these fields

$$D_\mu\psi_Q^{j,\alpha} \;=\; \partial_\mu\psi_Q^{j,\alpha} \;+\; \frac{ig_3}{2}\,G_\mu^a(\lambda^a)^j{}_k\psi_Q^{k,\alpha} \;+\; \frac{ig_2}{2}\,W_\mu^a(\tau^a)^\alpha{}_\beta\psi_Q^{j,\beta} \;+\; \frac{ig_1}{6}\,B_\mu\Psi_Q^{j,\alpha}. \qquad (122)$$

- Right-handed quarks of flavors u,c,t form color triplets but they are singlets of $SU(2)$ and have hypercharge $y = +\frac{2}{3}$, hence for these fields

$$D_\mu\psi_U^j \;=\; \partial_\mu\psi_U^j \;+\; \frac{ig_3}{2}\,G_\mu^a(\lambda^a)^j{}_k\psi_U^k \;+\; \frac{2ig_1}{3}\,B_\mu\Psi_Q^j\,. \qquad (123)$$

- Right-handed quarks of flavors d,s,b also form color triplets and $SU(2)$ singlets, but they have hypercharge $y = -\frac{1}{3}$, hence for these fields

$$D_\mu\psi_D^j \;=\; \partial_\mu\psi_D^j \;+\; \frac{ig_3}{2}\,G_\mu^a(\lambda^a)^j{}_k\psi_D^k \;-\; \frac{ig_1}{3}\,B_\mu\Psi_D^j\,. \qquad (124)$$

- Left-handed leptons are color-singlets but $SU(2)$ doublets (ν_e,e^-), (ν_μ,μ^-), (ν_τ,τ^-)

of hypercharge $y = -\frac{1}{2}$. Hence, for these fields

$$D_\mu \psi_L^\alpha \;=\; \partial_\mu \psi_L^\alpha \;+\; \frac{ig_2}{2}\, W_\mu^a (\tau^a)^\alpha{}_\beta \psi_L^\beta \;-\; \frac{ig_1}{2}\, B_\mu \Psi_L^\beta \,. \tag{125}$$

- Right-handed charged leptons e^-, μ^-, τ^- are singlets of both $SU(3)_{\text{color}}$ and $SU(2)$, and have hypercharge $y = -1$, hence

$$D_\mu \psi_E \;=\; \partial_\mu \psi_E \;-\; i g_1 B_\mu \psi_E \,. \tag{126}$$

◦ It is not known whether the right-handed neutrinos exist at all, but if they do exist, they are singlets of both $SU(3)$ and $SU(2)$ and have zero hypercharges. Thus, they do not couple to any gauge fields of the standard model and

$$D_\mu \psi_N \;=\; \partial_\mu \psi_N \;+\; 0. \tag{127}$$