Fluctuations in Superfluid Helium

Last supplementary lecture I wrote down the Landau–Ginzburg theory of superfluid helium and its classical field theory limit — a complex scalar field $\phi(\mathbf{x},t)$ with Hamiltonian

$$H \;=\; \int d^3\mathbf{x}\left(\frac{1}{2m}\,|\nabla\phi|^2 \;+\; \frac{\lambda}{2}\,|\phi|^4 \;-\; \mu|\phi|^2\right). \tag{1}$$

For positive chemical potential μ, the ground state of the superfluid has non-zero value of the scalar field, namely

$$|\phi|^2 \;=\; \bar{n}_s \;=\; \frac{\mu}{\lambda}. \tag{2}$$

The phase of ϕ is arbitrary, as long as it is the same at all $\mathbf{x}$ (in the non-moving superfluid), so without loss of generality we assume $\phi_{\text{ground}} = \sqrt{\bar{n}_s}$.

Let's study the small fluctuations around this ground state,

$$\phi(\mathbf{x},t) \;=\; \sqrt{\bar{n}_s} \;+\; \delta\phi(\mathbf{x},t). \tag{3}$$

In terms of the fluctuation $\delta\phi$, the Kinetic part of the Hamiltonian (1) becomes

$$\frac{1}{2m}\,|\nabla\phi|^2 \;=\; \frac{1}{2m}\,|\nabla\delta\phi|^2, \tag{4}$$

while for the potential part we have

$$\begin{aligned}
V(\phi^*,\phi) \;&=\; \frac{\lambda}{2}\,|\phi|^4 \;-\; \mu|\phi|^2 \;=\; \frac{\lambda}{2}\Big(\phi^*\phi \;-\; \bar{n}_s\Big)^2 \;+\; \text{constant}, &(5)\\
\phi^*\phi \;-\; \bar{n}_s \;&=\; \cancel{\bar{n}_s} \;+\; \sqrt{\bar{n}_s}\big(\delta\phi+\delta\phi^*\big) \;+\; |\delta\phi|^2 \;-\; \cancel{\bar{n}_s}\,, &(6)\\
V(\delta\phi^*,\delta\phi) \;&=\; \frac{\lambda\bar{n}_s}{2}\big(\delta\phi+\delta\phi^*\big)^2 \;+\; \lambda\sqrt{\bar{n}_s}\big(\delta\phi+\delta\phi^*\big)\times\delta\phi^*\delta\phi \;+\; \frac{\lambda}{2}\,|\delta\phi|^4.
\end{aligned}$$

Altogether,

$$H \;=\; H_{\text{free}} \;+\; H_{\text{int}} \;+\; \text{constant} \tag{7}$$

where

$$H_{\text{free}} \;=\; \int d^3\mathbf{x}\left(\frac{1}{2m}\,|\nabla\delta\phi|^2 \;+\; \frac{\lambda\bar{n}_s}{2}\big(\delta\phi+\delta\phi^*\big)^2\right) \tag{8}$$

while the H_{int} comprises the cubic and the quartic terms in the fluctuations $\delta\phi$ and $\delta\phi^*$.

In the quantum Landau–Ginzburg theory, the quantum field

$$\delta\psi^\dagger(\mathbf{x}) \;=\; \psi^\dagger(\mathbf{x}) \;-\; \sqrt{\bar n_s} \;=\; L^{-3/2}\sum_{\mathbf{k}\neq0} e^{i\mathbf{k}\mathbf{x}}\hat a^\dagger_{\mathbf{k}} \tag{9}$$

creates Helium atoms *outside the Bose–Einstein condensate* while the field

$$\delta\psi(\mathbf{x}) \;=\; \psi(\mathbf{x}) \;-\; \sqrt{\bar n_s} \;=\; L^{-3/2}\sum_{\mathbf{k}\neq0} e^{-i\mathbf{k}\mathbf{x}}\hat a_{\mathbf{k}} \tag{10}$$

annihilates such atoms, and the Hamiltonian operator follows from the classical Hamiltonian,

$$\hat H \;=\; \hat H_{\text{free}} \;+\; \text{constant} \;+\; \text{perturbations} \tag{11}$$

where

$$\begin{aligned} \hat H_{\text{free}} &= \int d^3\mathbf{x}\left(\frac{1}{2m}\nabla\delta\hat\psi^\dagger\cdot\nabla\delta\hat\psi \;+\; \lambda\bar n_s\big(\delta\hat\psi^\dagger\delta\hat\psi \;+\; \tfrac12\delta\hat\phi^2 \;+\; \tfrac12\delta\hat\psi^{\dagger2}\big)\right) \\ &= \sum_{\mathbf{k}\neq0}\left(\left(\frac{\mathbf{k}^2}{2m}+\lambda\bar n_s\right)\hat a^\dagger_{\mathbf{k}}\hat a_{\mathbf{k}} \;+\; \tfrac12\lambda\bar n_s\big(\hat a_{\mathbf{k}}\hat a_{-\mathbf{k}} + \hat a^\dagger_{\mathbf{k}}\hat a^\dagger_{-\mathbf{k}}\big)\right) \end{aligned} \tag{12}$$

To diagonalize this Hamiltonian — or more generally, any Hamiltonian of the form

$$\hat H \;=\; \sum_{\mathbf{k}}\Big(A_{\mathbf{k}}\hat a^\dagger_{\mathbf{k}}\hat a_{\mathbf{k}} \;+\; \tfrac12 B_{\mathbf{k}}\big(\hat a_{\mathbf{k}}\hat a_{-\mathbf{k}} + \hat a^\dagger_{\mathbf{k}}\hat a^\dagger_{-\mathbf{k}}\big)\Big), \tag{13}$$

(with real $A_{\mathbf{k}} = A_{-\mathbf{k}}$ and $B_{\mathbf{k}} = B_{-\mathbf{k}}$), we use the **Bogolyubov transform** *of the creation and annihilation operators:*

$$\begin{aligned} \hat b_{\mathbf{k}} &= \cosh(t_{\mathbf{k}})\times\hat a_{\mathbf{k}} \;+\; \sinh(t_{\mathbf{k}})\times\hat a^\dagger_{-\mathbf{k}}\,, \\ \hat b^\dagger_{\mathbf{k}} &= \cosh(t_{\mathbf{k}})\times\hat a^\dagger_{\mathbf{k}} \;+\; \sinh(t_{\mathbf{k}})\times\hat a_{-\mathbf{k}}\,. \end{aligned} \tag{14}$$

for some real parameters $t_{\mathbf{k}} = t_{-\mathbf{k}}$.

Lemma 1: For any real $t_{\mathbf{k}} = t_{-\mathbf{k}}$, the $\hat{b}_{\mathbf{k}}$ and $\hat{b}^\dagger_{\mathbf{k}}$ operators obey the same bosonic commutation relations as the $\hat{a}_{\mathbf{k}}$ and $\hat{a}^\dagger_{\mathbf{k}}$ operators,

$$\left[\hat{b}_{\mathbf{k}}, \hat{b}_{\mathbf{k}'}\right] = 0, \quad \left[\hat{b}^\dagger_{\mathbf{k}}, \hat{b}^\dagger_{\mathbf{k}'}\right] = 0, \quad \left[\hat{b}_{\mathbf{k}}, \hat{b}^\dagger_{\mathbf{k}'}\right] = \delta_{\mathbf{k},\mathbf{k}'} \,. \tag{15}$$

Lemma 2: For any Hamiltonian of the form (13) with real $A_{\mathbf{k}} = A_{-\mathbf{k}}$, real $B_{\mathbf{k}} = B_{-\mathbf{k}}$ and $|B_{\mathbf{k}}| < A_{\mathbf{k}}$, there is a Bogolyubov transform (14) with

$$t_{\mathbf{k}} = \tfrac{1}{2}\,\mathrm{artanh}\,\frac{B_{\mathbf{k}}}{A_{\mathbf{k}}}\,, \tag{16}$$

which leads to

$$\hat{H} = \sum_{\mathbf{k}} \omega(\mathbf{k}) \hat{b}^\dagger_{\mathbf{k}} \hat{b}_{\mathbf{k}} + \text{constant} \tag{17}$$

$$\text{for } \omega(\mathbf{k}) = \sqrt{A^2_{\mathbf{k}} - B^2_{\mathbf{k}}}\,. \tag{18}$$

Physically, the $\hat{b}^\dagger_{\mathbf{k}}$ operators create while the $\hat{b}_{\mathbf{k}}$ operators annihilate some kind of *quasi-particles* of energy $\omega(\mathbf{k})$, and the ground state of the Hamiltonian (17) is the *quasiparticle vacuum*, the state annihilated by all the $\hat{b}$ operators,

$$\forall\mathbf{k}, \quad \hat{b}_{\mathbf{k}}\,|\text{ground}\rangle = 0. \tag{19}$$

Lemma 3: In terms of the original $\hat{a}_{\mathbf{k}}$ and $\hat{a}^\dagger_{\mathbf{k}}$ operators and the state $|\text{BEC}\rangle$ annihilated by all the $\hat{a}_{\mathbf{k}}$ operators with $\mathbf{k} \neq 0$,

$$|\text{ground}\rangle = \exp\left(-\frac{1}{2}\sum_{\mathbf{k}} \tanh(t_{\mathbf{k}}) \hat{a}^\dagger_{\mathbf{k}} \hat{a}^\dagger_{-\mathbf{k}} - \text{const}\right) |\text{BEC}\rangle \tag{20}$$

where $|\text{BEC}\rangle$ is the state annihilated by all the $\hat{a}_{\mathbf{k}}$ operators for $\mathbf{k} \neq 0$. In the liquid helium context, $|\text{BEC}\rangle$ is the pure Bose–Einstein condensate state — all the atoms are in the $\mathbf{k} = 0$ mode so there are *no atoms in the other modes.* By comparison, the state (20) has a lot of atoms paired-up in $\pm\mathbf{k}$ modes, and indeed, the experiments with BEC condensates of ultra-cold atoms show more atoms in such $\pm\mathbf{k}$ pairs than the atoms in the $\mathbf{k} = 0$ condensate itself.

Lemma 4: The quasiparticle vacuum state (20) has zero net mechanical momentum, while the quasiparticles have definite momenta $\mathbf{k}$, thus

$$\hat{\mathbf{P}}_{\text{net}} = \sum_{\mathbf{k}} \mathbf{k}\hat{a}^\dagger_{\mathbf{k}}\hat{a}_{\mathbf{k}} = \sum_{\mathbf{k}} \mathbf{k}\hat{b}^\dagger_{\mathbf{k}}\hat{b}_{\mathbf{k}}\,. \tag{21}$$

I shall prove the Lemmas 1–4 later in these notes, but first let me apply them to the superfluid helium. In the Landau–Ginzburg theory, the Hamiltonian for the fluctuation fields — or rather the free part of that Hamiltonian — has form

$$\hat{H}_{\text{free}} = \sum_{\mathbf{k}\neq 0}\left(\left(\frac{\mathbf{k}^2}{2m} + \lambda\bar{n}_s\right)\hat{a}^\dagger_{\mathbf{k}}\hat{a}_{\mathbf{k}} + \tfrac{1}{2}\lambda\bar{n}_s\big(\hat{a}_{\mathbf{k}}\hat{a}_{-\mathbf{k}} + \hat{a}^\dagger_{\mathbf{k}}\hat{a}^\dagger_{-\mathbf{k}}\big)\right) \tag{22}$$

which is a special case of (13) with

$$A_{\mathbf{k}} = \frac{\mathbf{k}^2}{2m} + \lambda\bar{n}_s\,, \qquad B_{\mathbf{k}} = \lambda\bar{n}_s, \tag{23}$$

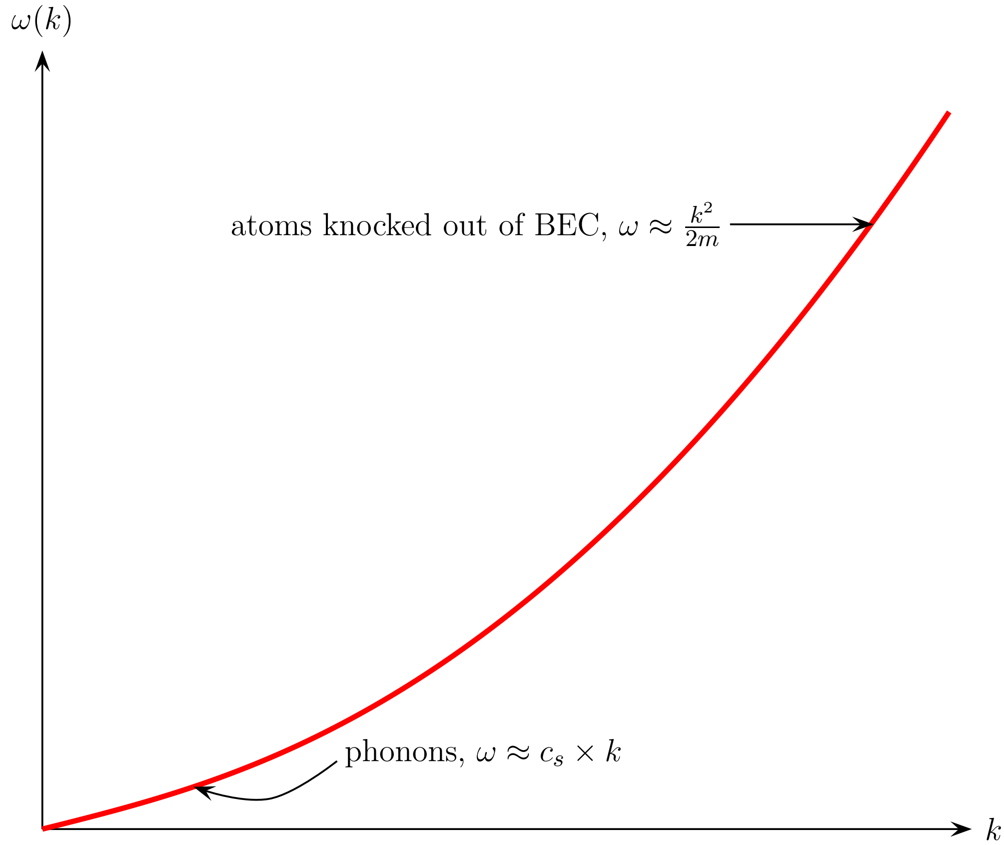
hence

$$t_{\mathbf{k}} = \frac{1}{2}\operatorname{artanh}\frac{2\lambda\bar{n}_s m}{2\lambda\bar{n}_2 m + k^2} \longrightarrow \begin{cases}\infty & \text{for small } k,\\ 0 & \text{for large } k.\end{cases} \tag{24}$$

while

$$\omega(\mathbf{k}) = \sqrt{\left(\frac{\mathbf{k}^2}{2m} + \lambda\bar{n}_s\right)^2 - (\lambda\bar{n}_s)^2} = k\times\sqrt{\frac{k^2}{4m^2} + \frac{\lambda\bar{n}_s}{m}}\,. \tag{25}$$

Graphically,



Lemma 5: Beyond the Landau–Ginzburg approximation, a finite-range two-body potential $V_2(\mathbf{x}-\mathbf{y})$ between helium atoms leads to the fluctuation Hamiltonian (or rather, its free part) of the form

$$\begin{aligned}\hat{H}_{\rm free} \;&=\; \frac{1}{2m}\int\!d^3\mathbf{x}\,\nabla\delta\hat{\phi}^\dagger\cdot\nabla\delta\hat{\psi}\\ &\qquad+\int\!d^3\mathbf{x}\int\!d^3\mathbf{y}\,V_2(\mathbf{x}-\mathbf{y})\times\Bigl(\hat{\psi}^\dagger(\mathbf{x})\hat{\psi}(\mathbf{y})\,+\,\tfrac12\hat{\psi}(\mathbf{x})\hat{\psi}(\mathbf{y})\,+\,\tfrac12\hat{\psi}^\dagger(\mathbf{x})\hat{\psi}^\dagger(\mathbf{y})\Bigr)\\ &=\sum_{\mathbf{k}\neq 0}\left(\left(\frac{\mathbf{k}^2}{2m}\,+\,\bar{n}_sW(\mathbf{k})\right)\hat{a}^\dagger_{\mathbf{k}}\hat{a}_{\mathbf{k}}\;+\;\tfrac12\bar{n}_sW(\mathbf{k})\bigl(\hat{a}_{\mathbf{k}}\hat{a}_{-\mathbf{k}}+\hat{a}^\dagger_{\mathbf{k}}\hat{a}^\dagger_{-\mathbf{k}}\bigr)\right)\end{aligned}\tag{26}$$

where $W(\mathbf{k})$ is the Fourier transform of the two-body potential,

$$W(\mathbf{k})\;=\;\int\!d^3\mathbf{x}\,e^{-i\mathbf{k}\mathbf{x}}V_2(\mathbf{x}).\tag{27}$$

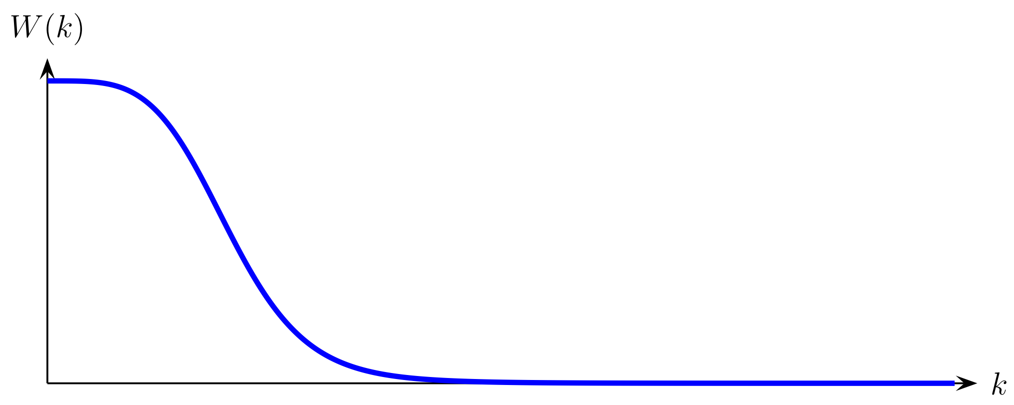
Consequently,

$$\hat{H}_{\text{free}} = \sum_{\mathbf{k}\neq 0} \omega(\mathbf{k}) \hat{b}^\dagger_{\mathbf{k}} \hat{b}_{\mathbf{k}} \tag{28}$$

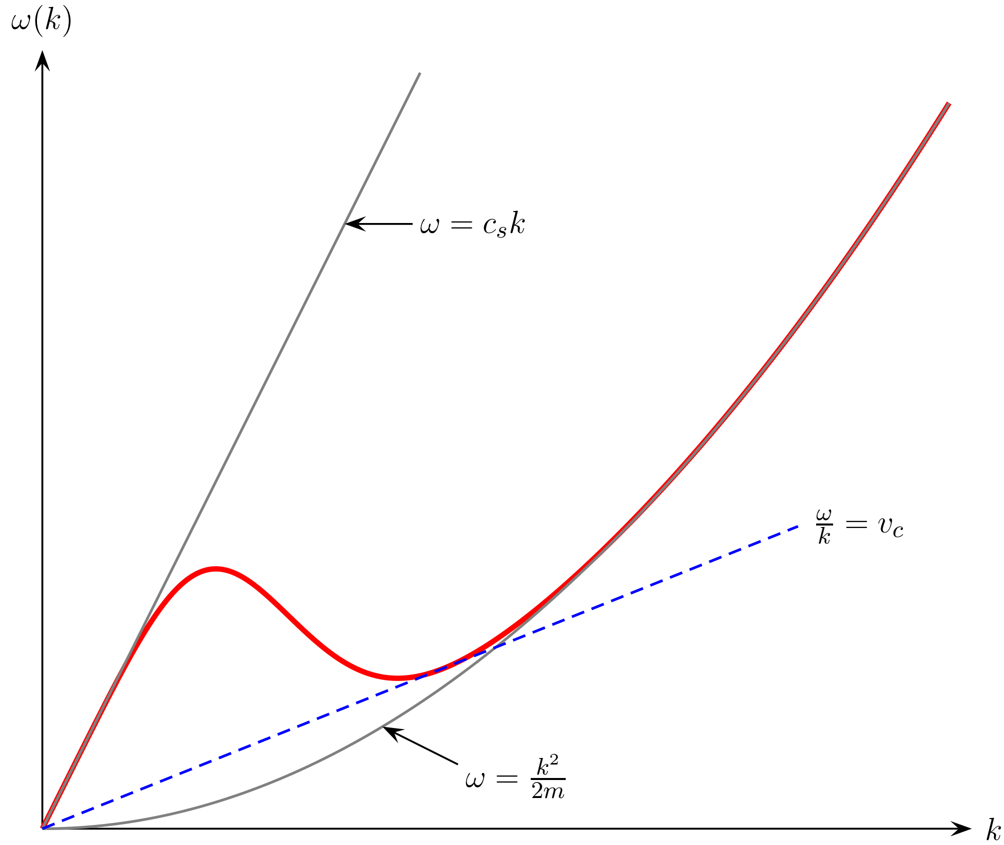
for quasiparticle energies

$$\omega(\mathbf{k}) = \sqrt{\left(\frac{\mathbf{k}^2}{2m} + \bar{n}_s W(\mathbf{k})\right)^2 - (\lambda \bar{n}_s)^2} = k \times \sqrt{\frac{k^2}{4m^2} + \frac{\bar{n}_s}{m} W(k)}\,. \tag{29}$$

For the helium atoms, the $W(\mathbf{k})$ drops off at large momenta,



hence the energy-momentum relation $\omega(k)$ for the quasiparticles — or equivalently, the wavenumber-frequency *dispersion relation* for the waves of small fluctuations — has a dip:



I shall explain the significance of this curve in the blackboard part of the supplementary lecture.

Proofs of the Lemmas

Lemma 1: the bosonic commutation relations (15) for the quasiparticle creation and annihilation operators. Starting from the bosonic commutation relations

$$[\hat{a}_{\mathbf{k}}, \hat{a}_{\mathbf{k}'}] \;=\; 0, \quad [\hat{a}^\dagger_{\mathbf{k}}, \hat{a}^\dagger_{\mathbf{k}'}] \;=\; 0, \quad [\hat{a}_{\mathbf{k}}, \hat{a}^\dagger_{\mathbf{k}'}] \;=\; \delta_{\mathbf{k},\mathbf{k}'} \tag{30}$$

for the operators creating and annihilating the helium atoms and treating eqs. (14) as the definitions of the $\hat{b}_{\mathbf{k}}$ and $\hat{b}^\dagger_{\mathbf{k}}$ operators, we immediately calculate

$$\begin{aligned}
[\hat{b}_{\mathbf{k}}, \hat{b}_{\mathbf{k}'}] \;&=\; \cosh(t_{\mathbf{k}})\sinh(t_{\mathbf{k}'}) \times \left([\hat{a}_{\mathbf{k}}, \hat{a}^\dagger_{-\mathbf{k}'}] = \delta_{\mathbf{k},-\mathbf{k}'}\right) \\
&\qquad +\; \sinh(t_{\mathbf{k}})\cosh(t_{\mathbf{k}'}) \times \left([\hat{a}^\dagger_{-\mathbf{k}}, \hat{a}_{\mathbf{k}'}] = -\delta_{-\mathbf{k},\mathbf{k}'}\right) \\
&=\; \delta_{\mathbf{k}',-\mathbf{k}} \times \Big(\cosh(t_{\mathbf{k}})\sinh(t_{\mathbf{k}'}) \;-\; \sinh(t_{\mathbf{k}})\cosh(t_{\mathbf{k}'}) \;=\; \sinh(t_{\mathbf{k}'} - t_{\mathbf{k}})\Big) \\
&=\; 0 \quad \text{because } t'_{\mathbf{k}} = t_{\mathbf{k}} \text{ when } \mathbf{k}' = -\mathbf{k}.
\end{aligned} \tag{31}$$

In the same way, $[\hat{b}^\dagger_{\mathbf{k}}, \hat{b}^\dagger_{\mathbf{k}'}] = 0$.

Finally,

$$
\begin{aligned}
[\hat{b}_{\mathbf{k}}, \hat{b}^\dagger_{\mathbf{k}'}] &= \cosh(t_{\mathbf{k}})\cosh(t_{\mathbf{k}'}) \times \big([\hat{a}_{\mathbf{k}}, \hat{a}^\dagger_{\mathbf{k}'}] = \delta_{\mathbf{k},\mathbf{k}'}\big) \\
&\quad + \sinh(t_{-\mathbf{k}})\sinh(t_{-\mathbf{k}'}) \times \big([\hat{a}^\dagger_{-\mathbf{k}}, \hat{a}_{-\mathbf{k}'}] = \delta_{-\mathbf{k},-\mathbf{k}'}\big) \\
&= \delta_{\mathbf{k},\mathbf{k}'} \times \Big(\cosh^2(t_{\mathbf{k}}) - \sinh^2(t_{-\mathbf{k}}) \ = \cosh^2(t_{\mathbf{k}}) - \sinh^2(t_{\mathbf{k}}) \ = \ 1\Big) \\
&= \delta_{\mathbf{k},\mathbf{k}'} \,.
\end{aligned}
\tag{32}
$$

Quod erat demonstrandum.

Lemma 2: bringing the Hamiltonian (13) to the form (17). Let's start by expressing the product $\hat{b}^\dagger_{\mathbf{k}}\hat{b}_{\mathbf{k}}$ in terms of the $\hat{a}^\dagger$ and $\hat{a}$ operators. Applying both definitions (14), we immediately obtain

$$
\begin{aligned}
\hat{b}^\dagger_{\mathbf{k}}\hat{b}_{\mathbf{k}} \ &= \ \cosh^2(t_{\mathbf{k}})\,\hat{a}^\dagger_{\mathbf{k}}\hat{a}_{\mathbf{k}} \ + \ \cosh(t_{\mathbf{k}})\sinh(t_{\mathbf{k}})\,(\hat{a}^\dagger_{\mathbf{k}}\hat{a}^\dagger_{-\mathbf{k}} \ + \ \hat{a}_{-\mathbf{k}}\hat{a}_{\mathbf{k}}) \\
&\quad + \ \sinh^2(t_{\mathbf{k}})\,(\hat{a}_{-\mathbf{k}}\hat{a}^\dagger_{-\mathbf{k}} \ = \ \hat{a}^\dagger_{-\mathbf{k}}\hat{a}_{-\mathbf{k}} + 1).
\end{aligned}
\tag{33}
$$

Likewise,

$$
\begin{aligned}
\hat{b}^\dagger_{-\mathbf{k}}\hat{b}_{-\mathbf{k}} \ &= \ \cosh^2(t_{-\mathbf{k}})\,\hat{a}^\dagger_{-\mathbf{k}}\hat{a}_{-\mathbf{k}} \ + \ \cosh(t_{-\mathbf{k}})\sinh(t_{\mathbf{k}})\,(\hat{a}^\dagger_{-\mathbf{k}}\hat{a}^\dagger_{\mathbf{k}} \ + \ \hat{a}_{\mathbf{k}}\hat{a}_{-\mathbf{k}}) \\
&\quad + \ \sinh^2(t_{-\mathbf{k}})\,(\hat{a}_{\mathbf{k}}\hat{a}^\dagger_{\mathbf{k}} \ = \ \hat{a}^\dagger_{\mathbf{k}}\hat{a}_{\mathbf{k}} + 1).
\end{aligned}
\tag{34}
$$

Assuming $t_{-\mathbf{k}} = t_{\mathbf{k}}$, we may combine

$$
\begin{aligned}
\hat{b}^\dagger_{\mathbf{k}}\hat{b}_{\mathbf{k}} \ + \ \hat{b}^\dagger_{-\mathbf{k}}\hat{b}_{-\mathbf{k}} \ &= \ \Big(\cosh^2(t_{\mathbf{k}}) + \sinh^2(t_{\mathbf{k}}) = \cosh(2t_{\mathbf{k}})\Big) \times (\hat{a}^\dagger_{\mathbf{k}}\hat{a}_{\mathbf{k}} + \hat{a}^\dagger_{-\mathbf{k}}\hat{a}_{-\mathbf{k}}) \\
&\quad + \ \Big(2\cosh(t_{\mathbf{k}})\sinh(t_{\mathbf{k}}) = \sinh(2t_{\mathbf{k}})\Big) \times (\hat{a}^\dagger_{\mathbf{k}}\hat{a}^\dagger_{-\mathbf{k}} \ + \ \hat{a}_{-\mathbf{k}}\hat{a}_{\mathbf{k}}) \ + \ \text{const.}
\end{aligned}
\tag{35}
$$

Now let's plug this formula into a Hamiltonian of the form (17) for some $\omega_{\mathbf{k}}$ and require that the result matches the original Hamiltonian (13). Assuming $\omega_{-\mathbf{k}} \equiv \omega_{\mathbf{k}}$, we obtain

$$
\begin{aligned}
\hat{H} \ &= \ \sum_{\mathbf{k}} \omega_{\mathbf{k}}\hat{b}^\dagger_{\mathbf{k}}\hat{b}_{\mathbf{k}} \ = \ \frac{1}{2}\sum_{\mathbf{k}} \omega_{\mathbf{k}}(\hat{b}^\dagger_{\mathbf{k}}\hat{b}_{\mathbf{k}} + \hat{b}^\dagger_{-\mathbf{k}}\hat{b}_{-\mathbf{k}}) \\
&= \sum_{\mathbf{k}} \omega_{\mathbf{k}}\cosh(2t_{\mathbf{k}})\hat{a}^\dagger_{\mathbf{k}}\hat{a}_{\mathbf{k}} \ + \ \tfrac{1}{2}\sum_{\mathbf{k}} \omega_{\mathbf{k}}\sinh(2t_{\mathbf{k}})\,\Big(\hat{a}^\dagger_{\mathbf{k}}\hat{a}^\dagger_{-\mathbf{k}} \ + \ \hat{a}_{-\mathbf{k}}\hat{a}_{\mathbf{k}}\Big) \ + \ \text{const.}
\end{aligned}
\tag{36}
$$

This formula must match (up to a constant) the original Hamiltonian (13), so we need to choose the parameters $\omega_{\mathbf{k}} = \omega_{-\mathbf{k}}$ and $t_{\mathbf{k}} = t_{-\mathbf{k}}$ such that

$$\omega_{\mathbf{k}} \cosh(2t_{\mathbf{k}}) \;=\; A_{\mathbf{k}} \quad \text{and} \quad \omega_{\mathbf{k}} \sinh(2t_{\mathbf{k}}) \;=\; B_{\mathbf{k}} \,. \tag{37}$$

These equations are easy to solve, and the solution exists as long as $A_{\mathbf{k}} = A_{-\mathbf{k}}$, $B_{\mathbf{k}} = B_{-\mathbf{k}}$, and $A_{\mathbf{k}} > |B_{\mathbf{k}}|$, namely

$$t_{\mathbf{k}} \;=\; \frac{1}{2}\,\mathrm{artanh}\,\frac{B_{\mathbf{k}}}{A_{\mathbf{k}}} \quad \text{and} \quad \omega_{\mathbf{k}} \;=\; \sqrt{A_{\mathbf{k}}^2 - B_{\mathbf{k}}^2}\,. \tag{38}$$

Quod erat demonstrandum.

Lemma 3: the ground state annihilated by all the $\hat{b}_{\mathbf{k}}$ operators. Let's focus on a single pair of opposite momenta $\pm\mathbf{k}$. In other words, consider the Hilbert space of two harmonic oscillators for the operators $\hat{a}_{\mathbf{k}}, \hat{a}^\dagger_{\mathbf{k}}, \hat{a}_{-\mathbf{k}}, \hat{a}^\dagger_{-\mathbf{k}}$. In that Hilbert space, consider the state

$$|\Psi\rangle \;=\; C \times \exp(-\tau \hat{a}^\dagger_{\mathbf{k}} \hat{a}^\dagger_{-\mathbf{k}})\,|0,0\rangle \tag{39}$$

where $\tau = \tanh(t_k)$ while $C = 1/\cosh(t_k)$ is the normalization coefficient. Expanding the exponent, we have

$$\begin{aligned} |\Psi\rangle \;&=\; C \times \sum_{n=0}^{\infty} \frac{(-\tau)^n}{n!}\,(\hat{a}^\dagger_{\mathbf{k}})^n (\hat{a}^\dagger_{-\mathbf{k}})^n\,|0,0\rangle \\ &=\; C \times \sum_n \frac{(-\tau)^n}{n!}\,(\sqrt{n!})^2\,|n,n\rangle \\ &=\; C \times \sum_n (-\tau)^n\,|n,n\rangle\,. \end{aligned} \tag{40}$$

Note normalization:

$$\langle\Psi|\Psi\rangle \;=\; C^2 \times \sum_n (-\tau)^{2n} \;=\; \frac{C^2}{1-\tau^2} \;=\; \frac{\bigl(1/\cosh(t_k)\bigr)^2}{1-\tanh^2(t_k)} \;=\; 1. \tag{41}$$

Let's act on the state $|\Psi\rangle$ with the quasiparticle annihilation operators $\hat{b}_{\mathbf{k}}$ and $\hat{b}_{-\mathbf{k}}$:

$$
\begin{aligned}
\hat{b}_{\mathbf{k}}\,|\Psi\rangle \;&=\; \cosh(t_k)\times\hat{a}_{\mathbf{k}}\,|\Psi\rangle \;+\; \sinh(t_k)\hat{a}^\dagger_{-\mathbf{k}}\,|\Psi\rangle \\
&=\; C\cosh(t_k)\times\sum_{n=1}^{\infty}(-\tau)^n\times\sqrt{n}\,|n-1,n\rangle \\
&\qquad +\; C\sinh(t_k)\times\sum_{n=0}^{\infty}(-\tau)^n\times\sqrt{n+1}\,|n,n+1\rangle \\
&\qquad \langle\!\langle\,\text{changing } n\to n+1 \text{ in the first sum only}\,\rangle\!\rangle \\
&=\; C\cosh(t_k)\times\sum_{n=0}^{\infty}(-\tau)^{n+1}\times\sqrt{n+1}\,|n,n+1\rangle \\
&\qquad +\; C\sinh(t_k)\times\sum_{n=0}^{\infty}(-\tau)^n\times\sqrt{n+1}\,|n,n+1\rangle \\
&\qquad \langle\!\langle\,\text{noting similarity of the two sums except for the power of } (-\tau)\,\rangle\!\rangle \\
&=\; C\Big(\cosh(t_k)\times(-\tau) \;+\; \sinh(t_k)\Big)\times\sum_{n=0}^{\infty}(-\tau)^n\times\sqrt{n+1}\,|n,n+1\rangle \\
&\qquad \langle\!\langle\,\text{plugging in } \tau=\tanh(t_k)\text{, hence } \cosh(t_k)\times(-\tau) \;+\; \sinh(t_k) \;=\; 0\,\rangle\!\rangle \\
&=\; C\times 0\times\sum\cdots \;=\; 0.
\end{aligned}
\tag{42}
$$

In exactly the same way we also have $\hat{b}_{-\mathbf{k}}\,|\Psi\rangle = 0$.

To generalize the construction of the $|\Psi\rangle$ state to the entire Fock space, we simply take a tensor product over all the distinct $(+\mathbf{k},-\mathbf{k})$ pairs. Thus, the state

$$
|\text{ground}\rangle \;=\; \bigotimes_{(+\mathbf{k},-\mathbf{k})\,\text{pairs}} |\Psi\rangle_{\mathbf{k}}
\tag{43}
$$

should be annihilated by all the $\hat{b}_{\mathbf{k}}$ operators, $\hat{b}_{\mathbf{k}}\,|\text{ground}\rangle = 0$.

To rewrite this state in the specific form (20), we simply note that

$$
\begin{aligned}
|\text{ground}\rangle \;&=\; \bigotimes_{(+\mathbf{k},-\mathbf{k})\,\text{pairs}} |\Psi\rangle_{\mathbf{k}} \;=\; \left(\prod_{(+\mathbf{k},-\mathbf{k})\,\text{pairs}} \frac{1}{\cosh(t_{\mathbf{k}})} \exp\bigl(-\tanh(t_{\mathbf{k}})\hat{a}^\dagger_{\mathbf{k}}\hat{a}^\dagger_{-\mathbf{k}}\bigr)\right)|\text{BEC}\rangle \\
&=\; C_{\text{net}} \times \exp\left(-\sum_{(+\mathbf{k},-\mathbf{k})\,\text{pairs}} \tanh(\mathbf{k})\hat{a}^\dagger_{\mathbf{k}}\hat{a}^\dagger_{-\mathbf{k}}\right)|\text{BEC}\rangle\,, \\
&=\; C_{\text{net}} \times \exp\left(-\frac{1}{2}\sum_{\mathbf{k}} \tanh(\mathbf{k})\hat{a}^\dagger_{\mathbf{k}}\hat{a}^\dagger_{-\mathbf{k}}\right)|\text{BEC}\rangle \\
&=\; \exp\left(-\frac{1}{2}\sum_{\mathbf{k}} \tanh(\mathbf{k})\hat{a}^\dagger_{\mathbf{k}}\hat{a}^\dagger_{-\mathbf{k}}\ \text{constant}\right)|\text{BEC}\rangle\,,
\end{aligned}
\tag{44}
$$

quod erat demonstrandum.

If you want the specific normalization constant inside the exponential, note the C_{net} factor on the second and third lines of eq. (44),

$$
C_{\text{net}} \;=\; \prod_{(+\mathbf{k},-\mathbf{k})\,\text{pairs}} \frac{1}{\cosh(t_k)} \;=\; \exp\left(-\frac{1}{2}\sum_{\mathbf{k}} \log(\cosh(t_k))\right), \tag{45}
$$

hence

$$
|\text{ground}\rangle \;=\; \exp\left(-\frac{1}{2}\sum_{\mathbf{k}}\Bigl(\tanh(\mathbf{k})\hat{a}^\dagger_{\mathbf{k}}\hat{a}^\dagger_{-\mathbf{k}} \,+\, \log(\cosh(t_k))\Bigr)\right)|\text{BEC}\rangle\,. \tag{46}
$$

Lemma 4: the net momentum operator

$$
\hat{\mathbf{P}}_{\text{net}} \;=\; \sum_{\mathbf{k}} \mathbf{k}\hat{a}^\dagger_{\mathbf{k}}\hat{a}_{\mathbf{k}}\,. \tag{47}
$$

Using eqs. (33) and (34) from the proof of Lemma 2 and $t_{-\mathbf{k}} = t_{\mathbf{k}}$, we immediately see that

$$
\hat{b}^\dagger_{\mathbf{k}}\hat{b}_{\mathbf{k}} \,-\, \hat{b}^\dagger_{-\mathbf{k}}\hat{b}_{-\mathbf{k}} \;=\; \Bigl(\cosh^2(t_{\mathbf{k}}) - \sinh^2(t_{\mathbf{k}}) = 1\Bigr) \times (\hat{a}^\dagger_{\mathbf{k}}\hat{a}_{\mathbf{k}} \,-\, \hat{a}^\dagger_{-\mathbf{k}}\hat{a}_{-\mathbf{k}}). \tag{48}
$$

Consequently, for the momentum operator (47) we have

$$\begin{aligned}
\hat{\mathbf{P}} &= \sum_{\mathbf{k}} \mathbf{k} \times \hat{a}^\dagger_{\mathbf{k}} \hat{a}_{\mathbf{k}} \;=\; \sum_{\mathbf{k}} (-\mathbf{k}) \times \hat{a}^\dagger_{-\mathbf{k}} \hat{a}_{-\mathbf{k}} \\
&= \frac{1}{2} \sum_{\mathbf{k}} \mathbf{k} \times (\hat{a}^\dagger_{\mathbf{k}} \hat{a}_{\mathbf{k}} \,-\, \hat{a}^\dagger_{-\mathbf{k}} \hat{a}_{-\mathbf{k}}) \\
&= \frac{1}{2} \sum_{\mathbf{k}} \mathbf{k} \times (\hat{b}^\dagger_{\mathbf{k}} \hat{b}_{\mathbf{k}} \,-\, \hat{b}^\dagger_{-\mathbf{k}} \hat{b}_{-\mathbf{k}}) \\
&= \sum_{\mathbf{k}} \mathbf{k} \times \hat{b}^\dagger_{\mathbf{k}} \hat{b}_{\mathbf{k}}\,.
\end{aligned} \tag{49}$$

Quod erat demonstrandum.

Lemma 5: the finite-range potential $V_2(\mathbf{x}-\mathbf{y})$ for the helium atoms. Consider the net potential operator

$$\hat{V} \;=\; \frac{1}{2} \int\! d^3\mathbf{x} \int\! d^3\mathbf{y}\, V_2(\mathbf{x}-\mathbf{y}) \times \hat{\psi}^\dagger(\mathbf{x}) \hat{\psi}^\dagger(\mathbf{y}) \hat{\psi}(\mathbf{y}) \hat{\psi}(\mathbf{x}). \tag{50}$$

In terms of the shifted fields $\delta\hat{\psi}(\mathbf{x}) = \hat{\psi}(\mathbf{x}) - \sqrt{\bar{n}_s}$ and $\delta\hat{\psi}^\dagger(\mathbf{x}) = \hat{\psi}^\dagger(\mathbf{x}) - \sqrt{\bar{n}_s}$, we have

$$\begin{aligned}
\hat{\psi}^\dagger(\mathbf{x}) \hat{\psi}^\dagger(\mathbf{y}) \hat{\psi}(\mathbf{y}) \hat{\psi}(\mathbf{x}) \;=\; & \bar{n}_s^2 \,+\, \bar{n}_s^{3/2} \Big(\delta\hat{\psi}^\dagger(\mathbf{x}) \,+\, \delta\hat{\psi}^\dagger(\mathbf{y}) \,+\, \delta\hat{\psi}(\mathbf{x}) \,+\, \delta\hat{\psi}(\mathbf{y}) \Big) \\
& +\, \bar{n}_s \Big(\delta\hat{\psi}^\dagger(\mathbf{x}) \delta\hat{\psi}(\mathbf{x}) \,+\, \delta\hat{\psi}^\dagger(\mathbf{y}) \delta\hat{\psi}(\mathbf{y}) \Big) \\
& +\, \bar{n}_s \Big(\delta\hat{\psi}^\dagger(\mathbf{x}) \delta\hat{\psi}(\mathbf{y}) \,+\, \delta\hat{\psi}^\dagger(\mathbf{y}) \delta\hat{\psi}(\mathbf{x}) \Big) \\
& +\, \bar{n}_s \Big(\delta\hat{\psi}^\dagger(\mathbf{x}) \delta\hat{\psi}^\dagger(\mathbf{y}) \,+\, \delta\hat{\psi}(\mathbf{y}) \delta\hat{\psi}(\mathbf{x}) \Big) \\
& +\, \text{cubic} \,+\, \text{quartic}.
\end{aligned} \tag{51}$$

The terms on the first two lines here depend only on the $\mathbf{x}$ or only on the $\mathbf{y}$, so when we plug them into the potential operator (50), we may immediately integrate over the other space position to obtain

$$[\text{@any fixed } \mathbf{y}] \int\! d^3\mathbf{x}\, V_2(\mathbf{x}-\mathbf{y}) \;=\; [\text{@any fixed } \mathbf{x}] \int\! d^3\mathbf{y}\, V_2(\mathbf{x}-\mathbf{y}) \;=\; W(0). \tag{52}$$

Consequently, integrating over the expansion (51) in the context of the potential (50) and making use of the $\mathbf{x} \leftrightarrow \mathbf{y}$ symmetry, we obtain

$$\begin{aligned}\hat{V} \;&=\; \bar{n}_s \times W(0) \times \int\!d^3\mathbf{x}\Big(\tfrac{1}{2}\bar{n}_s \;+\; \sqrt{\bar{n}_s}\big(\delta\hat{\psi}^\dagger(\mathbf{x}) + \delta\hat{\psi}(\mathbf{x})\big) \;+\; \delta\hat{\psi}^\dagger(\mathbf{x})\delta\hat{\psi}(\mathbf{x})\Big)\\ &\quad+\; \frac{\bar{n}_2}{2} \times \int\!d^3\mathbf{x}\int\!d^3\mathbf{y}\,V_2(\mathbf{x}-\mathbf{y}) \times \Big(2\delta\hat{\psi}^\dagger(\mathbf{x})\delta\hat{\psi}(\mathbf{y}) \;+\; \delta\hat{\psi}^\dagger(\mathbf{x})\delta\hat{\psi}^\dagger(\mathbf{y}) \;+\; \delta\hat{\psi}(\mathbf{y})\delta\hat{\psi}(\mathbf{x})\Big)\\ &\quad+\; \text{cubic} \;+\; \text{quartic}.\end{aligned} \tag{53}$$

Now consider the other non-derivative term in the Helium Hamiltonian

$$\hat{H}_{\text{net}} \;=\; \hat{K} \;+\; \hat{V} \;-\; \mu\hat{N}, \tag{54}$$

namely the chemical potential term,

$$\begin{aligned}-\mu\hat{N} \;&=\; -\mu\int\!d^3\mathbf{x}\,\hat{\psi}^\dagger(\mathbf{x})\hat{\psi}(\mathbf{x})\\ &=\; -\mu\int\!d^3\mathbf{x}\Big(\bar{n}_s \;+\; \sqrt{\bar{n}_s}\big(\delta\hat{\psi}(\mathbf{x}) + \delta\hat{\psi}^\dagger(\mathbf{x})\big) \;+\; \delta\hat{\psi}^\dagger(\mathbf{x})\delta\hat{\psi}(\mathbf{x})\Big)\end{aligned} \tag{55}$$

If we generalize the $\mu = \lambda\bar{n}_s$ formula of the Landau–Ginzburg theory to the

$$\mu \;=\; W(0) \times \bar{n}_s\,, \tag{56}$$

then the chemical potential term (55) cancels the top line of the two-body potential (53) (except for the constant part), hence

$$\begin{aligned}\hat{V} \;-\; \mu\hat{N} \;=\; &\frac{\bar{n}_2}{2}\int\!d^3\mathbf{x}\int\!d^3\mathbf{y}\,V_2(\mathbf{x}-\mathbf{y}) \times \Big(2\delta\hat{\psi}^\dagger(\mathbf{x})\delta\hat{\psi}(\mathbf{y}) \;+\; \delta\hat{\psi}^\dagger(\mathbf{x})\delta\hat{\psi}^\dagger(\mathbf{y}) \;+\; \delta\hat{\psi}(\mathbf{y})\delta\hat{\psi}(\mathbf{x})\Big)\\ &+\; \text{constant} \;+\text{cubic} \;+\; \text{quartic}.\end{aligned} \tag{57}$$

Thus altogether,

$$\hat{H} \;=\; \text{constant} \;+\; \hat{H}_{\text{free}} \;+\; \hat{H}_{\text{interactions}} \tag{58}$$

where

$$\begin{aligned}\hat{H}_{\text{free}} \;&=\; \frac{1}{2m}\int\!d^3\mathbf{x}\,\nabla\delta\hat{\psi}^\dagger(\mathbf{x})\cdot\nabla\delta\hat{\psi}(\mathbf{x})\\ &\quad+\; \frac{\bar{n}_2}{2}\int\!d^3\mathbf{x}\int\!d^3\mathbf{y}\,V_2(\mathbf{x}-\mathbf{y}) \times \Big(2\delta\hat{\psi}^\dagger(\mathbf{x})\delta\hat{\psi}(\mathbf{y}) \;+\; \delta\hat{\psi}^\dagger(\mathbf{x})\delta\hat{\psi}^\dagger(\mathbf{y}) \;+\; \delta\hat{\psi}(\mathbf{y})\delta\hat{\psi}(\mathbf{x})\Big).\end{aligned} \tag{59}$$

This completes the proof of the first part of the Lemma 5 — the top two lines of the eq. (26).

To prove the second part of the Lemma (the bottom line of eq. (26)) we simply Fourier transform from the shifted creation and annihilation fields to the creation and annihilation operators for modes $\mathbf{k} \neq 0$,

$$\delta\hat{\psi}^\dagger(\mathbf{x}) \;=\; L^{-3/2}\sum_{\mathbf{k}\neq 0} e^{+i\mathbf{kx}}\hat{a}^\dagger_{\mathbf{k}}\,, \qquad \delta\hat{\psi}(\mathbf{x}) \;=\; L^{-3/2}\sum_{\mathbf{k}\neq 0} e^{-i\mathbf{kx}}\hat{a}_{\mathbf{k}}\,. \tag{60}$$

Consequently,

$$\begin{aligned}\int\! d^3\mathbf{x}\int\! d^3\mathbf{y}\, V_2(\mathbf{x}-\mathbf{y}) \times \delta\hat{\psi}^\dagger(\mathbf{x})\delta\hat{\psi}(\mathbf{y}) \;=&\\ =\int\! d^3\mathbf{x}\int\! d^3\mathbf{y}\, V_2(\mathbf{x}-\mathbf{y}) \times L^{-3}\sum_{\mathbf{k},\mathbf{k}'} e^{i\mathbf{kx}-i\mathbf{k}'\mathbf{y}} \times \hat{a}^\dagger_{\mathbf{k}}\hat{a}_{\mathbf{k}'}&\\ =\sum_{\mathbf{k},\mathbf{k}'}\hat{a}^\dagger_{\mathbf{k}}\hat{a}_{\mathbf{k}'} \times L^{-3}\int\! d^3\mathbf{x}\int\! d^3\mathbf{y}\, V_2(\mathbf{x}-\mathbf{y}) \times e^{i\mathbf{kx}-i\mathbf{k}'\mathbf{y}}&\end{aligned} \tag{61}$$

where

$$\begin{aligned}L^{-3}\int\! d^3\mathbf{x}\int\! d^3\mathbf{y}\, V_2(\mathbf{x}-\mathbf{y}) \times e^{i\mathbf{kx}-i\mathbf{k}'\mathbf{y}} \;=&\\ =\; L^{-3}\int\! d^3\mathbf{y}\int\! d^3(\mathbf{z}=\mathbf{x}-\mathbf{y})\, V_2(\mathbf{z}) \times e^{i\mathbf{k}(\mathbf{y}+\mathbf{z})-i\mathbf{k}'\mathbf{y}}&\\ =\int\! d^3\mathbf{z}\, V_2(\mathbf{z})e^{i\mathbf{kz}} \times L^{-3}\int_{\text{box}}\! d^3\mathbf{y}\, e^{i\mathbf{ky}-i\mathbf{k}'\mathbf{y}}&\\ =\; W(\mathbf{k}) \times \delta_{\mathbf{k},\mathbf{k}'}\,,&\end{aligned} \tag{62}$$

hence

$$\int\! d^3\mathbf{x}\int\! d^3\mathbf{y}\, V_2(\mathbf{x}-\mathbf{y}) \times \delta\hat{\psi}^\dagger(\mathbf{x})\delta\hat{\psi}(\mathbf{y}) \;=\; \sum_{\mathbf{k}} W(\mathbf{k}) \times \hat{a}^\dagger_{\mathbf{k}}\hat{a}_{\mathbf{k}}\,. \tag{63}$$

In the same way we obtain

$$\begin{aligned}\int\! d^3\mathbf{x}\int\! d^3\mathbf{y}\, V_2(\mathbf{x}-\mathbf{y}) \times \delta\hat{\psi}^\dagger(\mathbf{x})\delta\hat{\psi}^\dagger(\mathbf{y}) \;=&\\ =\int\! d^3\mathbf{x}\int\! d^3\mathbf{y}\, V_2(\mathbf{x}-\mathbf{y}) \times L^{-3}\sum_{\mathbf{k},\mathbf{k}'} e^{i\mathbf{kx}-i\mathbf{k}'\mathbf{y}} \times \hat{a}^\dagger_{\mathbf{k}}\hat{a}^\dagger_{-\mathbf{k}}&\\ =\sum_{\mathbf{k}} W(\mathbf{k}) \times \hat{a}^\dagger_{\mathbf{k}}\hat{a}^\dagger_{-\mathbf{k}}&\end{aligned} \tag{64}$$

and likewise

$$\int d^3\mathbf{x} \int d^3\mathbf{y}\, V_2(\mathbf{x}-\mathbf{y}) \times \delta\hat{\psi}(\mathbf{x})\delta\hat{\psi}(\mathbf{y}) \;=\; \sum_{\mathbf{k}} W(\mathbf{k}) \times \hat{a}_{-\mathbf{k}}\hat{a}_{\mathbf{k}}\,. \tag{65}$$

Combining all these formulae with the gradient term in the Hamiltonian (59),

$$\hat{K} \;=\; \frac{1}{2m}\int d^3\mathbf{x}\, \nabla\psi^\dagger \cdot \nabla\psi \;=\; \frac{1}{2m}\int d^3\mathbf{x}\, \nabla\delta\psi^\dagger \cdot \nabla\delta\psi \;=\; \sum_{\mathbf{k}} \frac{\mathbf{k}^2}{2m}\, \hat{a}^\dagger_{\mathbf{k}}\hat{a}_{\mathbf{k}}\,, \tag{66}$$

we finally assemble all quadratic terms to

$$\hat{H}_{\text{free}} \;=\; \sum_{\mathbf{k}} \left(\left(\frac{\mathbf{k}^2}{2m} \,+\, W(\mathbf{k})\bar{n}_s \right) \hat{a}^\dagger_{\mathbf{k}}\hat{a}_{\mathbf{k}} \,+\, \tfrac{1}{2}W(\mathbf{k})\bar{n}_s \big(\hat{a}^\dagger_{\mathbf{k}}\hat{a}^\dagger_{-\mathbf{k}} \,+\, \hat{a}_{-\mathbf{k}}\hat{a}_{\mathbf{k}}\big) \right). \tag{67}$$

Quod erat demonstrandum.