1. Verify the integrals used by the Feynman's parameter trick and its generalizations:

$$\frac{1}{AB} = \int_0^1 \frac{d\xi}{[\xi A + (1-\xi)B]^2}\,, \tag{F.a}$$

$$\frac{1}{A^n B} = \int_0^1 \frac{n\xi^{n-1} d\xi}{[\xi A + (1-\xi)B]^{n+1}}\,, \tag{F.b}$$

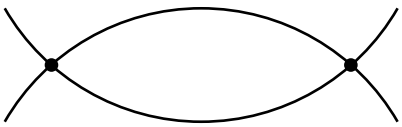
$$\frac{1}{A^n B^m} = \frac{(n+m-1)!}{(n-1)!(m-1)!} \times \int_0^1 \frac{\xi^{n-1}(1-\xi)^{m-1} d\xi}{[\xi A + (1-\xi)B]^{n+m}}\,, \tag{F.c}$$

$$\begin{aligned}\frac{1}{ABC} &= \int_0^1 d\xi \int_0^{1-\xi} \frac{2d\eta}{[\xi A + \eta B + (1-\xi-\eta)C]^3} \\ &\equiv \iiint\limits_{\xi,\eta,\zeta\geq 0} d\xi\, d\eta\, d\zeta\, \delta(\xi+\eta+\zeta-1) \times \frac{2}{[\xi A + \eta B + (1-\xi-\eta)C]^3}\,,\end{aligned} \tag{F.d}$$

$$\frac{1}{A_1 A_2 \cdots A_k} = \int\limits_{\xi_1,\ldots,\xi_k\geq 0} \cdots \int d^k\xi\, \delta(\xi_1 + \cdots + \xi_k - 1) \times \frac{(k-1)!}{[\xi_1 A_1 + \cdots + \xi_k A_k]^k}\,, \tag{F.e}$$

$$\begin{aligned}\frac{1}{A_1^{n_1} A_2^{n_2} \cdots A_k^{n_k}} &= \frac{(n_1 + \cdots + n_k - 1)!}{(n_1-1)! \cdots (n_k-1)!} \times \\ &\times \int\limits_{\xi_1,\ldots,\xi_k\geq 0} \cdots \int d^k\xi\, \delta(\xi_1 + \cdots + \xi_k - 1) \times \frac{\xi_1^{n_1-1} \cdots \xi_k^{n_k-1}}{[\xi_1 A_1 + \cdots + \xi_k A_k]^{(n_1+\cdots+n_k)}}\,.\end{aligned} \tag{F.f}$$

2. In class, we have evaluated the one-loop diagram



(1)

using the *hard-edge cutoff* as an ultraviolet regulator. Your task is to evaluate the same diagram using two other UV regulators: (1) Pauli–Villars, and (2) higher derivatives.

Show that all 3 regulators yield similar amplitudes of the form

$$\mathcal{M}(\text{diagram (1)}) \;=\; \frac{\lambda_{\text{bare}}^2}{32\pi^2} \times \left(\log \frac{\Lambda^2}{m^2} \;+\; C \;-\; J(t/m^2) \;+\; \text{negligible} \right) \tag{2}$$

where

$$J(t/m^2) \;= \int_0^1 d\xi \, \log \frac{m^2 - t \times \xi(1-\xi)}{m^2} \,, \tag{3}$$

'negligible' stands for terms that vanish as negative powers of the cutoff scale Λ for $\Lambda \to \infty$, and C is an $O(1)$ numeric constant that depends on the particular UV regulator:

$$C_{\text{hard edge}} \;\neq\; C_{\text{Paili Villars}} \;\neq\; C_{\text{higher derivative}} \,. \tag{4}$$

Fortunately, this regulator dependence can be canceled by adjusting the cutoff scale parameter Λ for each regulator: Let

$$\Lambda_{\text{HE}}^2 \times e^{C_{\text{HE}}} \;=\; \Lambda_{\text{PV}}^2 \times e^{C_{\text{PV}}} \;=\; \Lambda_{\text{HD}}^2 \times e^{C_{\text{HD}}}, \tag{5}$$

then all 3 regulators would yield exactly the same loop amplitude (2).

Note: the dimensional regularization also yields exactly the same amplitude (2), provided we identify the UV cutoff scale as

$$\Lambda_{\text{DR}}^2 \;=\; \mu^2 \times \exp\left(\frac{1}{\epsilon} = \frac{2}{4-D} \right) \tag{6}$$

and then set

$$\Lambda_{\text{DR}}^2 \times e^{C_{\text{DR}}} \;=\; \Lambda_{\text{HE}}^2 \times e^{C_{\text{HE}}} \;=\; \Lambda_{\text{PV}}^2 \times e^{C_{\text{PV}}} \;=\; \Lambda_{\text{HE}}^2 \times e^{C_{\text{HD}}} \tag{7}$$

for a suitable $O(1)$ numeric constant C_{DR}.

Hint: for the higher-derivative regulator, approximate the modified propagator as

$$\frac{i}{q^2 - m^2 - (q^4/\Lambda^2) + i\epsilon} \;\approx\; \frac{i}{q^2 - m^2 + i\epsilon} \times \frac{-\Lambda^2}{q^2 - \Lambda^2 + i\epsilon} \tag{8}$$

where the second factor differs from 1 only for very large momenta. Consequently, for the two propagators in the loop we may further approximate

$$\frac{-\Lambda^2}{q_1^2 - \Lambda^2 + i\epsilon} \;\approx\; \frac{-\Lambda^2}{q_2^2 - \Lambda^2 + i\epsilon} \;\approx\; \frac{-\Lambda^2}{(q_1 - \xi q_{\rm net})^2 - \Lambda^2 + i\epsilon}\,. \tag{9}$$