

1. Let's start with the Yukawa theory of a Dirac fermion field Ψ coupled to a real scalar field Φ according to

$$\mathcal{L} = \bar{\Psi}(i \not{\partial} - m_f)\Psi + \frac{1}{2}(\partial_\mu \Phi)^2 - \frac{1}{2}m_s^2 \Phi^2 + g\Phi \bar{\Psi}\Psi. \quad (1)$$

For $M_s > 2M_f$, the scalar particle becomes unstable: it decays into a fermion and an antifermion, $S \rightarrow f + \bar{f}$.

(a) Calculate the tree-level decay rate $\Gamma(S \rightarrow f + \bar{f})$.

(b) In class, we have calculated

$$\Sigma_\Phi^{1\text{loop}}(p^2) = \frac{12g^2}{16\pi^2} \int_0^1 d\xi \Delta(\xi) \times \left[\frac{1}{\epsilon} - \gamma_E + \frac{1}{3} + \log \frac{4\pi\mu^2}{\Delta(\xi)} \right] \quad (2)$$

$$\text{for } \Delta(\xi) = m_f^2 - \xi(1-\xi)p^2. \quad (3)$$

Show that for $p^2 > 4m_f^2$, this $\Sigma_\Phi(p^2)$ has an imaginary part and calculate it for $p^2 = M_s^2 + i\epsilon$.

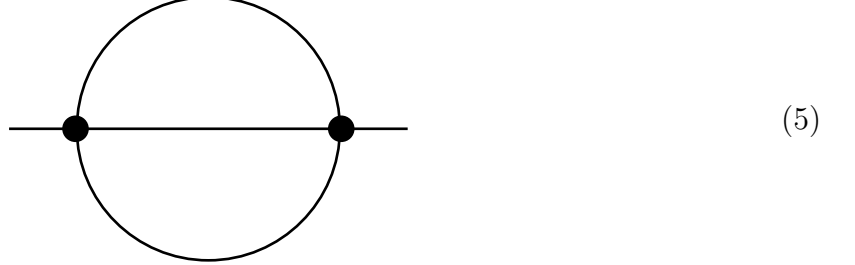
Note: at this level, you may neglect the difference between m_f^{bare} and M_f^{physical} .

(c) Verify that

$$\text{Im } \Sigma_\Phi^{1\text{loop}}(p^2 = M_s^2 + i\epsilon) = -M_s \Gamma^{\text{tree}}(S \rightarrow f + \bar{f}) \quad (4)$$

and explain this relation in terms of the optical theorem.

2. And now, a harder exercise about the scalar $\lambda\phi^4$ theory. As discussed in class, in this theory the field strength renormalization begins at the two-loop level. Specifically, the leading contribution to the $d\Sigma(p^2)/dp^2$ — and hence to the $Z - 1$ — comes from the two-loop 1PI diagram



Your task is to evaluate this contribution.

- (a) First, write the $\Sigma(p^2)$ from the diagram (5) as an integral over two independent loop momenta, say q_1^μ and q_2^μ , then use the Feynman's parameter trick — *cf.* eq. (F.d) of the [homework set 12](#) — to write the product of three propagators as

$$\iiint d\xi d\eta d\zeta \delta(\xi + \eta + \zeta - 1) \frac{2}{(\mathcal{D})^3} \quad (6)$$

where \mathcal{D} is a quadratic polynomial of the momenta q_1, q_2, p , and mass m with Feynman-parameter dependent coefficients.

Warning: Do not set $p^2 = m^2$ but keep p an independent variable.

- (b) Next, change the independent loop momentum variables from q_1 and q_2 to $k_1 = q_1 + \text{something} \times q_2 + \text{something} \times p$ and $k_2 = q_2 + \text{something} \times p$ to give \mathcal{D} a simpler form

$$\mathcal{D} = \alpha \times k_1^2 + \beta \times k_2^2 + \gamma \times p^2 - m^2 + i0 \quad (7)$$

for some (ξ, η, ζ) -dependent coefficients α, β, γ , for example

$$\alpha = (\xi + \zeta), \quad \beta = \frac{\xi\eta + \xi\zeta + \eta\zeta}{\xi + \zeta}, \quad \gamma = \frac{\xi\eta\zeta}{\xi\eta + \xi\zeta + \eta\zeta}. \quad (8)$$

Make sure the momentum shift has unit Jacobian $\partial(q_1, q_2)/\partial(k_1, k_2) = 1$.

(c) Express the derivative $d\Sigma(p^2)/dp^2$ in terms of

$$\iint d^4k_1 d^4k_2 \frac{1}{\mathcal{D}^4}. \quad (9)$$

Note that although this momentum integral diverges as $k_{1,2} \rightarrow \infty$, the divergence is logarithmic rather than quadratic.

(d) To evaluate the momentum integral (9), Wick-rotate the momenta k_1 and k_2 to the Euclidean space, and then use the dimensional regularization. Here are some useful formulæ for this calculation:

$$\frac{6}{A^4} = \int_0^\infty dt t^3 e^{-At}, \quad (10)$$

$$\int \frac{d^D k}{(2\pi)^D} e^{-ctk^2} = (4\pi ct)^{-D/2}, \quad (11)$$

$$\Gamma(2\epsilon)X^\epsilon = \frac{1}{2\epsilon} - \gamma_E + \frac{1}{2} \log X + O(\epsilon). \quad (12)$$

(e) Assemble your results as

$$\begin{aligned} \frac{d\Sigma(p^2)}{dp^2} = & -\frac{\lambda^2}{12(4\pi)^4} \iiint_{\xi, \eta, \zeta \geq 0} d\xi d\eta d\zeta \delta(\xi + \eta + \zeta - 1) \times \frac{\xi\eta\zeta}{(\xi\eta + \xi\zeta + \eta\zeta)^3} \times \\ & \times \left(\frac{1}{\epsilon} - 2\gamma_E + 2 \log \frac{4\pi\mu^2}{m^2} + \log \frac{(\xi\eta + \xi\zeta + \eta\zeta)^3}{(\xi\eta + \xi\zeta + \eta\zeta - \xi\eta\zeta(p^2/m^2))^2} \right). \end{aligned} \quad (13)$$

(f) Before you evaluate the Feynman parameter integral (13) — which looks like a frightful mess — make sure it does not introduce its own divergences. That is, without actually calculating the integrals

$$\iiint_{\xi, \eta, \zeta \geq 0} d\xi d\eta d\zeta \delta(\xi + \eta + \zeta - 1) \times \frac{\xi\eta\zeta}{(\xi\eta + \xi\zeta + \eta\zeta)^3}, \quad (14)$$

$$\iiint_{\xi, \eta, \zeta \geq 0} d\xi d\eta d\zeta \delta(\xi + \eta + \zeta - 1) \times \frac{\xi\eta\zeta}{(\xi\eta + \xi\zeta + \eta\zeta)^3} \times \log \frac{(\xi\eta + \xi\zeta + \eta\zeta)^3}{(\xi\eta + \xi\zeta + \eta\zeta - \xi\eta\zeta(p^2/m^2))^2},$$

make sure that they converge. Pay attentions to the boundaries of the parameter space and especially to the corners where $\xi, \eta \rightarrow 0$ while $\zeta \rightarrow 1$ (or $\xi, \zeta \rightarrow 0$, or $\eta, \zeta \rightarrow 0$).

- This calculation shows that

$$\frac{d\Sigma}{dp^2} = \frac{\text{constant}}{\epsilon} + \text{a_finite_function}(p^2) \quad (15)$$

and hence

$$\begin{aligned} \Sigma(p^2) = & (\text{a divergent constant}) + (\text{another divergent constant}) \times p^2 \\ & + \text{a_finite_function}(p^2) \end{aligned} \quad (16)$$

up to the two-loop order. In fact, this behavior persists to all loops, so all the divergences of $\Sigma(p^2)$ may be canceled with just two counterterms, δ^m and $\delta^Z \times p^2$.

For the purposes of calculating the field strength renormalization factor

$$Z = \left[1 - \frac{d\Sigma}{dp^2} \right]^{-1} \quad (17)$$

we need to evaluate the derivative $d\Sigma/dp^2$ at $p^2 = M_{\text{ph}}^2$ — the physical mass² of the scalar particle. However, to the leading non-trivial order in λ we may approximate $M_{\text{ph}}^2 \approx m_{\text{bare}}^2$ and set $p^2 = m^2$ in the Feynman-parameter integral (13). Consequently, the second integral (14) becomes a little simpler, although it is still a frightful mess.

- ★ Optional exercise: Evaluate the integrals (14) for $p^2 = m^2$ and show that

$$\begin{aligned} \iiint_{\xi, \eta, \zeta \geq 0} d\xi d\eta d\zeta \delta(\xi + \eta + \zeta - 1) \times \frac{\xi\eta\zeta}{(\xi\eta + \xi\zeta + \eta\zeta)^3} &= \frac{1}{2}, \\ \iiint_{\xi, \eta, \zeta \geq 0} d\xi d\eta d\zeta \delta(\xi + \eta + \zeta - 1) \times \frac{\xi\eta\zeta}{(\xi\eta + \xi\zeta + \eta\zeta)^3} \times \log \frac{(\xi\eta + \xi\zeta + \eta\zeta)^3}{(\xi\eta + \xi\zeta + \eta\zeta - \xi\eta\zeta)^2} &= -\frac{3}{4}. \end{aligned} \quad (18)$$

Do not try to do this calculation by hand — it would take way too much time. Instead, use *Mathematica* or equivalent software. To help it along, replace the (ξ, η, ζ) variables with (x, w) according to

$$\begin{aligned} \xi &= w \times x, \quad \eta = w \times (1 - x), \quad \zeta = 1 - w, \\ \iiint d\xi d\eta d\zeta \delta(\xi + \eta + \zeta - 1) &= \int_0^1 dx \int_0^1 dw w, \end{aligned} \quad (19)$$

then integrate over w first and over x second.

Alternatively, you may evaluate the integrals like this numerically. In this case, don't bother changing variables, just use a simple 2D grid spanning a triangle defined by $\xi + \eta + \zeta = 1$, $\xi, \eta, \zeta \geq 0$; modern computers can sum up a billion grid points in less than a minute. But watch out for singularities at the corners of the triangle.

- (g) Finally, assemble your results and calculate the field strength renormalization factor Z to the two-loop order.