1. Let's start with the electron mass renormalization in QED.

   (a) Calculate to one-loop order the infinite parts of the δ_2 and δm counterterm as functions of the gauge-fixing parameter ξ. Use the off-shell renormalization condition for $E \gg m_e$:

$$\Sigma_{\rm net}(\not p) \;=\; A(p^2)\times\not p \;+\; B(p^2)\times m; \quad A \;=\; B \;=\; 0 \text{ for } p^2=-E^2. \tag{1}$$

   The counterterms you obtain in part (a) should have form

$$\delta_2(E) \;=\; \frac{C_2(\xi)\alpha}{2\pi}\times\left(\frac{1}{\epsilon}\,+\,\log\frac{\mu^2}{E^2}\,+\,\text{const}\right), \tag{2}$$

$$\delta_m(E) \;=\; \frac{C_m(\xi)\alpha m}{2\pi}\times\left(\frac{1}{\epsilon}\,+\,\log\frac{\mu^2}{E^2}\,+\,\text{const}\right), \tag{3}$$

   for some ξ–dependent coefficients $C_2(\xi)$ and $C_m(\xi)$.

   (b) Check that the difference $C_m(\xi)-C_2(\xi)$ does not depend on ξ. If it does, go back to part (a) and check for mistakes.

   (c) Show that the dependence of the running electron's mass $m(E)$ on the energy scale E is governed by the equation

$$\frac{dm(E)}{d\log(E)} \;=\; 2\gamma_2\times\big(m(E)+\delta_m(E)\big)\;-\;\frac{d\delta_m(E)}{d\log(E)}\,. \tag{4}$$

   (d) Use eqs. (2) and (3) to show that to the one-loop order

$$\frac{dm(E)}{d\log(E)} \;=\; 2(C_m-C_2)\times\frac{\alpha(E)m(E)}{2\pi}\;+\;O(\alpha^2 m). \tag{5}$$

   Note: the running mass should be gauge invariant, that's why I asked you to check the ξ–independence of C_m-C_2 in part (b).

2. Next, consider the Wess–Zumino model, a QFT comprising a Majorana spinor field $\Psi(x)$, a real scalar field $\Phi_1(x)$, and a real pseudoscalar field $\Phi_2(x)$, all massless. The physical Lagrangian

$$\mathcal{L} \;=\; \frac{i}{2}\overline{\Psi}\,\partial\!\!\!/\Psi \;+\; \frac{1}{2}(\partial_\mu\Phi_1)^2 \;+\; \frac{1}{2}(\partial_\mu\Phi_2)^2 - \;\frac{g}{2}\overline{\Psi}(\Phi_1 + i\gamma^5\Phi_2)\Psi \;-\; \frac{\lambda}{8}\left(\Phi_1^2 + \Phi_2^2\right)^2, \quad (6)$$

has a global $U(1)$ axial symmetry, which acts as

$$\Psi \;\to\; \exp(i\theta\gamma^5)\Psi, \qquad (\Phi_1 + i\Phi_2) \;\to\; \exp(-2i\theta) \times (\Phi_1 + i\Phi_2). \quad (7)$$

Wess and Zumino found that for $\lambda = g^2$, the renormalization of this theory is particularly simple, but at first they did not know why. Salam and Strathdee realized there must be a symmetry behind this simplicity, and after working very hard to find it, they discovered the *supersymmetry.*

But this homework is about the basic renormalization theory rather than supersymmetry, and you do not need to know anything about supersymmetry to work it out.

(a) Argue that thanks to the chiral symmetry (7) of the Wess–Zumino model, the bare Lagrangian of the quantum theory needs only five counterterms to cancel all the divergences, namely δ^λ and δ^g for the couplings, δ^Z_ϕ and δ^M_ϕ for the bosonic fields, and δ^Z_ψ for the fermion. In particular, it does **not** need the mass counterterm δ^M_ψ for the fermion.

In general, the two-scalar 1PI amplitude has quadratic UV divergence which needs $\delta^M_\phi = O(\Lambda^2)$ to cancel. However, for $\lambda = g^2$ the net quadratic divergence cancels between loops of fermions and loops of pseudo/scalars. Instead, $\Sigma^{\text{net(loops)}}_\phi(p^2) = p^2 \times O(\log\Lambda^2/p^2)$, which needs only the $\delta^Z_\phi = O(\log\Lambda/E)$ counterterm to cancel but does **not** need the δ^M_ϕ counterterm.

(b) Verify that this is true at the one-loop level.

Note: Feynman rules for Majorana fermions are similar to those for Dirac fermions (same propagators, vertices, and external leg factors), but there is an extra factor $\frac{1}{2}$ for each closed fermionic loop. (*i.e.*, $-\frac{1}{2}\,\mathrm{tr}(\cdots)$ instead of $-\,\mathrm{tr}(\cdots)$).

Hint: mind the combinatorics for the loops of pseudo/scalar fields.

(c) Next, calculate the infinite parts of the other 4 counterterms at the one-loop level. Proceed similarly to homework #16, and do not hesitate to recycle similar calculations instead of redoing them from scratch.
Do not assume $\lambda = g^2$ at this stage.

(d) Calculate the anomalous dimensions of the scalar and fermionic fields to order $O(g^2, \lambda)$ and show that $\gamma_\phi = \gamma_\psi$.

Note: at the one-loop level this is true for any λ, but at the higher loop levels $\gamma_\phi = \gamma_\psi$ only when $\lambda = g^2$.

(e) Calculate the beta-functions $\beta_g(g, \lambda)$ and $\beta_\lambda(g, \lambda)$ to the one-loop order for general λ and g. Then show that

$$\text{for } \lambda = g^2, \quad \beta_\lambda(\lambda = g^2) \; = \; 2g \times \beta_g(\lambda = g^2). \tag{8}$$

Note: because of supersymmetry, this relation holds true to all orders of the perturbation theory. But in this exercise, you should check it at the one-loop level only.

(f) Show that the relation (8) implies that **if** $\lambda(E_0) = g^2(E_0)$ for any particular energy E_0, **then** $\lambda(E) = g^2(E)$ for all energies E.

⋆ Optional exercise, in case I explain the RG flow before this homework is due.
Consider the renormalization group flow in the (g^2, λ) plane. In the UV $\to$ IR direction, is the $\lambda = g^2$ line attractive or repulsive?