- The first two problems of this homework set are about non-abelian gauge theories while the last two problems are about path integrals in quantum mechanics.

⋆ In my notations, the A_μ and their components A^a_μ are the canonically normalized vector fields, while the $\mathcal{A}_\mu = gA_\mu$ and the $\mathcal{A}^a_\mu = gA^a_\mu$ are normalized by the symmetry action. Likewise, the tension fields $F_{\mu\nu}$ and their components $F^a_{\mu\nu}$ are canonically normalized while the $\mathcal{F}_{\mu\nu} = gF_{\mu\nu}$ and the $\mathcal{F}^a_{\mu\nu} = gF^a_{\mu\nu}$ are normalized by the symmetry action.

1. The first problem is about $SU(N)$ local symmetry and its most commonly used multiplets — the fundamental, the antifundamental, and the adjoint.

   - The fundamental multiplet is a set of of N fields (complex scalars or Dirac fermions) which transform as a column vector,

   $$\Psi'(x) \;=\; U(x)\Psi(x) \quad i.e. \quad \Psi'^i(x) \;=\; U^i_{\ j}(x)\Psi^j(x), \quad i,j = 1,2,\ldots,N \tag{1}$$

   where $U(x)$ is an x–dependent unitary $N \times N$ matrix, $\det U(x) \equiv 1$.

   - The antifundamental multiplet is the hermitian conjugate of the fundamental multiplet — a set of N conjugate fields $\Psi^*_i(x)$ arranged in a raw vector $\Psi^\dagger(x)$ and transforming according to

   $$\Psi'^\dagger(x) \;=\; \Psi^\dagger(x)U^\dagger(x) \quad i.e. \quad \Psi'_i(x) \;=\; \bigl(U^*(x)\bigr)_i^{\ j}\Psi_j(x). \tag{2}$$

   Note: for $N = 2$ the antifundamental doublet is equivalent to the fundamental doublet thanks to $\tau_2 U^* \tau_2 = U$. But for $N \geq 3$, the fundamental and the antifundamental multiplets are not equivalent to each other.

   - The adjoint multiplet of $SU(N)$ is a set of $N^2 - 1$ real fields $\Phi^a(x)$ — one field for each generator of the group — combined into a traceless hermitian $N \times N$ matrix

   $$\Phi(x) \;=\; \sum_a \Phi^a(x) \times \frac{\lambda^a}{2} \tag{3}$$

which transforms according to

$$\Phi'(x) \;=\; U(x)\Phi(x)U^\dagger(x). \tag{4}$$

Note that this transformation law preserves the $\Phi^\dagger = \Phi$ and $\mathrm{tr}(\Phi) = 0$ conditions. The covariant derivatives D_μ act on these kinds of multiplets according to

$$D_\mu\Psi(x) \;=\; \partial_\mu\Psi(x) \;+\; i\mathcal{A}_\mu(x)\Psi(x), \tag{5}$$

$$D_\mu\Psi^\dagger(x) \;=\; \partial_\mu\Psi^\dagger(x) \;-\; i\Psi^\dagger(x)\mathcal{A}_\mu(x), \tag{6}$$

$$D_\mu\Phi(x) \;=\; \partial_\mu\Phi(x) \;+\; i[\mathcal{A}_\mu(x),\Phi(x)] \;\equiv\; \partial_\mu\Phi(x) \;+\; i\mathcal{A}_\mu(x)\Phi(x) \;-\; i\Phi(x)\mathcal{A}_\mu(x). \tag{7}$$

In class we checked that the derivatives (5) are indeed covariant.

(a) Verify the covariance of the derivatives (6) and (7) under finite local symmetries $U(x)$.

(b) Verify the Leibniz rule for the covariant derivatives of matrix products. Let $\Phi(x)$ and $\Xi(x)$ be two adjoint multiplets while $\Psi(x)$ is a fundamental multiplet and $\Psi^\dagger(x)$ is its hermitian conjugate (a row vector of Ψ^*_i). Show that

$$\begin{aligned} D_\mu(\Phi\Xi) \;&=\; (D_\mu\Phi)\Xi \;+\; \Phi(D_\mu\Xi),\\ D_\mu(\Phi\Psi) \;&=\; (D_\mu\Phi)\Psi \;+\; \Phi(D_\mu\Psi),\\ D_\mu(\Psi^\dagger\Xi) \;&=\; (D_\mu\Psi^\dagger)\Xi \;+\; \Psi^\dagger(D_\mu\Xi). \end{aligned} \tag{8}$$

(c) Show that for an adjoint multiplet $\Phi(x)$,

$$[D_\mu, D_\nu]\Phi(x) \;=\; i[\mathcal{F}_{\mu\nu}(x),\Phi(x)] \;=\; ig[F_{\mu\nu}(x),\Phi(x)] \tag{9}$$

or in components $[D_\mu, D_\nu]\Phi^a(x) = -gf^{abc}F^b_{\mu\nu}(x)\Phi^c(x)$.

In class, I have argued (using covariant derivatives) that the tension fields $\mathcal{F}_{\mu\nu}(x)$ themselves transform according to eq. (4). In other words, the $\mathcal{F}^a_{\mu\nu}(x)$ form an adjoint multiplet of the $SU(N)$ symmetry group.

(d) Verify the $\mathcal{F}'_{\mu\nu}(x) = U(x)\mathcal{F}_{\mu\nu}(x)U^\dagger(x)$ transformation law directly from the definition $\mathcal{F}_{\mu\nu} \stackrel{\text{def}}{=} \partial_\mu\mathcal{A}_\nu - \partial_\mu\mathcal{A}_\nu + i[\mathcal{A}_\mu,\mathcal{A}_\nu]$ and from the non-abelian gauge transform formula for the $\mathcal{A}_\mu$ fields.

(e) Verify the Bianchi identity for the non-abelian tension fields $\mathcal{F}_{\mu\nu}(x)$:

$$D_\lambda \mathcal{F}_{\mu\nu} \;+\; D_\mu \mathcal{F}_{\nu\lambda} \;+\; D_\nu \mathcal{F}_{\lambda\mu} = 0. \tag{10}$$

Note the covariant derivatives in this equation.

Finally, consider the $SU(N)$ Yang–Mills theory — the non-abelian gauge theory that does not have any fields except $\mathcal{A}^a(x)$ and $\mathcal{F}^a(x)$; its Lagrangian is

$$\mathcal{L}_{\rm YM} \;=\; -\frac{1}{2g^2}\,\mathrm{tr}\big(\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu}\big) \;=\; -\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu}. \tag{11}$$

(f) Show that the Euler–Lagrange field equations for the Yang–Mills theory can be written in covariant form as $D_\mu \mathcal{F}^{\mu\nu} = 0$.
Hint: first show that for an infinitesimal variation $\delta\mathcal{A}_\mu(x)$ of the non-abelian gauge fields, the tension fields vary according to $\delta\mathcal{F}_{\mu\nu}(x) = D_\mu\delta\mathcal{A}_\nu(x) - D_\nu\delta\mathcal{A}_\mu(x)$.

2. Continuing the previous problem, consider an $SU(N)$ gauge theory in which N^2-1 vector fields $A^a_\mu(x)$ interact with some "matter" fields $\phi_\alpha(x)$,

$$\mathcal{L} \;=\; -\frac{1}{2g^2}\,\mathrm{tr}\big(\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu}\big) \;+\; \mathcal{L}_{\rm mat}(\phi, D_\mu\phi). \tag{12}$$

For the moment, let me keep the matter fields completely generic — they can be scalars, or vectors, or spinors, or whatever, and form any kind of a multiplet of the local $SU(N)$ symmetry as long as such multiplet is complete and non-trivial. All we need to know right now is that there are well-defined covariant derivatives $D_\mu\phi$ that depend on the gauge fields A^a_μ, which give rise to the currents

$$J^{a\mu} \;=\; -\frac{\partial\mathcal{L}_{\rm mat}}{\partial A^a_\mu}\,. \tag{13}$$

Collectively, these N^2-1 currents should form an adjoint multiplet $J^\mu = \sum_a(\frac{1}{2}\lambda^a)J^{a\mu}$ of the $SU(N)$ symmetry.

(a) Show that in this theory the equation of motion for the A^a_μ fields are $D_\mu F^{a\mu\nu} = J^{a\nu}$ and that consistency of these equations requires the currents to be *covariantly conserved,*

$$D_\mu J^\mu \;=\; \partial_\mu J^\mu \;+\; ig[A_\mu, J^\mu] \;=\; 0, \tag{14}$$

or in components,

$$D_\mu J^{\mu a} \;=\; \partial_\mu J^{a\mu} \;-\; g f^{abc} A^b_\mu J^{c\mu} \;=\; 0. \tag{15}$$

Note: a covariantly conserved current does *not* lead to a conserved charge, $(d/dt)\int d^3\mathbf{x}\, J^{a0}(\mathbf{x},t) \neq 0$!

Now consider a simple example of matter fields — a fundamental multiplet $\Psi(x)$ of N Dirac fermions $\Psi^i(x)$, with a Lagrangian

$$\mathcal{L}_{\rm mat} \;=\; \overline{\Psi}\big(i\gamma^\mu D_\mu \,-\, m\big)\Psi, \qquad \mathcal{L}_{\rm net} \;=\; \mathcal{L}_{\rm mat} \;-\; \frac{1}{2g^2}\,{\rm tr}\big(\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu}\big). \tag{16}$$

(b) Derive the $SU(N)$ currents $J^{a\mu}$ for these fermions and verify that under the $SU(N)$ symmetries they transform covariantly into each other as members of an adjoint multiplet. That is, the $N\times N$ matrix $J^\mu = \sum_a(\frac{1}{2}\lambda^a)J^{a\mu}$ transforms according to eq. (4). Hint: for any complex N–vectors ξ^i and η^j,

$$\sum_a \big(\eta^\dagger\lambda^a\xi\big) \times \big(\lambda^a\big)^i{}_j \;=\; 2\,\eta^*_j\xi^i \;-\; \frac{2}{N}\big(\eta^\dagger\xi\big)\times\delta^i_j\,.. \tag{17}$$

(c) Finally, verify the covariant conservation $D_\mu J^{a\mu} \;=\; 0$ of these currents when the fermionic fields $\Psi^i(x)$ and $\overline{\Psi}_i(x)$ obey their equations of motion.

3. Next, a reading assignment: my notes about properly discretized path integral for the harmonic oscillator.

4. Finally, a simple exercise in using path integrals. Consider a 1D particle living on a circle of radius R, or equivalently a 1D particle in a box of length $L = 2\pi R$ with periodic boundary conditions where moving past the $x = L$ point brings you back to $x = 0$. In other words, the particle's position $x(t)$ is defined modulo L.

   The particle has no potential energy, only the non-relativistic kinetic energy $p^2/2M$.

   (a) As a particle moves from some point $x_1 \pmod{L}$ at time $t_1 = 0$ to some other point $x_2 \pmod{L}$ at time $t_2 = T$, it may travel directly from x_1 to x_2, or it may take a few turns around the circle before ending at the x_2. Show that the space of all such paths on a circle is isomorphic to the space of all paths on an infinite line which begin at fixed x_1 at time t_1 and end at time t_2 at any one of the points $x_2' = x_2 + nL$ where $n = 0, \pm 1, \pm 2, \ldots$ is any whole number.

   Then use path integrals to relate the evolution kernels for the circle and for the infinite line (over the same time interval $t_2 - t_1 = T$) as

$$U_{\text{circle}}(x_2; x_1) \;=\; \sum_{n=-\infty}^{+\infty} U_{\text{line}}(x_2 + nL; x_1). \tag{18}$$

The next question uses Poisson's resummation formula: If a function $F(n)$ of integer n can be analytically continued to a function $F(\nu)$ of arbitrary real ν, then

$$\sum_{n=-\infty}^{+\infty} F(n) \;=\; \int\! d\nu\, F(\nu) \times \sum_{n=-\infty}^{+\infty} \delta(\nu - n) \;=\; \sum_{\ell=-\infty}^{+\infty} \int\! d\nu\, F(\nu) \times e^{2\pi i \ell \nu}. \tag{19}$$

   (b) The free particle living on an infinite 1D line has evolution kernel

$$U_{\text{line}}(x_2; x_1) \;=\; \sqrt{\frac{M}{2\pi i \hbar T}} \times \exp\left(+\frac{iM(x_2 - x_1)^2}{2\hbar T}\right). \tag{20}$$

   Plug this free kernel into eq. (18) and use Poisson's formula to sum over n.

(c) Verify that the resulting evolution kernel for the particle one the circle agrees with the usual QM formula

$$U_{\text{box}}(x_2; x_1) \ = \ \sum_p L^{-1/2} e^{ipx_2/\hbar} \times e^{-iT(p^2/2M)/\hbar} \times L^{-1/2} e^{-ipx_1/\hbar} \tag{21}$$

where p takes circle-quantized values

$$p \ = \ \frac{2\pi\hbar}{L} \times \text{integer}. \tag{22}$$