1. First, a modified textbook problem 9.2(c–e) about the Euclidean path integrals at finite temperature. Questions (a–d) below concern a free scalar field, questions (e–f) concern free fermionic fields, and question (g) is about the free electromagnetic field.

   (a) Consider a free scalar field in 3+1 dimensions at finite temperature T. Use Euclidean path integral to calculate the partition function and hence the Helmholtz free energy. Show that formally

$$\mathcal{F}(T) \;=\; \frac{T}{2}\times \operatorname{Tr}\log\bigl(-\partial_E^2+m^2\bigr) \tag{1}$$

   where the ∂_E^2 operator acts on functions$(x_1,x_2,x_3,x_4)_E$ which are periodic in the Euclidean time x_4 with period $\beta=1/T$.

   (b) Write down the trace in eq. (1) as a momentum space sum/integral. Then use the Poisson resummation formula — which appeared in the previous homework set#20 as eq. (19) — to show that

$$\mathcal{F}(T) \;=\; \text{const} \;+\; \frac{1}{2}\sum_{\ell=-\infty}^{+\infty}\int\!\frac{d^4p_E}{(2\pi)^4}\,\exp(i\ell\beta p_4)\times\log(p_E^2+m^2) \tag{2}$$

$$=\; \mathcal{F}(0) \;+\; \sum_{\ell=1}^{\infty}\int\!\frac{d^4p_E}{(2\pi)^4}\,\exp(i\ell\beta p_4)\times\log(p_E^2+m^2). \tag{3}$$

   (c) To evaluate the $\int dp_4$ integral in eq. (3), move the integration contour from the real axis to the two 'banks' of a branch cut. Show that

$$\int\limits_{-\infty}^{+\infty}\!\frac{dp_4}{2\pi}\,\exp(i\ell\beta p_4)\times\log(p_4^2+E^2) \;=\; -\frac{\exp(-\ell\beta E)}{\ell\beta}\,. \tag{4}$$

   (d) Finally, use eqs. (3) and (4) to show that the free energy of a free scalar field above the zero-point energy is

$$\mathcal{F}(T)\,-\,\mathcal{F}(0) \;=\; \int\!\frac{d^3\mathbf{p}}{(2\pi)^3}\,T\,\log\Bigl(1-e^{-\beta E_p}\Bigr) \;=\; \int\!\frac{d^3\mathbf{p}}{(2\pi)^3}\,\Bigl(\mathcal{F}^{\text{harmonic}}_{\text{oscillator}}(T,E_p)\;-\;\tfrac{1}{2}E_p\Bigr)\,. \tag{5}$$

Next, consider a free fermion $0+1$ dimensions, basically a two-level system in Quantum Mechanics. In the Hamiltonian formulation this means

$$\hat{H} \;=\; \omega\hat{\psi}^\dagger\hat{\psi} \quad \text{where} \quad \{\hat{\psi},\hat{\psi}^\dagger\} \;=\; 1 \quad \text{and} \quad \omega = \text{constant} > 0, \tag{6}$$

while in the Lagrangian formulation, $\psi(t)$ and $\psi^*(t)$ are Grassmann-number-valued functions of the time and

$$L_E \;=\; \psi^* \times \frac{d\psi}{dt_E} \;+\; \omega \times \psi^*\psi. \tag{7}$$

(e) Use the path integral to calculate the partition function for both periodic and antiperiodic boundary conditions for the fermionic variables in the Euclidean time, $\psi(t_E+\beta) = \pm\psi(t_E)$. Show that the periodic conditions lead to an unphysical partition function, while the antiperiodic conditions lead to the correct partition function of a two-level system.

(f) Now apply the lesson of part (e) to a Dirac fermionic field in $3+1$ dimensions. Calculate the partition function and hence the free energy using the Euclidean path integral over Dirac fields which are antiperiodic in the Euclidean time, $\Psi(\mathbf{x},x_4+\beta) = \Psi(\mathbf{x},x_4)$.

Finally, consider the free electromagnetic field $A_\mu(x)$. At finite temperature, the $A^\mu(x)$ — just like any other bosonic field — is periodic in the Euclidean time, $A^\mu(\mathbf{x},x_4+\beta) = A^\mu(\mathbf{x},x_4)$.

(g) Use the path integral — and mind the gauge-fixing and the Fadde'ev–Popov determinant — to show that formally, the EM free energy is

$$\mathcal{F}(T) \;=\; T \times \operatorname{Tr}\log\bigl(-\partial_E^2\bigr). \tag{8}$$

(h) Recycle arguments from parts (a–d) to show that eq. (8) leads to

$$\mathcal{F}(T) \;-\; \mathcal{F}(0) \;=\; \int\!\frac{d^3\mathbf{p}}{(2\pi)^3}\, 2T \times \left(1 - e^{-\beta|p|}\right). \tag{9}$$

2. Second, a problem about the scalar QCD, or more generally a non-abelian gauge theory with some gauge group G and complex scalar fields $\Phi^i(x)$ in some multiplet (r) of G.

   (a) Write down the physical Lagrangian of this theory, the complete bare Lagrangian of the quantum theory in the Feynman gauge, and the Feynman rules.

   Now consider the annihilation process $\Phi + \Phi^* \to 2$ gauge bosons. At the tree level, there are four Feynman diagrams contributing to this process.

   (b) Draw the diagrams and write down the tree-level annihilation amplitude.

   As discussed in class, amplitudes involving the non-abelian gauge fields satisfy a weak form of the Ward identity: *On-shell Amplitudes involving* **a** *longitudinally polarized gauge bosons vanish, provided all the other gauge bosons are transversely polarized.* In other words,

$$\begin{aligned} \mathcal{M} \;&\equiv\; e_1^{\mu_1} e_2^{\mu_2} \cdots e_n^{\mu_n}\, \mathcal{M}_{\mu_1\mu_2\cdots\mu_n}(\text{momenta}) \;=\; 0 \\ \text{when}\quad e_1^{\mu} \propto k_1^{\mu} \quad &\text{but}\quad e_2^{\nu} k_{2\nu} \;=\; \cdots \;=\; e_n^{\nu} k_{n\nu} \;=\; 0. \end{aligned}$$

   (c) Verify this identity for the scalar annihilation amplitude: Show that IF $e_2^{\nu} k_{2\nu} = 0$ THEN $K_{1\mu}\mathcal{M}^{\mu\nu} e_{2\nu} = 0$.

   Similar to hat we had in class for the quark-antiquark annihilations, there are non-zero amplitudes for the scalar 'quark' and 'antiquark' annihilating into a pair of longitudinal gluons or a ghost-antighost pair, but the crossections for these two unphysical processes cancel each other.

   (d) Take both final-state gluons are longitudinally polarized; specifically, assume null polarization vectors $e_1^{\mu} = (1, +\mathbf{n}_1)/\sqrt{2}$ for the first gluon and $e_2^{\nu} = (1, -\mathbf{n})/\sqrt{2}$ for the second gluons.
   Calculate the tree-level annihilation amplitude $\Phi + \Phi^* \to g_L + g_L$ for these polarizations.

   (e) Next, calculate the tree amplitude for the $\Phi + \Phi^* \to \mathrm{gh} + \overline{\mathrm{gh}}$. There is only one tree graph for this process, so evaluating it should not be hard.

(f) Compare the two un-physical amplitudes and show that the corresponding partial cross-sections cancel each other, thus

$$\frac{d\sigma_{\rm net}}{d\Omega} \;=\; \frac{d\sigma_{\rm physical}}{d\Omega}\,. \tag{10}$$