1. Let's start with a bit of group theory. Consider a generic simple non-abelian compact Lie group G and its generators T^a. For a suitable normalization of the generators,

$$\mathrm{tr}_{(r)}(T^aT^b) \;\equiv\; \mathrm{tr}\left(T^a_{(r)}T^b_{(r)}\right) \;=\; R(r)\delta^{ab} \tag{1}$$

where the trace is taken over any complete multiplet (r) — irreducible or reducible, it does not matter — and $T^a_{(r)}$ is the matrix representing the generator T^a in that multiplet. The coefficient $R(r)$ in eq. (1) depends on the multiplet (r) but it's the same for all generators T^a and T^b. The $R(r)$ is called the *index* of the multiplet (r).

The (quadratic) Casimir operator $C_2 = \sum_a T^aT^a$ commutes with all the generators, $\forall b,\ [C_2, T^b] = 0$. Consequently, when we restrict this operator to any *irreducible* multiplet (r) of the group G, it becomes a unit matrix times some number $C(r)$. In other words,

$$\text{for an irreducible } (r), \quad \sum_a T^a_{(r)}T^a_{(r)} \;=\; C(r)\times \mathbf{1}_{(r)}\,. \tag{2}$$

For example, for the isospin group $SU(2)$, the Casimir operator is $C_2 = \vec{I}^2$, the irreducible multiplets have definite isospin $I = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \ldots$, and $C(I) = I(I+1)$.

(a) Show that for any irreducible multiplet (r),

$$\frac{R(r)}{C(r)} \;=\; \frac{\dim(r)}{\dim(G)}\,. \tag{3}$$

In particular, for the $SU(2)$ group, this formula gives $R(I) = \frac{1}{3}I(I+1)(2I+1)$.

(b) Suppose the first three generators T^1, T^2, and T^3 of G generate an $SU(2)$ subgroup, thus

$$[T^1, T^2] \;=\; iT^3, \quad [T^2, T^3] \;=\; iT^1, \quad [T^3, T^1] \;=\; iT^2. \tag{4}$$

Show that if a multiplet (r) of G decomposes into several $SU(2)$ multiplets of isospins

$I_1, I_2, \ldots, I_n$, then

$$R(r) = \sum_{i=1}^{n} \tfrac{1}{3} I_i(I_i+1)(2I_i+1). \qquad (5)$$

(c) Now consider the $SU(N)$ group with an obvious $SU(2)$ subgroup of matrices acting only on the first two components of a complex N-vector. This complex N-vector is called the fundamental multiplet (of the $SU(N)$) and denoted (N) or $\mathbf{N}$. As far as the $SU(2)$ subgroup is concerned, (N) comprises one doublet and $N-2$ singlets, hence

$$R(N) = \frac{1}{2} \quad \text{and} \quad C(N) = \frac{N^2-1}{2N}. \qquad (6)$$

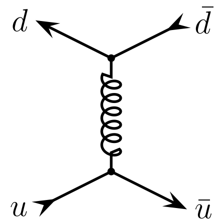
Show that the adjoint multiplet of the $SU(N)$ decomposes into one $SU(2)$ triplet, $2(N-2)$ doublets, and $(N-2)^2$ singlets, therefore

$$R(\text{adj}) = C(\text{adj}) \equiv C(G) = N. \qquad (7)$$

Hint: $(N) \times (\overline{N}) = (\text{adj}) + (1)$.

(d) The symmetric and the anti-symmetric 2–index tensors form irreducible multiplets of the $SU(N)$ group. Find out the decomposition of these multiplets under the $SU(2) \subset SU(N)$ and calculate their respective indices R and Casimirs C.

2. Next, let's apply this group theory to physics. Consider quark-antiquark pair production in QCD, specifically $u\bar{u} \to d\bar{d}$. There is only one tree diagram contributing to this process,



(8)

Evaluate this diagram, then sum/average the $|\mathcal{M}|^2$ over both spins and *colors* of the final/initial particles to calculate the total cross section. For simplicity, you may neglect the quark masses.

Note that the diagram (8) looks exactly like the QED pair production process $e^-e^+ \to$ virtual $\gamma \to \mu^-\mu^+$, so you can re-use the QED formula for summing/averaging over the spins, *cf.* my notes on Dirac traceology from the Fall semester, page 11. But in QCD, you should also sum/average over the colors of all the quarks, and that's the whole point of this exercise.

3. For another exercise of group theory in gauge theories, let's go back to scalar QCD from problem **2** of the previous homework set. Again, we consider tree-level annohilation of a scalar 'quark' Φ^i and an 'antiquark' Φ^*_j into a pair of gauge bosons with adjoint colors a and b. But this time, we focus on the group theory and on the physical cross-sections rather than the Ward identities.

   (a) Take the annihilation amplitude from part (b) of problem (21.1), focus on its color dependence, and rewrite it in the form

   $$\mathcal{M}(j+i\to a+b) \;=\; F\times\{T^a,T^b\}^i{}_j \;+\; iG\times[T^a,T^b]^i{}_j \tag{9}$$

   where F and G are some functions of all the momenta and of the two vectors' polarizations. Give explicit formulae for F and G.

   (b) Next, let us sum the $|\mathcal{M}|^2$ over the gauge boson's colors and average over the scalars' colors. Show that

   $$\frac{1}{\dim^2(r)}\sum_{ij}\sum_{ab}|\mathcal{M}|^2 \;=\; \frac{C(r)}{\dim(r)}\times\Big(\big(4C(r)-C(\mathrm{adj})\big)\times|F|^2+C(\mathrm{adj})\times|G|^2\Big). \tag{10}$$

   In particular, for scalars in the fundamental representation of the $SU(N)$ gauge group,

   $$\frac{1}{N^2}\sum_{ij}\sum_{ab}|\mathcal{M}|^2 \;=\; \frac{N^2-1}{2N^2}\left(\frac{N^2-2}{N}\times|F|^2+N\times|G|^2\right). \tag{11}$$

   (c) Evaluate F and G in the center of mass frame, where the vector particles' polarizations $e^\mu_{1,2}=(0,\mathbf{e}_{1,2})$ are purely spatial and transverse to the vectors' momenta $\pm\mathbf{k}$. For simplicity, use planar rather than circular polarizations.

   (d) Assemble your results and calculate the (polarized, partial) cross-section for the annihilation process.

4. Finally, let's evaluate a few one-loop diagrams. In class, I calculated the (infinite parts of the) δ_2 and δ_1 counterterms for the quarks. Your task is to calculate the analogous $\delta_2^{(\mathrm{gh})}$ and $\delta_1^{(\mathrm{gh})}$ counterterms for the *ghosts fields.*

   (a) Draw one-loop diagrams whose divergences are cancelled by the respective conterterms $\delta_2^{(\mathrm{gh})}$ and $\delta_1^{(\mathrm{gh})}$, and calculate the group factors for each diagrams.

   (b) Calculate the momentum integrals for the diagrams. Focus on the UV divergences and ignore the finite parts of the integrals.

   (c) Assemble your results and show that the *difference* $\delta_1^{(\mathrm{gh})} - \delta_2^{(\mathrm{gh})}$ for the ghosts is exactly the same as the $\delta_1 - \delta_2$ difference for the quarks.