1. Consider the three gauge couplings of the $SU(3)\times SU(2)\times U(1)$ Standard Model and their one-loop beta-functions

$$\beta_1^{1\,\text{loop}} = \frac{b_1 g_1^3}{16\pi^2}\,,\quad \beta_2^{1\,\text{loop}} = \frac{b_2 g_2^3}{16\pi^2}\,,\quad \beta_3^{1\,\text{loop}} = \frac{b_3 g_3^3}{16\pi^2}\,. \tag{1}$$

In this exercise, you do not need to calculate these beta-function from scratch by evaluating the UV divergences of a bunch of loop diagrams. Instead, use eqs. (119) and (121–2) from my notes on QCD beta-function (pages 24–25).

(a) Calculate the b_1, b_2, b_3 coefficients it for the minimal version of the Standard Model: the $SU(3)\times SU(2)\times U(1)$ gauge fields, one Higgs doublet, three families of quarks and leptons, and nothing else.

⋆ FYI, each family comprises 8 left-handed Weyl fields in the $(\mathbf{3},\mathbf{2},y=+\frac{1}{6})$ and $(\mathbf{1},\mathbf{2},y=-\frac{1}{2})$ multiplets of the gauge symmetry and 7 right-handed Weyl fermions in the $(\mathbf{3},\mathbf{1},y=+\frac{2}{3})$, $(\mathbf{3},\mathbf{1},y=-\frac{1}{3})$, and $(\mathbf{1},\mathbf{1},y=-1)$ multiplets.

(b) Re-calculate the b_1, b_2, b_3 for the MSSM — the Minimal Supersymmetric Standard Model. FYI, here is complete list of the MSSM fields:

- ∘ The $SU(3)\times SU(2)\times U(1)$ gauge fields, same as the non-SUSY SM.
- For each vector field there is a Majorana fermion (gaugino) with similar $SU(3)\times SU(2)\times U(1)$ quantum numbers. Altogether, there is an adjoint multiplet of gauginos for each factor of the gauge symmetry.
- ∘ 3 families of quarks and leptons, same as the non-SUSY SM.
- For each Weyl fermion — left-handed or right-handed — in these three families, the MSSM also have a complex scalar field (squark or slepton) with similar $SU(3)\times SU(2)\times U(1)$ quantum numbers. Altogether, this makes 45 complex scalar fields in similar multiplets to the quarks and leptons.
- The Higgs sector of the MSSM comprises *two* $SU(2)$ doublets of complex scalars accompanied by one $SU(2)$ doublet of Dirac fermions (the higgsinos); all these doublets have $y=\frac{1}{2}$.

— There are all kinds of Yukawa and ϕ^4 interactions between the MSSM fields, but you do not need them for the one-loop calculation of the gauge couplings' beta-functions.

In Grand Unified Theories

$$\alpha_3 \;=\; \alpha_2 \;=\; \tfrac{5}{3}\alpha_1 \;=\; \alpha_{\rm GUT} \quad \text{at the GUT scale.} \tag{2}$$

At lower energy scales $E \ll M_{\rm GUT}$ the SM couplings are given (lo the leading one-loop order) by the Georgi–Quinn–Weinberg equations

$$\begin{aligned} \frac{1}{\alpha_3(E)} &= \frac{1}{\alpha_{\rm GUT}} + b_3 \times \frac{1}{2\pi}\log\frac{M_{\rm GUT}}{E}\,, \\ \frac{1}{\alpha_2(E)} &= \frac{1}{\alpha_{\rm GUT}} + b_2 \times \frac{1}{2\pi}\log\frac{M_{\rm GUT}}{E}\,, \\ \frac{1}{\alpha_1(E)} &= \frac{5/3}{\alpha_{\rm GUT}} + b_1 \times \frac{1}{2\pi}\log\frac{M_{\rm GUT}}{E}\,. \end{aligned} \tag{3}$$

(c) Derive these equations from eqs. (1).

(d) The experimental data interpreted in terms of the $\overline{\rm MS}$ gauge couplings at $E = M_{\rm top} \approx$ 173 GeV and translated to the $\overline{\rm MS}$ give

$$\frac{1}{\alpha_3(M_Z)} \approx 9.23, \qquad \frac{1}{\alpha_2(M_Z)} \approx 29.97, \qquad \frac{1}{\alpha_1(M_Z)} \approx 97.76. \tag{4}$$

Check that these couplings are consistent with eqs. (3) for the MSSM but not for the non-SUSY minimal Standard Model. For the MSSM, calculate the Grand Unification scale $M_{\rm GUT}$ and the unified gauge coupling $\alpha_{\rm GUT}$.

Although all the additional particles of the MSSM are heavier than $M_{|rmtop}$, for this exercise you should ignore the thresholds due to these masses. Instead, use the b_3, b_2, b_1 coefficients of the massless theory — the minimal SM or the MSSM — for all energies between the $M_{\rm top}$ and the $M_{\rm GUT}$.

2. Next, a reading assignment: \$16.7 of *Peskin & Schroeder* about the "magnetic anti-screening" explanation of the asymptotic freedom in QCD.

3. And another reading assignment: §19.3 of *Peskin & Schroeder* about the chiral symmetry of QCD and the pions.

   Chapter 19 of *Weinberg* has a deeper discussion of pions (and Goldstone bosons in general); you are advised to read it, but not necessarily this week.

4. The pions are pseudo-Goldstone bosons of the spontaneously broken chiral symmetry of QCD, so they can be created or annihilated by the appropriate axial currents. In particular, for the charged pions

$$\langle\text{vacuum}|\,\overline{\Psi}_d\gamma^5\gamma^\alpha\Psi_u\left|\pi^+(p)\right\rangle \;=\; \langle\text{vacuum}|\,\overline{\Psi}_u\gamma^5\gamma^\alpha\Psi_d\left|\pi^-(p)\right\rangle \;=\; f_\pi\times p^\alpha \tag{5}$$

   for $f_\pi\approx 93$ MeV. The f_π is called the *pion decay constant* because it controls the decay rate of the charged pions, mostly into muons and neutrinos, $\pi^+\to\mu^+\nu_\mu$ and $\pi^-\to\mu^-\bar\nu_\mu$. In this exercise, we shall see how this works.

   The weak interactions at energies $O(M_\pi)\ll M_W$ are governed by the Fermi's current-current effective Lagrangian

$$\mathcal{L} \;=\; -2\sqrt{2}G_F J_L^{+\alpha}J_{L\alpha}^{-} \tag{6}$$

   where $L_L^{\pm\alpha}=\frac{1}{2}(J_V^{\pm\alpha}-J_A^{\pm\alpha})$ are the left-handed charged currents. In terms of the quark and lepton fields,

$$\begin{aligned} J_L^{+\alpha} &= \tfrac{1}{2}\overline{\Psi}(\nu_\mu)(1-\gamma^5)\gamma^\alpha\Psi(\mu) \;+\; \cos\theta_c\times\tfrac{1}{2}\overline{\Psi}(u)(1-\gamma^5)\gamma^\alpha\Psi(d) \;+\; \cdots,\\ J_L^{-\alpha} &= \tfrac{1}{2}\overline{\Psi}(\mu)(1-\gamma^5)\gamma^\alpha\Psi(\nu_\mu) \;+\; \cos\theta_c\times\tfrac{1}{2}\overline{\Psi}(d)(1-\gamma^5)\gamma^\alpha\Psi(u) \;+\; \cdots, \end{aligned} \tag{7}$$

   where the $\cdots$ stand for other fermions of the Standard Model, and $\theta_c\approx 13^\circ$ is the Cabibbo angle. For the pion decay process, the axial part one of the charged currents annihilates the charged pion according to eq. (6) while the other charged current creates the lepton pair.

   (a) Show that the tree-level pion decay amplitude is

$$\mathcal{M}(\pi^+\to\mu^+\nu_\mu) \;=\; \frac{G_f f_\pi\cos\theta_c}{\sqrt{2}}\times p^\alpha(\pi)\times\bar u(\nu_\mu)(1-\gamma^5)\gamma_\alpha v(\mu^+). \tag{8}$$

   (b) Sum over the fermion spins and calculate the decay rate $\Gamma(\pi^+\to\mu^+\nu_\mu)$. FYI, $f_\pi\approx$ 93 MeV, $M_\pi\approx 140$ MeV, $M_\mu\approx 106$ MeV, and $G_F\approx 1.17\cdot 10^{-5}\,\text{GeV}^{-2}$.

(c) The charged pions decay to muons much more often than they decay to electrons,

$$\frac{\Gamma(\pi^+ \to e^+ \nu_e)}{\Gamma(\pi^+ \to \mu^+ \nu_\mu)} = \frac{M_e^2}{M_\mu^2} \frac{(1 - (M_e/M_\pi)^2)^2}{(1 - (M_\mu/M_\pi)^2)^2} \approx 1.2 \cdot 10^{-4}. \tag{9}$$

Derive this formula, then explain this preference for the heavier final-state lepton in terms of mismatch between lepton's chirality and helicity.