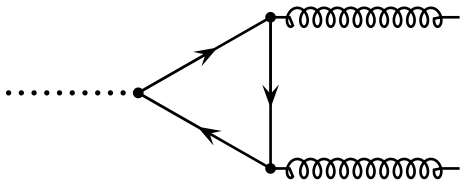
1. Consider the axial anomaly in a non-abelian gauge theory, for example QCD with N_f massless quark flavors,

$$J_A^\mu = \sum_{i,f} \overline{\Psi}_{if} \gamma^5 \gamma^\mu \Psi^{if}, \qquad \partial_\mu J_A^\mu = -\frac{N_f g^2}{16\pi^2} \epsilon^{\alpha\beta\mu\nu} \operatorname{tr}\Bigl(F_{\alpha\beta} F_{\mu\nu}\Bigr) \tag{1}$$

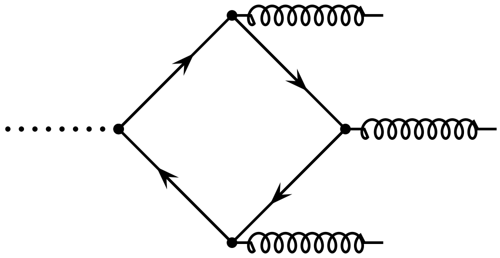
where $F_{\mu\nu}$ is the non-abelian gauge field strength.

(a) Expand the right hand side of eq. (1) into 2–gluon, 3–gluon, and 4–gluon terms and show that the 4–gluon term vanishes identically.
Hint: Use the cyclic symmetry of the trace.

The two-gluon anomaly term obtains from the triangle diagrams

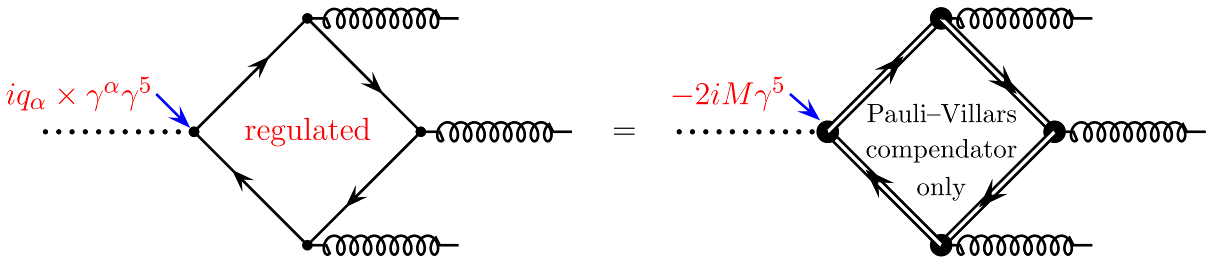
 + gluon permutation. (2)

This works exactly as discussed in class for the QED, except in QCD we should trace $F_{\alpha\beta} F_{\gamma\delta}$ over the quark colors and flavors. But in QCD there is also the three-gluon anomaly (*cf.* part (a)) which obtains from the quadrangle diagrams

 + gluon permutations. (3)

Since the quadrangle diagrams suffer from linear UV divergences, we need to regulate them, so let's use the Pauli–Villars regulator.

(b) Show that



+ terms which cancel after summing over gluon permutations (4)

(c) Finally, evaluate the the quadrangle diagrams for the Pauli–Villars regulators and derive the three-gluon anomaly in QCD.

2. Next, a reading assignment: §22.2–3 of *Weinberg* about the chiral anomaly. Pay particular attention to the Jacobian of the fermion path integral and to regularization of the functional trace.

3. In any *even* spacetime dimension $d = 2n$, a massless Dirac fermion has an axial symmetry $\Psi(x) \to \exp(i\theta\Gamma)\Psi(x)$ where Γ generalizes the γ^5. Classically, the axial current $J^\mu_A = \overline{\Psi}\Gamma\gamma^\mu\Psi$ is conserved, but when the fermion is coupled to a gauge field — abelian or non-abelian — the axial symmetry is broken by the anomaly and

$$\partial_\mu J^\mu_A \;=\; -\frac{2}{n!}\left(\frac{g}{4\pi}\right)^n \epsilon^{\alpha_1\beta_1\alpha_2\beta_2\cdots\alpha_n\beta_n}\,\mathrm{tr}\Big(F_{\alpha_1\beta_1}F_{\alpha_2\beta_2}\cdots F_{\alpha_n\beta_n}\Big). \tag{5}$$

Generalize Weinberg's calculation of the anomaly via Jacobian of the fermionic path integral to any even spacetime dimension $d = 2n$.

For your information, in $2n$ Euclidean dimensions $\{\gamma^\mu,\gamma^\nu\} = +2\delta^{\mu\nu}$, $\Gamma = i^{n-2}\gamma^1\gamma^2\cdots\gamma^{2n}$, $\{\Gamma,\gamma^\mu\} = 0$, $\Gamma^2 = +1$, and for any $2n = d$ matrices $\gamma^\alpha,\ldots,\gamma^\omega$, $\mathrm{tr}(\Gamma\gamma^\alpha\gamma^\beta\cdots\gamma^\omega) = 2^n i^{2-n}\epsilon^{\alpha\beta\cdots\omega}$.