Spin–Statistics Theorem

Relativistic causality requires quantum fields at two spacetime points x and y separated by a space-like interval $(x-y)^2 < 0$ to either commute or anticommute with each other. The **spin–statistics theorem** says that the fields of integral spins commute (and therefore must be quantized as bosons) while the fields of half-integral spin anticommute (and therefore must be quantized as fermions). The spin-statistics theorem applies to all quantum field theories which have:

1. Special relativity, *i.e.* Lorentz invariance and relativistic causality.

2. Positive energies of all particles.

3. Hilbert space with positive norms of all states.*

The theorem is valid for both free or interacting quantum field theories,† and in any spacetime dimension $d > 2$. In these notes I shall prove the theorem for the free fields in four dimensions and outline its generalization to $d \neq 4$; proving the theorem for the interactive fields is too complicated for this class.

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I have mentioned earlier in class (and you have worked out the details in homework#4, problem 3) that relativistic causality does not allow for quantum mechanics of a single particle or a fixed number of particles. Instead, one should use a quantum field theory which allows for arbitrary numbers of particles and antiparticles.

So let us consider a general Lorentz multiplet of free quantum fields $\hat{\phi}_A$ whose quanta have spin j and mass M. Free fields satisfy some kind of linear equations of motion which

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$\star$ This is automatic when all states are physical. But covariant quantisation of EM and other gauge fields expands the formal Fock space to include quanta of the un-physical polarizations, and also of the 'ghost' fields which cancel them. The ghost quanta have negative norms and wrong spin-statistics combinations (scalar fermions).

$\dagger$ In strongly interacting field theories, the physical particles are often bound states rather then quanta of the fields one starts with. For example, in QCD the particles are mesons and baryons rather than quarks or gluons. In such cases one should treat particles as quanta of some effective field theories, for example, a theory of pseudoscalar fields for pions and Dirac fields for nucleons.

have plane-wave solutions with $p^2 = M^2$. Let $p^0 = +E_{\mathbf{p}} = +\sqrt{\mathbf{p}^2 + M^2}$ and let

$$e^{-ipx} f_A(\mathbf{p}, s) \quad \text{and} \quad e^{+ipx} h_A(\mathbf{p}, s) \tag{1}$$

be respectively the positive-frequency and negative-frequency plane-wave solutions. The s here labels different wave polarizations for same p^μ; it corresponds to particle spin states (for $M > 0$) or helicities (for $M = 0$). Note: for the EM fields and other fields subject to gauge symmetries, we should consider only the physical polarizations and ignore the gauge artefacts, and also let the $\hat{\phi}_A(x)$ be the gauge-invariant tension field $\hat{F}^{\mu\nu}(x)$ rather than the gauge-dependent potential fields $\hat{A}^\mu(x)$.

The relation between the particles' spin and the anti/commutativity of the fields follows from the sums

$$\mathcal{F}_{AB}(p) \stackrel{\text{def}}{=} \sum_s f_A(\mathbf{p}, s)\, f_B^*(\mathbf{p}, s) \quad \text{and} \quad \mathcal{H}_{AB}(p) \stackrel{\text{def}}{=} \sum_s h_A(\mathbf{p}, s)\, h_B^*(\mathbf{p}, s) \tag{2}$$

which satisfy two important lemmas:

Lemma 1: Both $\mathcal{F}_{AB}(p)$ and $\mathcal{H}_{AB}(p)$ can be analytically continued to off-shell momenta (with $p^0 \neq E_{\mathbf{p}}$) as *polynomials* in the four components of the p^μ.

Lemma 2: Those polynomials are related to each other as

$$\begin{aligned} \mathcal{H}_{AB}(-p^\mu) &= +\mathcal{F}_{AB}(+p^\mu) \quad \text{for particles of integral spin,} \\ \mathcal{H}_{AB}(-p^\mu) &= -\mathcal{F}_{AB}(+p^\mu) \quad \text{for particles of half-integral spin.} \end{aligned} \tag{3}$$

I shall prove the two lemmas later in these notes. Right now, I want to show how they lead to the spin-statistics theorem.

A free quantum field is a superposition of plane-wave solutions with operatorial coefficients, thus

$$\begin{aligned} \hat{\phi}_A(x) &= \int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{1}{2E_{\mathbf{p}}} \sum_s \left[e^{-ipx} f_A(\mathbf{p}, s)\, \hat{a}(\mathbf{p}, s) \;+\; e^{+ipx} h_A(\mathbf{p}, s)\, \hat{b}^\dagger(\mathbf{p}, s) \right]_{p^0 = +E_{\mathbf{p}}}, \\ \hat{\phi}_B^\dagger(y) &= \int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{1}{2E_{\mathbf{p}}} \sum_s \left[e^{-ipy} h_B^*(\mathbf{p}, s)\, \hat{b}(\mathbf{p}, s) \;+\; e^{+ipy} f_B^*(\mathbf{p}, s)\, \hat{a}^\dagger(\mathbf{p}, s) \right]_{p^0 = +E_{\mathbf{p}}}. \end{aligned} \tag{4}$$

(Without loss of generality I assume complex fields and charged particles; for the neutral particles we would have $\hat{b} \equiv \hat{a}$ and $\hat{b}^\dagger \equiv \hat{a}^\dagger$.) *Regardless of statistics*, positive particle energies require $\hat{a}^\dagger(p,s)$ and $\hat{b}^\dagger(p,s)$ to be creation operators while $\hat{a}(p,s)$ and $\hat{b}(p,s)$ are annihilation operators, thus

$$\hat{a}^\dagger(\mathbf{p},s)\,|0\rangle \;=\; |1(\mathbf{p},s,+)\rangle\,, \quad \hat{b}^\dagger(\mathbf{p},s)\,|0\rangle \;=\; |1(\mathbf{p},s,-)\rangle\,, \quad \hat{a}(\mathbf{p},s)\,|0\rangle \;=\; \hat{b}(\mathbf{p},s)\,|0\rangle \;=\; 0. \tag{5}$$

Hence, in a Fock space of positive-definite norm,

$$\langle 0|\,\hat{a}(\mathbf{p},s)\,\hat{a}^\dagger(\mathbf{p}',s')\,|0\rangle \;=\; \langle 0|\,\hat{b}(\mathbf{p},s)\,\hat{b}^\dagger(\mathbf{p}',s')\,|0\rangle \;=\; +2E_{\mathbf{p}}(2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{p}')\delta_{s,s'}\,, \tag{6}$$

while all the other "vacuum sandwiches" of two creation / annihilation operators vanish identically. Therefore, *regardless of particles' statistics*, vacuum expectation values of products of two fields at distinct points x and y are given by

$$\langle 0|\,\hat{\phi}_A(x)\,\hat{\phi}^\dagger_B(y)\,|0\rangle \;=\; +\int\frac{d^3\mathbf{p}}{(2\pi)^3}\,\frac{e^{-ip(x-y)}}{2E_{\mathbf{p}}}\times\sum_s f_A(\mathbf{p},s)\,f^*_B(\mathbf{p},s) \tag{7}$$

and

$$\langle 0|\,\hat{\phi}^\dagger_B(y)\,\hat{\phi}_A(x)\,|0\rangle \;=\; +\int\frac{d^3\mathbf{p}}{(2\pi)^3}\,\frac{e^{+ip(x-y)}}{2E_{\mathbf{p}}}\times\sum_s h_A(\mathbf{p},s)\,h^*_B(\mathbf{p},s). \tag{8}$$

At this point, let's use the spin sums (2) and their polynomial dependence on the particle's 4–momenta (Lemma 1) to calculate

$$\langle 0|\,\hat{\phi}_A(x)\,\hat{\phi}^\dagger_B(y)\,|0\rangle \;=\; \int\frac{d^3\mathbf{p}}{(2\pi)^3}\,\frac{1}{2E_{\mathbf{p}}}\;e^{-ip(x-y)}\mathcal{F}_{AB}(p)\Big|_{p^0=+E_{\mathbf{p}}} \;=\; \mathcal{F}_{AB}(+i\partial_x)\,D(x-y) \tag{9}$$

where

$$D(x-y) \;=\; \int\frac{d^3\mathbf{p}}{(2\pi)^3}\,\frac{1}{2E_{\mathbf{p}}}\;e^{-ip(x-y)}\Big|_{p^0=+E_{\mathbf{p}}}$$

and $\mathcal{F}_{AB}(+i\partial_x)$ is a differential operator constructed as an appropriate polynomial of the $i\partial/\partial x_\mu$ instead of the p^μ. Likewise

$$\langle 0|\,\hat{\phi}^\dagger_B(y)\,\hat{\phi}_A(x)\,|0\rangle \;=\; \int\frac{d^3\mathbf{p}}{(2\pi)^3}\,\frac{1}{2E_{\mathbf{p}}}\;e^{+ip(x-y)}\mathcal{H}_{AB}(p)\Big|_{p^0=+E_{\mathbf{p}}} \;=\; \mathcal{H}_{AB}(-i\partial_x)\,D(y-x). \tag{10}$$

As explained in class, for a space-like interval $x-y$, $D(y-x) = +D(x-y)$. At the same time, the differential operators $\mathcal{F}_{AB}(+i\partial_x)$ and $\mathcal{H}_{AB}(-i\partial_x)$ are related to each other according to Lemma 2 (eqs. (3)). Therefore, regardless of particles' statistics, for $(x-y)^2 < 0$

$$\begin{aligned} \langle 0|\,\hat\phi_A(x)\,\hat\phi^\dagger_B(y)\,|0\rangle &= +\,\langle 0|\,\hat\phi^\dagger_B(y)\,\hat\phi_A(x)\,|0\rangle \quad \text{for particles of integral spin,} \\ \langle 0|\,\hat\phi_A(x)\,\hat\phi^\dagger_B(y)\,|0\rangle &= -\,\langle 0|\,\hat\phi^\dagger_B(y)\,\hat\phi_A(x)\,|0\rangle \quad \text{for particles of half-integral spin.} \end{aligned} \tag{11}$$

On the other hands, the *relativistic causality* requires for $(x-y)^2 < 0$

$$\left.\begin{aligned} \hat\phi_A(x)\,\hat\phi^\dagger_B(y) &= +\hat\phi^\dagger_B(y)\,\hat\phi_A(x) \quad \text{for bosonic fields,} \\ \hat\phi_A(x)\,\hat\phi^\dagger_B(y) &= -\hat\phi^\dagger_B(y)\,\hat\phi_A(x) \quad \text{for fermionic fields,} \end{aligned}\right\}\ \text{regardless of particle's spin.} \tag{12}$$

And the only way eqs. (11) and (12) can both hold true at the same time if *all particles of integral spin are bosons and all particles of half-integral spin are fermions*.

Indeed, for bosonic particles, all the creation and annihilation operators commute with each other except for

$$\begin{aligned} [\hat a(\mathbf{p},s), \hat a^\dagger(\mathbf{p}',s')] &= +2E_{\mathbf{p}}\,(2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{p}')\delta_{s,s'}\,, \\ [\hat b^\dagger(\mathbf{p},s), \hat b(\mathbf{p}',s')] &= -2E_{\mathbf{p}}\,(2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{p}')\delta_{s,s'}\,, \end{aligned} \tag{13}$$

and therefore the quantum fields commute or do not commute according to

$$\begin{aligned} \left[\hat\phi_A(x), \hat\phi^\dagger_B(y)\right] &= \int \frac{d^3\mathbf{p}}{(2\pi)^3}\,\frac{1}{2E_{\mathbf{p}}} \sum_s \left(e^{-ip(x-y)} f_A(p,s) f^*_B(p,s) - e^{-ip(x-y)} h_A(p,s) h^*_B(p,s)\right) \\ &= \mathcal{F}_{AB}(i\partial_x)\,D(x-y) - \mathcal{H}_{AB}(-i\partial_x)\,D(y-x) \\ &= \mathcal{F}_{AB}(i\partial_x)\,\big(D(x-y) - (-1)^{2j}\,D(y-x)\big) \end{aligned} \tag{14}$$

where j is the particle's spin, *cf.* eq. (11). For particles of integral spin, this commutator duly vanishes when points x and y are separated by a space-like distance. But for particles of half-integral spin, the two terms on the last line of eq. (14) add up instead of canceling each other, and the fields $\hat\phi_A(x)$ and $\hat\phi^\dagger_B(y)$ fail to commute — which violates relativistic causality. To avoid this violation, *the bosonic particles must have integral spins only.*

Likewise, for the fermionic particles, all the creation and annihilation operators anticommute with each other except for

$$\begin{aligned} \{\hat{a}(\mathbf{p},s),\hat{a}^{\dagger}(\mathbf{p}',s')\} &= +2E_{\mathbf{p}}\,(2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{p}')\delta_{s,s'}\,, \\ \{\hat{b}^{\dagger}(\mathbf{p},s),\hat{b}(\mathbf{p}',s')\} &= +2E_{\mathbf{p}}\,(2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{p}')\delta_{s,s'}\,, \end{aligned} \tag{15}$$

and therefore the quantum fields anticommute or do not anticommute according to

$$\begin{aligned} \left\{\hat{\phi}_A(x),\hat{\phi}_B^{\dagger}(y)\right\} &= \int\frac{d^3\mathbf{p}}{(2\pi)^3}\,\frac{1}{2E_{\mathbf{p}}}\sum_s\left(e^{-ip(x-y)}f_A(p,s)f_B^*(p,s)+e^{-ip(x-y)}h_A(p,s)h_B^*(p,s)\right) \\ &= \mathcal{F}_{AB}(i\partial_x)\,D(x-y)+\mathcal{H}_{AB}(-i\partial_x)\,D(y-x) \\ &= \mathcal{F}_{AB}(i\partial_x)\left(D(x-y)+(-1)^{2j}\,D(y-x)\right). \end{aligned} \tag{16}$$

This anticommutator vanishes when $(x-y)^2<0$ for half-integral j but not for integral j. Hence, to maintain relativistic causality, *the fermionic particles must have half-integral spins only.*

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The spin-statistics theorem works in spacetime dimensions $d\neq 4$ modulo generalization of the term 'spin'. By Wigner theorem, the massive particles form 'spin' multiplets of the $SO(d-1)$ group of space rotations while the massless particles form 'helicity' multiplets of the $SO(d-2)$ group of transverse rotations. For $d>4$, all such multiplets fall into two classes: The single-valued tensor multiplets for which $R(2\pi)=+1$, and the double-valued spinor multiplets for which $R(2\pi)=-1$. The relation between spin sums (2) follows this distinction:

$$\mathcal{H}_{AB}(-p^{\mu}) = \mathcal{F}_{AB}(+p^{\mu})\times R(2\pi), \tag{17}$$

which generalizes Lemma 2 to higher dimensions. The statistics follow the sign in eq. (17), thus *particles invariant under 2π rotations must be bosons while particles for which $R(2\pi)=-1$ must be fermions*.

For $D=3$ (two space dimensions) the situation is more complicated because the $SO(2)$ group of space rotations is abelian. Its multiplets are singlets of definite angular momentum

m_j, but in principle, this angular momentum does not have to be integer or half integer. Instead, it could be fractional, or even irrational, so a 2π rotation could multiply quanta by some complex phase $R(2\pi) = e^{2\pi i m_j} \neq \pm 1$. Such quanta are neither bosons nor fermions but *anyons* obeying fractional statistics: $|\alpha,\beta\rangle = |\beta,\alpha\rangle \times e^{\pm 2\pi i m_j}$, where the sign depends on how the two particles are exchanged in two space dimensions. Note however that even in this case, the statistics follows the spin m_j.

In condensed matter, anyons exists as 2D quasiparticles in thin layers of semiconductors in a magnetic field, and they play an important role in fractional quantum Hall effect. But in relativistic theories in $2+1$ dimensions, one cannot make anyons out of free or weakly interacting quantum fields. Indeed, the fields transform as finite multiplets of the non-abelian $SO(2,1)$ Lorentz group, which restricts their transformations under 2π space rotations to $R(2\pi) : \hat{\phi}_A \mapsto \pm\hat{\phi}_A$. Consequently, if the fields are linear combinations of creation and annihilation operators as in eqs. (4) — or even approximately linear combinations — then the operators and hence the particles must have $R(2\pi) = \pm 1$ and therefore either integer of half-integer m_j. In either case, the spin-statistics theorem works as usual: *Particles with integral m_j are bosons while particles with half-integral m_j are fermions.*

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At this point, I am done with the spins-statistics theorem itself. But to complete my argument, I need to prove the two lemmas about the spin sums (2), and that would take some group theory.

Let me start with the Lorentz symmetries of the field multiplet $\hat{\phi}_A(x)$,

$$x'^\mu = L^\mu_{\ \nu} x^\nu, \qquad \hat{\phi}'_A(x') = \sum_B M_A^{\ B}(L) \times \hat{\phi}_B(x). \tag{18}$$

The plane waves (1) of these fields transform according to

$$\begin{aligned} f_A(p,s) \ \mapsto\ f_A(Lp,s) &= \sum_B M_A^{\ B}(L) \sum_{s'} C_s^{\ s'}(L,p) \times f_B(p,s'), \\ h_A(p,s) \ \mapsto\ h_A(Lp,s) &= \sum_B M_A^{\ B}(L) \sum_{s'} C_s^{\ s'}(L,p) \times h_B(p,s'), \end{aligned} \tag{19}$$

where $C_s^{\ s'}(L,p)$ are some unitary matrices acting on polarization states s. When we take the spin sums (2), those matrices cancel out, and we get

$$\begin{aligned}\mathcal{F}_{AB}(Lp) &= \sum_{C,D} M_A^{\ C}(L)\, M_B^{*D}(L) \times \mathcal{F}_{CD}(p),\\ \mathcal{H}_{AB}(Lp) &= \sum_{C,D} M_A^{\ C}(L)\, M_B^{*D}(L) \times \mathcal{H}_{CD}(p).\end{aligned} \tag{20}$$

In other words, *the spin sums $\mathcal{F}_{AB}$ and $\mathcal{H}_{AB}$ are Lorentz-covariant functions of the momentum p.*

Covariant functions of vectors or tensors are governed by the Wigner–Eckard theorem and its generalizations.⋆ As an illustration, consider a matrix $Q_{ab}(\mathbf{v})$ of functions of a 3D vector $\mathbf{v}$ where the indices a and b run over components of some multiplet of the rotation symmetry $SO(3)$. The multiplet must be complete but may be reducible, thus $a, b \in (j_1) \oplus (j_2) \oplus \cdots$. If the matrix $Q_{ab}(\mathbf{v})$ transforms covariantly under rotations R, *i.e.*

$$Q_{ab}(R\mathbf{v}) = \sum_{c,d} M_a^{\ c}(R)\, M_b^{\ d}(R) \times Q_{cd}(\mathbf{v}), \tag{21}$$

then Wigner–Eckard theorem tells us that

$$Q_{ab}(\mathbf{v} = v\mathbf{n}) = \sum_{\ell=|j(a)-j(b)|}^{j(a)+j(b)} q_\ell(v) \sum_{m=-\ell}^{+\ell} v^\ell Y_{\ell,m}(\mathbf{n}) \times \mathrm{Clebbsch}(a, b|\ell, m), \tag{22}$$

where $q_\ell(v)$ depend only on ℓ and the magnitude v of the vector. The spherical harmonics $Y_{\ell,m}(\theta, \phi)$ are homogeneous polynomials (degree ℓ) of $\cos\theta$ and $\sin\theta \times e^{\pm i\phi}$, which makes $v^\ell \times Y_{\ell,m}(\mathbf{n})$ homogeneous (degree ℓ) polynomials of the Cartesian components v_x, v_y and v_z. Consequently, for the vector of fixed magnitude $\mathbf{v}^2 = v^2$ — which makes the $q_\ell(v)$ coefficients into constants — all the matrix elements $Q_{a,b}(\mathbf{v})$ become polynomials of (v_x, v_y, v_z) comprising terms of net degree ℓ ranging from $|j(a) - j(b)|$ to $j(a) + j(b)$.

⋆ From the mathematical point of view, the Wigner–Eckard theorem is about covariant functions of vectors or tensors in various representations of the rotation group. In QM textbooks, it is usually stated in terms of matrix elements of vector or tensor operators between states of given angular momenta.

In $d > 3$ dimensions — Euclidean or Minkowski — Wigner–Eckard theory gives us formulae similar to eq. (22), except for more complicated indexologies of spin or Lorentz multiplets and their members. In particular, in 3 + 1 Minkowski dimensions, the Lorentz algebra amounts to two non-hermitian angular momenta $\mathbf{J}^+$ and $\mathbf{J}^-$,† so instead of $|j,m\rangle$ states we have $|j^+,m^+,j^-,m^-\rangle$. Also, the 4D vectors have $j^+ = j^- = \frac{1}{2}$, so the 4D spherical harmonics (or rather hyperboloid harmonics in Minkowski spacetime) all have $j^+ = j^- = J$, but J takes both integer and half-integer values. Consequently, the Wigner–Eckard theorem for Lorentz-covariant functions $\mathcal{F}_{AB}$ and $\mathcal{H}_{AB}$ of the 4-momentum p^μ says:

$$\begin{aligned}
\mathcal{F}_{AB}(p) &= \sum_{J=J_{\min}}^{J_{\max}} f_J(M) \sum_{\substack{-J\le m^+\le J\\ -J\le m^-\le J}} M^{2J}\mathcal{Y}_{J,m^+,m^-}(p^\mu/M)\times \text{Clebbsch}(A,B|J,m^+,J,m^-),\\
\mathcal{H}_{AB}(p) &= \sum_{J=J_{\min}}^{J_{\max}} h_J(M) \sum_{\substack{-J\le m^+\le J\\ -J\le m^-\le J}} M^{2J}\mathcal{Y}_{J,m^+,m^-}(p^\mu/M)\times \text{Clebbsch}(A,B|J,m^+,J,m^-),
\end{aligned} \tag{24}$$

for some functions $f_J(M)$ and $h_J(M)$ of the particle mass $M = \sqrt{p^2}$. Similar to 3D spherical harmonics, the 4D hyperboloid harmonics $M^{2J}\mathcal{Y}_{J,m^+,m^-}(p^\mu/M)$ are homogeneous polynomials of degree $2J$ in (p^0,p^1,p^2,p^3). Consequently, *for fixed mass M and on-shell momenta*, all the f_J and h_J are constants and *the spin sums $\mathcal{F}_{AB}(p)$ and $\mathcal{H}_{AB}(p)$ are polynomial functions of $p^{x,y,z}$ and $E = \sqrt{\mathbf{p}^2 + M^2}$*.

In other Minkowski dimensions, there are formulae similar to eqs. (24), except that there are more m-like indices, the summation ranges are different, and the Clebbsch–Gordan coefficients are messier than in 3D or 4D. But in all dimensions, the hyperboloid harmonics $\mathcal{Y}_{J,m,\ldots,m}(n^\mu)$ are homogeneous polynomials of some degree in $(n^0,n^1,\ldots,n^{d-1})$. Consequently, for fixed mass M and on-shell momenta, $\mathcal{F}_{AB}(p)$ and $\mathcal{H}_{AB}(p)$ can be written as polynomial functions of the $(p^1,\ldots,p^{d-1})$ and $E = \sqrt{\mathbf{p}^2 + M^2}$.

† In terms of the usual Lorentz generators $J^{ij} = \epsilon^{ijk}J^k$ of rotations and $J^{i0} = -J^{0i} = K^i$ of the boosts, $\mathbf{J}^\pm = \frac{1}{2}(\mathbf{J} \pm i\mathbf{K})$. The $\mathbf{J}^\pm$ operators are not hermitian, but they do have commutation relation of two independent angular momenta,

$$[J^\pm_i, J^\pm_j] = i\epsilon_{ijk}J^\pm_k, \qquad [J^\pm_i, J^\mp_j] = 0. \tag{23}$$

Once we have the $\mathcal{F}_{AB}(p)$ and $\mathcal{H}_{AB}(p)$ written as polynomials of the d components of p^μ, we may analytically continue them as polynomials to arbitrary off-shell momenta ($p^0 \neq E_{\mathbf{p}}$), or even to complex momenta. The coefficients of such polynomials may be non-polynomial functions of M^2 — for example, $(\not{p} \pm M)_{\alpha\beta}$ for the Dirac spinor fields, or $(-g^{\mu\nu} + M^{-2}p^\mu p^\nu)$ for the massive vector fields — but that's OK because off shell M^2 is just a constant unrelated to the p^μ.

Technically, the off-shell continuations of the polynomials $\mathcal{F}_{AB}(p^\mu)$ and $\mathcal{H}_{AB}(p^\mu)$ are ambiguous modulo terms of the form $(p^2 - M^2) \times$ some polynomial of p^μ, because all such terms vanish identically on-shell. But physically, this ambiguity is irrelevant to any 'vacuum sandwiches' of two fields or (anti) commutators of fields. For example, consider eq. (9): If I change the analytic continuation of the $\mathcal{F}_{AB}(p)$ to the off-shell momenta by $(p^2 - M^2) \times$ some polynomial of p^μ, the differential operator $\mathcal{F}_{AB}(i\partial_x)$ will change by $(\partial_x^2 + M^2) \times$ some differential operator. But this change will have no effect on the $\mathcal{F}_{AB}(i\partial_x)D(x-y)$ on the right hand side of eq. (9) because $(\partial_x^2 + M^2)D(x-y) = 0$. Likewise, ambiguity in analytically continuing the $\mathcal{H}_{AB}(p^\mu)$ to off-shell momenta makes no difference to the right hand side of eq. (10).

This completes my proof of **Lemma 1**: *In any dimension, the spin sums $\mathcal{F}_{AB}(p)$ and $\mathcal{H}_{AB}(p)$ may be analytically continued to off-shell momenta (or even complex momenta) as polynomials in the p^μ. The continuation is ambiguous modulo polynomials proportional to the $(p^2 - M^2)$, but this ambiguity is physically irrelevant.*

Finally, I need to prove **Lemma 2**. For simplicity, I shall work in $d = 3 + 1$ dimensions only, although the Lemma works in all Minkowski dimensions according to eq. (17). The quantum fields $\hat{\phi}_A(x)$ form some kind of a Lorentz multiplet; allowing for its reducibility, we generally have $A \in (j_1^+, j_1^-) \oplus (j_2^+, j_2^-) \oplus \cdots$. Now consider the Wigner–Eckard eqs. (24) for indices belonging to particular irreducible multiplets, $A \in (j_A^+, j_A^-)$ and $B \in (j_B^+, j_B^-)$. For every hyperboloid harmonic $\mathcal{Y}_{J,m^+,m^-}$ which contributes to the $\mathcal{F}_{AB}(p)$ and $\mathcal{H}_{AB}(p)$, the angular momenta (J, j_A^+, j_B^+) should satisfy the triangle inequality, and so should the (J, j_A^-, j_B^-). Hence, the summation over J ranges

$$\begin{aligned} \text{from} \quad J_{\min} &= \max\left(|j_A^+ - j_B^+|, |j_A^- - j_B^-|\right) \\ \text{to} \quad J_{\max} &= \min\left((j_A^+ + j_B^+), (j_a^- + j_B^-)\right), \end{aligned} \tag{25}$$

and for each J, both m^+ and m^- range from $-J$ to $+J$. Moreover, J, m^+, and m^- are either all integers or all half-integers according to

$$(-1)^{2J} = (-1)^{2m^+} = (-1)^{2m^-} = (-1)^{2j_A^+} \times (-1)^{2j_B^+} = (-1)^{2j_A^-} \times (-1)^{2j_B^-}. \quad (26)$$

Now, the hyperboloid harmonics $\mathcal{Y}_{J,m^+,m^-}(p^\mu/M)$ are *homogeneous* polynomials of degree $2J$, and according to eq. (26), all the harmonics contributing to any particular matrix element $\mathcal{F}_{AB}$ or $\mathcal{H}_{AB}$ have similar $2J$ modulo 2. Therefore, each matrix element $\mathcal{F}_{AB}$ or $\mathcal{H}_{AB}$ is either an even polynomial of the p^μ or an odd polynomial, and when we analytically continue such polynomials via off-shell momenta to $-p^\mu = (-E, -\mathbf{p})$, we find that

$$\begin{aligned} \mathcal{F}_{AB}(-p^\mu) &= \mathcal{F}_{AB}(+p^\mu) \times (-1)^{2j_A^+}(-1)^{2j_B^+}, \\ \mathcal{H}_{AB}(-p^\mu) &= \mathcal{H}_{AB}(+p^\mu) \times (-1)^{2j_A^+}(-1)^{2j_B^+}, \end{aligned} \quad (27)$$

These sign relations provide the first half of our proof. The second half is based on the CPT theorem which states that simultaneous reversal of all charges (C), of space parity (P), and of time's direction (T) is always an exact symmetry of any quantum field theory. This symmetry acts on quantum fields according to

$$\begin{aligned} \mathbf{CPT}: \ \hat{\phi}_A(x) &\mapsto \hat{\phi}_A^\dagger(-x) \times (-1)^{2J_A^-}, \\ \mathbf{CPT}: \ \hat{\phi}_A^\dagger(x) &\mapsto \hat{\phi}_A(-x) \times (-1)^{2J_A^+}, \end{aligned} \quad (28)$$

where the $(-1)^{2J_A^-}$ sign in the first line is the (j_A^+, j_A^-) representation of the proper-but-not-orthochronous Lorentz transform $\mathbf{PT}: x^\mu \to -x^\mu$.[⋆] In the second line, this sign is changed to $(-1)^{2J_A^+}$ because hermitian conjugation exchanges $\mathbf{J}^+ \leftrightarrow \mathbf{J}^-$ and hence $j^+ \leftrightarrow j^-$ of any field: If $\hat{\phi}_A \in (j^+, j^-)$, then $\hat{\phi}_A^\dagger \in (j^-, j^+)$. Applying eqs. (28) to the plane waves (1) and re-interpreting the results in terms of CPT action on the particles — preserving E and $\mathbf{p}$

⋆ For example, Dirac spinor fields comprise a reducible Lorentz multiplet $(j^+ = \frac{1}{2}, j^- = 0) \oplus (j^+ = 0, j^- = \frac{1}{2})$. The $(-1)^{2j_A^-}$ acts on this multiplet as the $-\gamma^5$ matrix $\begin{pmatrix} +1 & 0 \\ 0 & -1 \end{pmatrix}$ in the Weyl basis. And indeed, the CPT symmetry acts on Dirac fields according to $\hat{\Psi}'(x) = -\gamma^5 \hat{\Psi}^*(-x)$.

but reversing spins and charges — we find that

$$h_A(+\mathbf{p},-s) \;=\; f_A(+\mathbf{p},+s)\times(-1)^{2J_A^-}, \qquad h_A^\dagger(+\mathbf{p},-s) \;=\; f_A^\dagger(+\mathbf{p},+s)\times(-1)^{2J_A^+}. \quad (29)$$

This gives us a relation between the positive-frequency and the negative-frequency plane waves, and consequently between the two spin sums (2):

$$\mathcal{H}_{AB}(p) \;=\; \mathcal{F}_{AB}(p)\times(-1)^{2J_A^-}(-1)^{2J_B^+}. \quad (30)$$

I have derived this relation for the actual spin sums and hence for the on-shell momenta only. Analytic continuation of the $\mathcal{F}_{AB}(p^\mu)$ and $\mathcal{H}_{AB}(p^\mu)$ to the off-shell momenta is ambiguous modulo polynomials of p^μ proportional to the (p^2-M^2), and this ambiguity may spoil the relation (30) for the off-shell momenta. But fortunately, the ambiguities of this kind are physically irrelevant (*cf.* the argument two pages above), and so without loss of generality, we can impose the relation (30) to the off-shell $\mathcal{F}_{AB}(p^\mu)$ and $\mathcal{H}_{AB}(p^\mu)$.

Now, combining the sign relations (27) and (30), we get

$$\mathcal{H}_{AB}(-p^\mu) \;=\; \mathcal{F}_{AB}(+p^\mu)\times(-1)^{2j_A^+}(-1)^{2j_A^-}. \quad (31)$$

Consider the sign factor on the right hand side: Although different field indices A may belong to different (j^+,j^-) Spin(3,1) multiplets, the net sign $(-1)^{2j_A^+}(-1)^{2j_A^-}$ has to be the same for all A because it determines how the fields transform under 2π rotations. Also, particles' spin j follows from adding j^+ and j^- of the fields as angular momenta, hence

$$\forall A:\quad \left|j_A^+-j_A^-\right| \;\le\; j \;\le j_A^+ + j_A^- \quad\text{and}\quad (-1)^{2j_A^+}(-1)^{2j_A^-} \;=\; (-1)^{2j}. \quad (32)$$

Consequently, eqs. (31) become

$$\mathcal{H}_{AB}(-p^\mu) \;=\; (-1)^{2j}\times\mathcal{F}_{AB}(+p^\mu), \quad (33)$$

or equivalently eqs. (11) of **Lemma 2**. This completes my proof.