RELATIVISTIC ENERGY AND MOMENTUM

Non-relativistically, the momentum and the energy of a free particle are related to its velocity $\mathbf{v}$ as

$$\mathbf{p} = m\mathbf{v}, \quad E = \text{const} + \tfrac{1}{2}m\mathbf{v}^2, \tag{1}$$

where m is the particle's mass. In special relativity, the relations are similar for particles moving much slower than light, but for fast particles there are more complicated formulae

$$\mathbf{p} = \gamma m\mathbf{v} = \frac{m\mathbf{v}}{\sqrt{1-(\mathbf{v}/c)^2}} = m\mathbf{v}\left(1 + \frac{\mathbf{v}^2}{2c^2} + O\left(\frac{v^4}{c^4}\right)\right) \tag{2}$$

and

$$E = \gamma mc^2 = mc^2 + \tfrac{1}{2}m\mathbf{v}^2 + \frac{3mv^4}{8c^2} + O\left(\frac{mv^6}{c^4}\right). \tag{3}$$

The m in these formulae is the *rest mass*, *i.e.* the mass of the particle in its rest frame. Nowadays, when we say *mass* we mean the rest mass, but in the early days of the relativity theory the name mass was commonly used for the *relativistic inertia* or *relativistic mass*

$$\mathcal{M}(v) = \gamma(v) \times m; \tag{4}$$

in terms of this relativistic mass

$$\mathbf{p} = \mathcal{M}(v)\mathbf{v} \quad \text{and} \quad E = \mathcal{M}(v)c^2. \tag{5}$$

In particular, Einstein's famous equation $E = \mathcal{M}c^2$ was written in terms of the relativistic mass $\mathcal{M}(v)$ rather than the rest mass m!

Derivation of relativistic momentum and energy.

Before we explore the consequences of the relativistic formulae (2) for the momentum and the energy, let's derive them from the conservation laws. In any collision of particles, the net momentum is conserved,

$$\sum_i^{\text{particles}} \mathbf{p}_i^{\text{after}} = \sum_i^{\text{particles}} \mathbf{p}_i^{\text{before}}, \tag{6}$$

and in an elastic collision the net kinetic energy is also conserved,

$$\sum_i^{\text{particles}} K_i^{\text{after}} = \sum_i^{\text{particles}} K_i^{\text{before}}, \tag{7}$$

Moreover, these conservation laws must work in any inertial frame of reference! Alas, plugging non-relativistic formulae (1) for the particles' energies and momenta into these conservation laws make them invariant under the Galilean boosts $\mathbf{v}' = \mathbf{u} + \mathbf{v}$, but not under the Lorentz boosts which act non-linearly on the velocities.

To repair the law of momentum conservation, we need to change Newton's formula $\mathbf{p} = m\mathbf{v}$ to

$$\mathbf{p} = \mathcal{M}(v)\mathbf{v} \tag{8}$$

with some velocity-dependent inertia $\mathcal{M}(v)$, although by the rotational symmetry of the 3-space it should depend only on the speed $v = |\mathbf{v}|$ but not on the velocity's direction. For the moment, all we know is that $\mathcal{M}$ is some analytic function of $(\mathbf{v}/c)^2$; we shall determine its exact form from the Lorentz invariance of the momentum conservation in collisions.

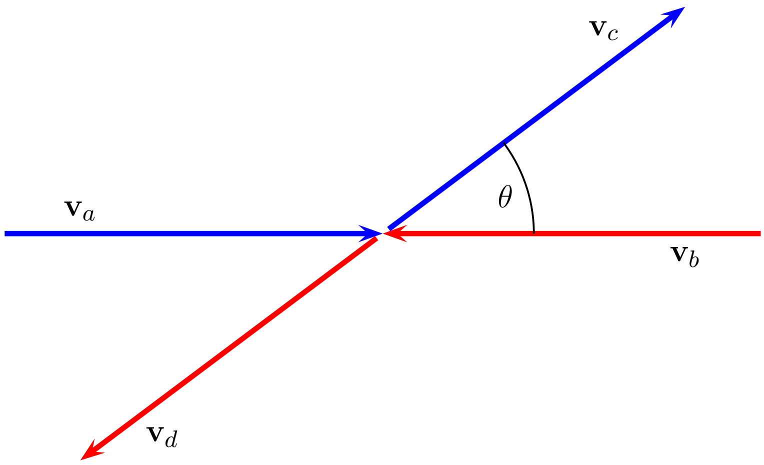
Indeed, consider an elastic collision of two similar particles in the center-of-mass frame. Relativistically, this frame is defined as the frame where the net momentum is zero before the collision and hence also after the collision. Thus, in the CM frame, the two particles collide head-on — so their momenta is equal in magnitude and opposite in direction, — and after the collision they also fly in opposite directions. In terms of velocities,

$$\begin{aligned} \mathcal{M}(v_a)\mathbf{v}_a \;+\; \mathcal{M}(v_b)\mathbf{v}_b \;&=\; 0 \quad \text{before the collision,} \\ \mathcal{M}(v_c)\mathbf{v}_c \;+\; \mathcal{M}(v_d)\mathbf{v}_d \;&=\; 0 \quad \text{after the collision,} \end{aligned} \tag{9}$$

and since $\mathcal{M}(-\mathbf{v}) = \mathcal{M}(+\mathbf{v})$ it follows that

$$\mathbf{v}_b \;=\; -\mathbf{v}_a\,, \quad \mathbf{v}_c \;=\; -\mathbf{v}_d\,, \tag{10}$$

or graphically



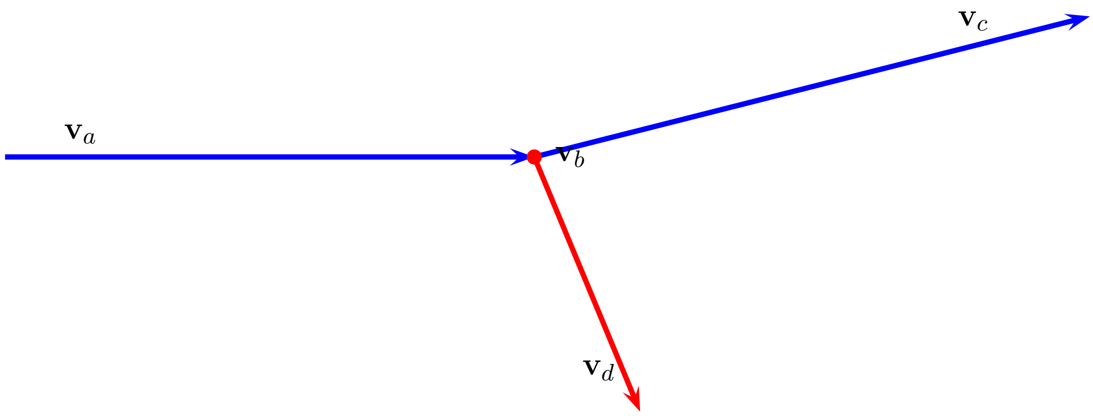
Also, assuming the kinetic energy $K(\mathbf{v})$ is some monotonically increasing function of the speed $|\mathbf{v}|$ which does not depend on the velocity's direction, conservation of energy in an elastic collision makes the particle's speeds after the collision equal to their speeds before the collision, thus

$$|\mathbf{v}_a| \;=\; |\mathbf{v}_b| \;=\; |\mathbf{v}_c| \;=\; |\mathbf{v}_d| \;=\; v. \tag{11}$$

Now let's consider the same collision in the lab frame where one of the particles was at rest before the collisions. Lorentz-boosting all the velocities by $\mathbf{u} = -\mathbf{v}_b = +\mathbf{v}_a$, we find (in the coordinates where x axis points in the direction of $\mathbf{v}_a$)

$$\begin{aligned}
v_{a,x}^{\rm lab} &= \frac{2v}{1+\beta^2}\,, \\
v_{a,y}^{\rm lab} &= v_{b,x}^{\rm lab} \;=\; v_{b,y}^{\rm lab} \;=\; 0, \\
v_{c,x}^{\rm lab} &= \frac{v(1+\cos\theta)}{1+\beta^2\cos\theta}\,, \\
v_{c,y}^{\rm lab} &= \frac{v\sin\theta}{\gamma(1+\beta^2\cos\theta)}\,, \\
v_{d,x}^{\rm lab} &= \frac{v(1-\cos\theta)}{1-\beta^2\cos\theta}\,, \\
v_{d,y}^{\rm lab} &= \frac{-v\sin\theta}{\gamma(1-\beta^2\cos\theta)}\,,
\end{aligned} \tag{12}$$

or graphically



Note that after the collision $v_{d,y}^{\rm lab} \neq -v_{c,y}^{\rm lab}$, so to assure momentum conservation in the y direction we must have velocity dependent relativistic inertia $\mathcal{M}(v)$; specifically, we need

$$\mathcal{M}(v_d^{\rm lab}) \times \frac{-v\sin\theta}{\gamma(1-\beta^2\cos\theta)} \;+\; \mathcal{M}(v_c^{\rm lab}) \times \frac{v\sin\theta}{\gamma(1+\beta^2\cos\theta)} \;=\; 0 \tag{13}$$

and hence

$$\frac{\mathcal{M}(v_c^{\rm lab})}{\mathcal{M}(v_d^{\rm lab})} \;=\; \frac{1+\beta^2\cos\theta}{1-\beta^2\cos\theta}\,. \tag{14}$$

In particular, consider a grazing collision with a very small scattering angle θ. In the limit of $\theta \to 0$, we have

$$\mathbf{v}_d^{\mathrm{lab}} \;\to\; 0 \quad \text{while} \quad \mathbf{v}_d^{\mathrm{lab}} \;\to\; \mathbf{v}_a^{\mathrm{lab}}, \tag{15}$$

so eq. (14) becomes

$$\frac{\mathcal{M}(v_a^{\mathrm{lab}})}{\mathcal{M}(0)} \;=\; \frac{1+\beta^2}{1-\beta^2}\,. \tag{16}$$

Moreover, the expression on the RHS here is nothing but $\gamma(v_a^{\mathrm{lab}})$; indeed,

$$\frac{1}{\gamma^2(v_a^{\mathrm{lab}})} \;=\; 1\;-\;(v_a^{\mathrm{lab}}/c)^2 \;=\; 1\;-\;\left(\frac{2\beta}{1+\beta^2}\right)^2 \;=\; \left(\frac{1-\beta^2}{1+\beta^2}\right)^2 \quad\Longrightarrow\quad \gamma(v_a^{\mathrm{lab}}) \;=\; \frac{1+\beta^2}{1-\beta^2}\,. \tag{17}$$

Thus, momentum conservation requires

$$\frac{\mathcal{M}(v_a^{\mathrm{lab}})}{\mathcal{M}(0)} \;=\; \gamma(v_a^{\mathrm{lab}}) \tag{18}$$

and hence for any other speed v'

$$\mathcal{M}(v') \;=\; \gamma(v')\times\mathcal{M}(0). \tag{19}$$

Given this formula for the relativistic inertia $\mathcal{M}(v)$ — and hence the momentum

$$\mathbf{p} \;=\; \mathcal{M}(v)\mathbf{v} \;=\; \gamma(v)m_{\mathrm{rest}}\mathbf{v} \tag{20}$$

— the Second Law of Newton becomes

$$\mathbf{F} \;=\; \frac{d\mathbf{p}}{dt} \;=\; m_{\mathrm{rest}}\,\frac{d(\gamma\mathbf{v})}{dt} \tag{21}$$

In terms of the acceleration $\mathbf{a} = d\mathbf{v}/dt$,

$$\frac{d(\gamma\mathbf{v})}{dt} \;=\; \gamma\mathbf{a} \;+\; \left(\frac{d\gamma}{dt} \;=\; \gamma^3\times\frac{(\mathbf{v}\cdot\mathbf{a})}{c^2}\right)\mathbf{v} \;=\; \gamma\mathbf{a}_\perp \;+\; (\gamma+\gamma^3\beta^2 \;=\; \gamma^3)\mathbf{a}_\parallel\,, \tag{22}$$

hence

$$\mathbf{a}_\perp \;=\; \frac{\mathbf{F}_\perp}{\gamma m_{\mathrm{rest}}} \quad \text{but} \quad \mathbf{a}_\parallel \;=\; \frac{\mathbf{F}_\parallel}{\gamma^3 m_{\mathrm{rest}}}\,. \tag{23}$$

In the early days of the special relativity theory, the γm_{rest} was called the *transverse relativistic mass* while $\gamma^3 m_{\mathrm{rest}}$ was called the *longitudinal relativistic mass*. This proliferation of different things called some kind of mass was rather confusing, so eventually the name *mass* was reserved for the rest mass $m = \mathcal{M}(0)$.

Now consider the relativistic kinetic energy $K(v)$. Since it depends only on the speed $|\mathbf{v}|$, we may write it as a function of $\gamma(v)$,

$$K(v) \;=\; f(\gamma(v)) \tag{24}$$

for example, the non-relativistic kinetic energy can be written as

$$K_{\text{non-rel}}(v) \;=\; \frac{mc^2}{2}\left(1 \;-\; \frac{1}{\gamma^2(v)}\right). \tag{25}$$

In an elastic collision, the net kinetic energy is conserved, $K_c + K_d = K_a + K_b$, so in the lab frame we should have

$$f(\gamma_c^{\text{lab}}) \;+\; f(\gamma_d^{\text{lab}})) \;=\; f(\gamma_a^{\text{lab}}) \;+\; f(\gamma_b^{\text{lab}}) \tag{26}$$

But in the lab frame $\gamma_b^{\text{lab}} = 1$ and we have already calculated

$$\gamma_a^{\text{lab}} \;=\; \frac{1+\beta^2}{1-\beta^2}\,. \tag{27}$$

In a similar fashion

$$\gamma_c^{\text{lab}} \;=\; \frac{1+\beta^2\cos\theta}{1-\beta^2}\,, \quad \gamma_d^{\text{lab}} \;=\; \frac{1-\beta^2\cos\theta}{1-\beta^2}\,; \tag{28}$$

indeed

$$\begin{aligned} \frac{1}{\gamma^2(v_c^{\text{lab}})} \;&=\; 1 \;-\; \frac{(v_{c,x}^{\text{lab}})^2 + (v_{c,y}^{\text{lab}})^2}{c^2} \;=\; 1 \;-\; \frac{\beta^2(1+\cos\theta)^2}{(1+\beta^2\cos\theta)^2} \;-\; \frac{\beta^2\sin^2\theta}{\gamma^2(1+\beta^2\cos\theta)^2} \\ &=\; \frac{(1+\beta^2\cos\theta)^2 \;-\; \beta^2(1+\cos\theta)^2 \;-\; \beta^2(1-\beta^2)\sin^2\theta}{(1+\beta^2\cos\theta)^2} \end{aligned} \tag{29}$$

where

$$
\begin{aligned}
\text{the numerator} \;&=\; (1+\beta^2\cos\theta)^2 \;-\; \beta^2(1+\cos\theta)^2 \;-\; \beta^2(1-\beta^2)\sin^2\theta \\
&=\; 1 \;+\; 2\beta^2\cos\theta \;+\; \beta^4\cos^4\theta \\
&\qquad -\; \beta^2 \;-\; 2\beta^2\cos\theta \;-\; \beta^2\cos^2\theta \;-\; \beta^2(1-\beta^2)\sin^2\theta \\
&=\; 1 \;-\; \beta^2 \;-\; \beta^2(1-\beta^2)\cos^2\theta \;-\; \beta^2(1-\beta^2)\sin^2\theta \\
&=\; 1 \;-\; \beta^2 \;-\; \beta^2(1-\beta^2) \;=\; (1-\beta^2)^2
\end{aligned}
\tag{30}
$$

,

thus

$$
\frac{1}{\gamma^2(v_c^{\mathrm{lab}})} \;=\; \left(\frac{1-\beta^2}{1+\beta^2\cos\theta}\right)^2 \quad\Longrightarrow\quad \gamma_c^{\mathrm{lab}} \;=\; \frac{1+\beta^2\cos\theta}{1-\beta^2}\,, \tag{31}
$$

and likewise

$$
\gamma_d^{\mathrm{lab}} \;=\; \frac{1-\beta^2\cos\theta}{1-\beta^2}\,. \tag{32}
$$

With these formulae for the $\gamma_{a,b,c,d}$ in the lab frame, the kinetic energy conservation in the elastic collision requires

$$
f\left(\frac{1+\beta^2\cos\theta}{1-\beta^2}\right) \;+\; f\left(\frac{1-\beta^2\cos\theta}{1-\beta^2}\right) \;=\; f\left(\frac{1+\beta^2}{1-\beta^2}\right) \;+\; f(1), \tag{33}
$$

and this equality must hold for any angle θ and any $\beta \leq 1$. Mathematically, the only analytic function which obeys this requirements is the linear function

$$
f(\gamma) \;=\; A\times\gamma \;+\; B \tag{34}
$$

for some constants A and B. In terms of the kinetic energy, this means

$$
K(v) \;=\; A\times\gamma(v) \;+\; B, \tag{35}
$$

and in order to agree with the non-relativistic limit of the kinetic energy, we need $A = mc^2$

and $B = -A$, thus

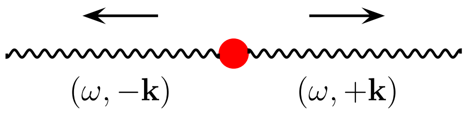
$$K(v) \;=\; mc^2 \times (\gamma(v) - 1) \;=\; \tfrac{1}{2}mv^2 \;+\; O\left(\frac{mv^4}{c^2}\right). \tag{36}$$

Or in terms of the total energy of the particle,

$$E \;=\; mc^2 \times \gamma(v) \;+\; \text{const} \;=\; \mathcal{M}(v)c^2 \;+\; \text{const}. \tag{37}$$

To find the constant term here, we need to consider an inelastic process in which the net energy of all kinds is conserved but the kinetic energy is not. And as Einstein found out, the mass is also not conserved!

To see how this works consider a nucleus emitting two photons of equal frequencies in opposite directions,



The two photons have opposite momenta $\pm\mathbf{k}$, so their net momentum is zero and there is no recoil — the nucleus initially at rest remains at rest. However, its internal energy U drops by the energy of the two photons,

$$\Delta U \;=\; U_1 \;-\; U_2 \;=\; 2\hbar\omega \tag{38}$$

Now consider the same process in a different frame where the nucleus moves with velocity $\mathbf{v}$ in the direction of one of the photons. In this frame, the photons have different frequencies due to Doppler effect,

$$\omega_1 \;=\; \gamma(1+\beta) \times \omega, \quad \omega_2 \;=\; \gamma(1-\beta) \times \omega, \tag{39}$$

and hence larger net energy

$$\hbar\omega_1 \;+\; \hbar\omega_2 \;=\; 2\hbar\omega\times\gamma. \tag{40}$$

This energy comes from the net kinetic + internal energy of the nucleus $E = K + U$, hence

$$\Delta E \;=\; E_1 \;-\; E_2 \;=\; \Delta K \;+\; \Delta U \;=\; \gamma\times 2\hbar\omega. \tag{41}$$

But the internal energy change is only $\Delta U = 2\hbar\omega$, so the kinetic energy must also change by

$$\Delta K \;=\; (\gamma-1)\times 2\hbar\omega. \tag{42}$$

On the other hand, we know that in its original frame the nucleus does not recoil, so in any other frame its velocity should also stay constant, whatever it was. And the only way to change the kinetic energy of a nucleus without changing its velocity is by changing its mass,

$$m_2 \;=\; m_1 \;-\; \Delta m, \tag{43}$$

such that

$$(\gamma-1)c^2\times\Delta m \;=\; \Delta K \;=\; (\gamma-1)\times 2\hbar\omega, \tag{44}$$

thus

$$\Delta m \;=\; \frac{2\hbar\omega}{c^2} \tag{45}$$

Note velocity independence of this formula!

In terms of the internal energy U of the nucleus — *i.e.*, the energy it has when it's not moving —

$$\Delta m \;=\; \frac{\Delta U}{c^2}\,, \tag{46}$$

or in terms of the net energy $E = K + U$,

$$\Delta E \;=\; c^2\times\Delta\mathcal{M}(v). \tag{47}$$

The same formula applies to other inelastic processes, even when there is a recoil, and this have lead Einstein to his famous formula

$$E = \mathcal{M}(v) \times c^2 = \gamma(v) mc^2 = \frac{mc^2}{\sqrt{1-(v/c)^2}}. \tag{48}$$

In particular, a particle at rest has a tremendous hidden energy $E_0 = mc^2$. In any inelastic process, this energy increases or decreases, and this leads to an increase or decrease of the net mass of the system.

In general, any change of net energy changes the net mass of the system, even in such mundane non-relativistic processes as pool-ball collisions or chemical reactions, although the resulting Δm is too small to measure. But in nuclear reactions Δm is typically of the order $10^{-3} \times m$, and that can be easily measured by a mass spectrometer. In fact, once can calculate the energy releases or consumed in some nuclear reaction by simply looking up the masses of initial and final nuclei and calculating the Δm. For example, in the deuterium-tritium fusion reaction

$$D_1^2 + T_1^3 = \mathrm{He}_2^4 + n_0^1, \tag{49}$$

the masses are

$$\begin{aligned} m(D) &= 2.014\,102 \text{ u}, \\ m(T) &= 3,016\,049 \text{ u}, \\ m(\mathrm{He}^4) &= 4,002\,602 \text{ u}, \\ m(n) &= 1.008,\,665 \text{ u}, \end{aligned} \tag{50}$$

so in the fusion reaction the net mass is reduced by

$$\Delta m = 18,884 \cdot 10^{-3} \text{ u}, \tag{51}$$

where u is the atomic mass unit,

$$\text{u} = 1.660\,539 \cdot 20^{-27} \text{ kg} = 931.494 \text{ MeV}/c^2. \tag{52}$$

Consequently, the fusion reaction should release energy

$$\Delta E = (18,884 \cdot 10^{-3} \text{ u}) \times c^2 = 17.59 \text{ MeV}.$$

and the experimentally measured fusion energy indeed agrees with this value.

In a more extreme example, an electron and a positron can annihilate each other, so their entire hidden energy $2 \times m_e c^2$ is converted to the energy of the photons produced in the annihilation. On the other hand, when highly energetic elementary particles collide, their kinetic energies can be converted to the mass of some heavy new particles. For example, at the Large Hadron Collider at CERN, the protons are accelerated till their γ factor reaches about 7000; in other words, their kinetic energy is 7000 larger than the rest energy $m_p c^2$. In GeV units, each proton has energy about 6500 GeV, and when two protons collide, a notable fraction of the net 13,000 GeV energy is converted to the masses of many particles particles created in the collision. Some of these particles can be much heavier than the original proton, for example the Higgs particle has mass $M_H = 126\ \mathrm{GeV}/c^2$.

ENERGY–MOMENTUM 4–VECTOR.

The relativistic energy and momentum of a free particle form a Lorentz 4–vector

$$p^\mu \;=\; \left(\frac{E}{c},\mathbf{p}\right) \;=\; mu^\mu \tag{53}$$

where u^μ is the 4–velocity of the particle. Indeed, in components

$$p^i \;=\; \gamma v^i m \;=\; u^i m \quad\text{and}\quad p^0 \;=\; \frac{E}{c} \;=\; \gamma cm \;=\; u^0 m. \tag{54}$$

Consequently, the net energy and momentum of any multi-particle system also form a 4–vector

$$P^\mu_{\rm net} \;=\; \left(\frac{E_{\rm net}}{c}\,,\mathbf{P}_{\rm net}\right) \;=\; \sum_a^{\rm particles} m_a u^\mu_a \tag{55}$$

which transforms under Lorentz symmetries as any other 4–vector,

$$\begin{aligned}
E'^{\,\rm net} &= \gamma(E^{\rm net} - vP^{\rm net}_{\|}),\\
P'^{\,\rm net}_{\|} &= \gamma\left(P^{\rm net}_{\|} - \frac{v}{c^2}\,E^{\rm net}\right),\\
\mathbf{P}'^{\,\rm net}_{\perp} &= \mathbf{P}^{\rm net}_{\perp}\,.
\end{aligned} \tag{56}$$

Therefore, *if the energy and the momentum are conserved in one frame of reference, then they are also conserved in any other reference frame!*

Now consider the Lorentz square of the energy-momentum 4–vector p^μ. For a single particle we have

$$p^\mu p_\mu \;=\; \frac{E^2}{c^2}\;-\;\mathbf{p}^2\;=\;(\gamma mc)^2\;-\;(\gamma m\mathbf{v})^2\;=\;m^2\gamma^2(c^2-\mathbf{v}^2)\;=\;m^2c^2, \tag{57}$$

or in terms of the particle's 4–velocity u^μ,

$$p^\mu \;=\; mu^\mu \quad\Longrightarrow\quad p^\mu p_\mu \;=\; m^2\times u^\mu u_\mu \;=\; m^2c^2. \tag{58}$$

Either way, the energy and the momentum of a particle are related as

$$E^2 \;=\; c^2\mathbf{p}^2 \;+\; m^2c^4. \tag{59}$$

In the 4–momenum space, this formula defines a hyperbolic hypersurface called the *mass shell*.

In the non-relativistic limit, the mass shell condition becomes

$$E \;=\; mc^2 \;+\; \frac{p^2}{2m} \;-\; \frac{p^4}{8m^3c^2} \;+\; \cdots, \tag{60}$$

while in the ultra-relativistic limit $\gamma \gg 1$ and hence $|\mathbf{p}| \gg mc$ we get

$$E \;=\; c|\mathbf{p}| \;+\; \frac{m^2c^3}{2|\mathbf{p}|} \;+\; \cdots. \tag{61}$$

In particular, for the massless particles like the photons

$$E \;=\; c|\mathbf{p}|, \tag{62}$$

in perfect agreement with $\omega = c|\mathbf{k}|$ for the EM waves in vacuum and the quantum formulae $E = \hbar\omega$ and $\mathbf{p} = \hbar\mathbf{k}$.

In general, massless particles (*i.e.*, particles having zero *rest mass*) must move with the speed of light — otherwise, they would have zero energy and zero momentum — while particles with non-zero rest masses must move slower than light. Among the presently known particles, only the photon is exactly massless. The neutrinos do have masses, although they are very small — less then eV/c^2 – and since we cannot detect neutrinos with energies much less than an MeV, all the neutrinos we have ever detected were ultra-relativistic with $\gamma > 10^6$, and their speeds were experimentally indistinguishable from the speed of light. The only reason we know about the neutrino masses is because of the quantum oscillations between the 3 neutrino species!

To see how the neutrino oscillations work, note that in quantum mechanics, eq. (61) for the energy of an ultra-relativistic particle becomes the Hamiltonian operator

$$\hat{H} \approx c|\mathbf{p}| + \frac{c^3}{2|\mathbf{p}|}\hat{M}^2 \tag{63}$$

where $\hat{M}^2$ is the 3×3 matrix in the Hilbert space of the neutrino species. This matrix is non-diagonal in the basis in which we make and detect neutrinos, and this causes neutrinos to oscillate from one species to another. Specifically, if the neutrino is created at time $t = 0$ in a state vector $|\psi_0\rangle$, then after it has traveled through long distance L in time $t = L/c$, its state vector becomes

$$|\psi\rangle = e^{i\,\text{some overall phase}} \times \exp\left(i\,\frac{c^3\hbar L}{2E}\,\hat{M}^2\right)|\psi_0\rangle\,. \tag{64}$$

The matrix exponential of a non-diagonal matrix is itself non-diagonal, and the off-diagonal matrix elements of this exponential are probability amplitudes for changing the neutrino's species. For more information, please see the Wikipedia article on neutrino oscillations.

Next consider the net 4–momentum of two colliding particles — which is also the net momentum of all the collision products,

$$P^\mu = p_1^\mu + p_2^\mu = \sum_i p_i'^\mu\,. \tag{65}$$

Let

$$s \;=\; c^2 P_\mu P^\mu; \tag{66}$$

as a Lorentz square of a 4–vectors, it has the same value in all reference frames. In the center of mass frame, the net 3–momentum is zero, hence

$$s \;=\; E^2_{\rm net} \;-\; c^2\mathbf{P}^2_{\rm net} \;=\; E^2_{\rm net}\,, \tag{67}$$

so $\sqrt{s}$ is the collision energy in the center of mass frame,

$$E_{\rm cm} \;=\; \sqrt{s}. \tag{68}$$

In any other frame, the net energy is larger,

$$E^2_{\rm net} \;=\; s^2 \;+\; c^2\mathbf{P}^2_{\rm net} \;=\; E^2_{\rm cm} \;+\; c^2\mathbf{P}^2_{\rm net} \;>\; E^2_{\rm cm}\,, \tag{69}$$

but the extra energy is completely tied up the the conserved net momentum, so it does nothing but make the system's center of mass move at constant velocity

$$\mathbf{v} \;=\; \frac{c^2\mathbf{P}_{\rm net}}{E_{\rm net}}\,. \tag{70}$$

The only energy available for making new particles, — or anything else besides the center of mass motion — is the $E_{\rm cm} = \sqrt{s}$.

Let's calculate the center-of-mass energy in the lab frame, where 1 particle moves with energy E_1 and momentum $\mathbf{p}_1$ while the second particle of mass M_2 is at rest. In this frame

$$\begin{aligned} s \;&=\; c^2(p_1+p_2)^2 \;=\; (E_1+M_2c^2)^2 \;-\; c^2\mathbf{p}_1^2 \\ &=\; E_1^2 \;+\; 2E_1M_2c^2 \;+\; M_2^2c^4 \;-\; c^2\mathbf{p}_1^2 \;=\; M_1^2c^4 \;+\; 2E_1M_2c^2 \;+\; M_2^2c^4 \\ &=\; (M_1+M_2)^2c^4 \;+\; 2(E_1-M_1c^2)\times M_2c^2, \end{aligned} \tag{71}$$

hence in the non-relativistic limit

$$E_{\rm cm} \;=\; \sqrt{s} \;\approx\; (M_1+M_2)c^2 \;+\; (E_1-M_1c^2)\times\frac{M_2}{M_1+M_2}\,, \tag{72}$$

or in terms of the kinetic energies $K = E - Mc^2$,

$$K_{\rm cm} \;=\; K_1 \times \frac{M_2}{M_1 + M_2}\,. \tag{73}$$

On the other hand, in the ultra-relativistic limit of $E_1 \gg M_1c^2$, we get

$$E_{\rm cm} \;\approx\; \sqrt{2E_1 \times M_2c^2} \;\ll\; E_1\,. \tag{74}$$

In particle physics, there are two types of collision experiments: the fixed-target experiments, in which a beam of accelerated particles collides with a solid or liquid target, and the collider experiments in which two particle beams moving in opposite directions are focused on some point where they collide with each other. In the fixed target experiments, all the accelerated particles collide with some particle in the target, which makes for a much higher collision rate than in a collider where most accelerated particles miss each other. On the other hand, in a collider the entire energy of the two accelerated particles is available in the CM frame for discovering new physics, while in a fixed-target experiment most of the energy is wasted on the center of mass motion, and only a small fraction $E_{\rm cm} \ll E_1$ goes towards the interesting physics.

For example, the oldest proton-proton collider at CERN — the PS, which started working back in 1959 — had two 28 GeV proton beams colliding head on, so the CM-frame energy available for discovering new physics was 56 GeV. To reach the same energy at a fixed-target experiment — a proton beam hitting a tank of liquid hydrogen — we need a proton beam of energy

$$E_1 \;=\; \frac{s = E_{\rm cm}^2}{2M_pc^2} \;=\; 1670\ {\rm GeV}, \tag{75}$$

and no accelerator had reached this energy level until 2010, half a century after the PS. Ironically, the first accelerator producing proton beams with energy higher than (75) was the LHC at CERN which is also a collider. Today, the CM-frame collision energy at LHS is about 13,000 GeV; to reach this energy at a fixed target experiment, we would need a proton beam of energy

$$E_1 \;\approx\; 90 \cdot 10^6\ {\rm GeV}, \tag{76}$$

and we would be lucky to reach this energy level in another half-a-century.

Let me conclude this section with a few words about relativistic kinematics of particle collisions — elastic or inelastic — or decays. There are several equations relating the energies and momenta of all initial-state and final state particles involved in a collision or decay. First, the the net energy-momentum must be conserved; in a covariant form,

$$\sum_{i}^{\text{initial}} p_i^\mu = \sum_{f}^{\text{final}} p_f'^\mu. \tag{77}$$

Second, for every initial-state or final-state particle, it's energy and momentum must be related by eq. (59), or in covariant form

$$\forall i: \; p_i^2 = m_i^2 c^2 \quad \text{and} \quad \forall f: \; p_f'^2 = m_f^2 c^2. \tag{78}$$

In many situations, these relations allow us to find the final particles' energies, or at least establish the relations between their energies and directions of motion. The homework set#12 has a problem on this very subject.

In these notes, let me work out a simpler example, the decay of a pion into a muon and a neutrino,

$$\pi^+ \to \mu^+ + \nu_\mu. \tag{79}$$

By energy-momentum conservation

$$p_\pi^\alpha = p_\mu^\alpha + p_\nu^\alpha \tag{80}$$

(where we use α for the Lorentz vector index since μ, ν, and π are used up as the species labels), hence

$$\begin{aligned} p_\nu^2 &= (p_\pi - p_\mu)^2 = p_\pi^2 + p_\mu^2 - 2p_\pi \cdot p_\mu, \\ p_\mu^2 &= (p_\pi - p_\nu)^2 = p_\pi^2 + p_\nu^2 - 2p_\pi \cdot p_\nu. \end{aligned} \tag{81}$$

Now let's use the *mass shell* conditions

$$p_\pi^2 = M_\pi^2 c^2, \quad p_\mu^2 = M_\mu^2 c^2, \quad p_\nu^2 = M_\nu^2 c^2 \approx 0; \tag{82}$$

plugging them into eqs. (81), we get

$$2p_\pi\cdot p_\mu\ =\ M_\pi^2c^2\ +\ M_\mu^2c^2\quad\text{while}\quad 2p_\pi\cdot p_\nu\ =\ M_\pi^2c^2\ -\ M_\mu^2c^2. \tag{83}$$

Now, in the rest frame of the initial pion

$$p_\pi\cdot p_\mu\ =\ (M_\pi c)\times(E_\mu/c)\ -\ \mathbf{p}_\pi\cdot\mathbf{p}_\mu\ =\ M_\pi\times E_\mu\ -\ 0, \tag{84}$$

and likewise

$$p_\pi\cdot p_\nu\ =\ M_\pi\times E_\nu, \tag{85}$$

so eqs. (83) give us the energies of the muon and the neutrino as

$$E_\mu\ =\ \frac{M_\pi^2+M_\mu^2}{2M_\pi}\times c^2,\quad E_\nu\ =\ \frac{M_\pi^2-M_\mu^2}{2M_\pi}\times c^2.$$

Numerically, $M_\pi c^2=139$ MeV, $M_\mu c^2=105$ MeV, hence after the decay $E_\mu=109$ MeV and $E_\nu=30$ MeV.

Relativistic Lagrangian

Consider a relativistic particle moving along some worldline $x^\mu(\tau)$. The action functional S[worldline] should be invariant under all symmetries of the theory, so for a relativistic particle S should be invariant under the Lorentz symmetries. The simplest Lorentz invariant functional of a worldline — and the only such such functional which does not involve higher derivatives — is the net proper time along the worldline,

$$S\ =\ A\times\int\limits_{\text{worldline}} d\tau \tag{86}$$

for some constant coefficient A. In a moment, we shall see that getting the right energy and momentum of the particle calls for $A=-mc^2$.

Indeed, let's express the action (86) in the Lagrangian form

$$S = \int dt\, L(\mathbf{x}, \mathbf{v}). \tag{87}$$

Since $d\tau = dt/\gamma$, it follows that the Lagrangian of a free relativistic particle has form

$$L(\mathbf{v}) = \frac{A}{\gamma(\mathbf{v})} = A \times \sqrt{1 - (\mathbf{v}/c)^2}\,. \tag{88}$$

Consequently, the canonical momentum of the particle is

$$\mathbf{p}_{\text{can}} = \frac{\partial L}{\partial \mathbf{v}} = -\frac{A}{c^2} \frac{\mathbf{v}}{\sqrt{1 - (\mathbf{v}/c)^2}} = -\frac{A}{c^2}\,\gamma \mathbf{v}. \tag{89}$$

Since the momentum should point in the same direction as the velocity, we need a negative A. Specifically, if we let $A = -mc^2$, then the canonical momentum becomes the relativistic momentum $\mathbf{p} = \gamma m \mathbf{v}$. Thus,

$$\text{the action } \mathrm{S} = -mc^2 \int_{\text{worldline}} d\tau, \tag{90}$$

$$\text{the Lagrangian } \mathrm{L} = -mc^2 \sqrt{1 - v^2/c^2}, \tag{91}$$

$$\text{the momentum } \mathbf{p} = +m\gamma\mathbf{v} = \frac{m\mathbf{v}}{\sqrt{1 - (v/c)^2}}\,. \tag{92}$$

Next, consider the relativistic energy and hence the Hamiltonian stemming from the Lagrangian (91). Given the canonical momentum (92), the energy function obtains as

$$E = \mathbf{v}\cdot\mathbf{p} - L = \gamma m \mathbf{v}^2 + \frac{mc^2}{\gamma} = \frac{mc^2}{\gamma}\left(\gamma^2\beta^2 + 1 = \gamma^2\right) = mc^2 \times \gamma(\mathbf{v}), \tag{93}$$

and this is precisely the relativistic energy we have obtained earlier in these notes. As to the Hamiltonian, we need to re-express this energy as a function of the canonical momentum rather than the velocity, as as we have seen before,

$$E^2(\mathbf{v}) \;=\; c^2\mathbf{p}^2(\mathbf{v}) \;+\; (mc^2)^2, \tag{94}$$

thus the Hamiltonian

$$H(\mathbf{p}) \;=\; +\sqrt{c^2\mathbf{p}^2 \;+\; (mc^2)^2}\,. \tag{95}$$

Charged relativistic particle in EM background

Now consider a charged particle interacting with some electromagnetic fields. In these notes, we are concerned with the particle's motion rather than the EM fields it produces, so let's treat the EM fields and potentials as a fixed background.

For a non-relativistic charged particle, the Lagrangian is

$$L(\mathbf{x},\mathbf{v}) \;=\; \frac{m\mathbf{v}^2}{2} \;-\; q\Phi(\mathbf{x}) \;+\; \frac{q}{c}\,\mathbf{A}(\mathbf{x})\cdot\mathbf{v}, \tag{96}$$

so we may write the net action as

$$S \;=\; S_{\text{free}} \;+\; S_{\text{EM}} \tag{97}$$

where S_{free} is the action of a free non-relativistic particle, while

$$\begin{aligned} S_{\text{EM}} \;&=\; -\frac{q}{c}\int dt\,\Big(c\Phi(\mathbf{x}) \;-\; \mathbf{A}(\mathbf{x})\cdot\mathbf{v}\Big) \;=\; -\frac{q}{c}\int\Big(c\Phi(\mathbf{x})dt \;-\; \mathbf{A}(\mathbf{x})\cdot d\mathbf{x}(t)\Big) \\ &=\; -\frac{q}{c}\int\limits_{\text{worldline}} A_\mu(x(\tau))\,dx^\mu(\tau). \end{aligned} \tag{98}$$

This action for interaction with the EM fields is manifestly Lorentz invariant, so in a relativistic theory we should keep exactly as in eq. (98) without any changes. On the other hand, the free-particle action for a relativistic particle should be changed to (90), so the net action becomes

$$S \;=\; \int\limits_{\text{worldline}} \Big(-mc^2\,d\tau \;-\; \frac{q}{c}\,A_\mu(x(\tau))\,dx^\mu(\tau)\Big)\,. \tag{99}$$

From this action we may derive the equation of motion for the charged particle in a manifestly covariant form. Instead of going through the Euler–Lagrange formalism which breaks Lorentz symmetry by treating the time t as a special variable, let's use the minimal action principle. That is, let's consider infinitesimal variations of the particle's worldline,

$$x^\mu(\tau) \;\to\; x^\mu(\tau) \;+\; \delta x^\mu(\tau), \tag{100}$$

calculate the first infinitesimal variation of the action (99), and demand that it vanishes for any $\delta x^\mu(\tau)$. Or rather, for any $\delta x^\mu(\tau)$ which vanishes at the beginning at the end of the worldline since the starting and the ending points should be fixed when minimizing the action; this will allow us to integrate by parts without worrying about the boundary terms.

Under infinitesimal variations of the path,

$$\delta(c^2 d\tau^2) \;=\; \delta(dx_\mu dx^\mu) \;=\; 2dx_\mu \times \delta dx^\mu \;=\; 2u_\mu d\tau \times \delta dx^\mu, \tag{101}$$

hence

$$c^2\delta(d\tau) \;=\; u_\mu \times \delta dx^\mu\,. \tag{102}$$

At the same time,

$$\begin{aligned}\delta\big(A_\mu(x)\times dx^\mu\big) \;&=\; A_\mu(x)\times\delta dx^\mu \;+\; (\partial_\nu A_\mu(x)\,\delta x^\nu)\times dx^\mu \\ &\qquad \langle\!\langle \text{renaming indices } \mu\leftrightarrow\nu \text{ in the second term} \rangle\!\rangle \\ &=\; A_\mu(x)\times\delta dx^\mu \;+\; (\partial_\mu A_\nu(x))dx^\nu\times\delta x^\mu,\end{aligned} \tag{103}$$

hence altogether,

$$\delta S \;=\; \int\limits_{\text{worldline}} \Big(-mu_\mu\times\delta dx^\mu \;-\; \frac{q}{c}\,A_\mu(x)\times\delta dx^\mu \;-\; \frac{q}{c}\,\partial_\mu A_\nu(x)dx^\nu\times\delta x^\mu\Big)\,. \tag{104}$$

In this formula, everything is a function of τ along the worldline, either directly or via $x(\tau)$, so let's integrate by parts every term containing the $\delta dx^\mu(\tau)$:

$$\begin{aligned}-mu_\mu\times\delta dx^\mu \;&=\; \text{total derivative} \;+\; d(mu_\mu = p_\mu)\times\delta x^\mu \\ &=\; \text{total derivative} \;+\; \frac{dp_\mu}{d\tau}\,d\tau\times\delta x^\mu,\end{aligned} \tag{105}$$

$$-\frac{q}{c}\,A_\mu(x)\times\delta dx^\mu \;=\; \text{total derivative} \;+\; \frac{q}{c}\Big(d(A_\mu(x(\tau))\;=\;(\partial_\nu A_\mu)\,dx^\nu\Big)\times\delta x^\mu$$
$$=\; \text{total derivative} \;+\; \frac{q}{c}\,(\partial_\nu A_\mu)u^\nu d\tau\times\delta x^\mu, \tag{106}$$

hence

$$\Delta S \;=\; \int\limits_{\text{worldline}} d\tau\,\delta x^\mu(\tau)\times\left(\frac{dp_\mu}{d\tau}\,+\,\frac{q}{c}\,(\partial_\nu A_\mu)\times u^\nu\,-\,\frac{q}{c}\,(\partial_\mu A_\nu)\times u^\nu\right). \tag{107}$$

The worldline obeying the equations of motion minimizes the action, so we should have $\delta S=0$ for any $\delta x^\mu(\tau)$, which calls for

$$\frac{dp_\mu}{d\tau}\;+\;\frac{q}{c}\,\big(\partial_\nu A_\mu\;-\;\partial_\nu A_\nu\;=\;F_{\nu\mu}\big)\times u^\nu\;=\;0 \tag{108}$$

and hence

$$\frac{dp_\mu}{d\tau}\;=\;-\frac{q}{c}\,F_{\nu\mu}u^\nu\;=\;+\frac{q}{c}\,F_{\mu\nu}u^\nu. \tag{109}$$

This gives us the charged particle's equation of motion in a manifestly covariant form; with raised indices, it becomes

$$\frac{dp^\mu}{d\tau}\;=\;\frac{q}{c}\,F^{\mu\nu}u_\nu\,. \tag{110}$$

Note consistency of this equation with constancy of $u^\mu u_\mu=c^2$, which requires

$$u_\mu\times\frac{du^\mu}{d\tau}\;=\;0. \tag{111}$$

Indeed, for the equation (110) we get

$$u_\mu\times\frac{du^\mu}{d\tau}\;=\;u_\mu\times\frac{q}{mc}\,F^{\mu\nu}\,u_\nu\;=\;\frac{q}{mc}\times u_\mu u_\nu F^{\mu\nu}, \tag{112}$$

which vanishes by the antisymmetry of the $F^{\mu\nu}$.

To conclude these notes, let me spell out the covariant equation of motion (110) in 3D terms. For $\mu = i = 1,2,3$,

$$F^{i\nu}u_\nu \;=\; F^{ij}u_j \,+\, F^{i0}u_0 \;=\; (-\epsilon^{ijk}B^k)\times(-\gamma v_j) \,+\, E^i\times(\gamma c) \;=\; \gamma\big(c\mathbf{E}+\mathbf{v}\times\mathbf{B}\big)^i, \tag{113}$$

hence on the RHS of eq. (110),

$$\frac{q}{c}\,F^{i\nu}u_\nu \;=\; \gamma q\left(\mathbf{E}+\frac{\mathbf{v}}{c}\times\mathbf{B}\right)^i. \tag{114}$$

Apart from the overall factor γ, this is the net electric+magnetic force acting on the charged particle. At the same time, on the LHS of eq. (110),

$$\frac{dp^i}{d\tau} \;=\; \gamma\,\frac{dp^i}{dt}\,, \tag{115}$$

hence dropping the factors of γ on both sides of equation, we recover the Newtonian equation of motion

$$\frac{d\mathbf{p}}{dt} \;=\; \mathbf{F} \;=\; q\left(\mathbf{E}+\frac{\mathbf{v}}{c}\times\mathbf{B}\right). \tag{116}$$

Note: there are no relativistic corrections to the EM force on the particle. The only relativistic effect here is the modified relation between the force and the acceleration.

Finally, consider the $\mu = 0$ component of the covariant equation of motion (110). For $\mu = 0$, on the LHS

$$\frac{dp^0}{d\tau} \;=\; \gamma\frac{dp^0}{dt} \;=\; \frac{\gamma}{c}\,\frac{dE}{dt}\,, \tag{117}$$

while on the RHS

$$\begin{aligned}\frac{q}{c}\,F^{0\nu}u_\nu \;&=\; \frac{q}{c}\,F^{0j}u_j \qquad \langle\!\langle\,\text{since } F^{00}=0\,\rangle\!\rangle\\ &=\; \frac{q}{c}\,(-E^j)\times(-\gamma v_j) \;=\; +\frac{\gamma q}{c}\,\mathbf{E}\cdot\mathbf{v}\,,\end{aligned} \tag{118}$$

so dropping the overall factors γ/c on both sides of the equation, we arrive at

$$\frac{dE}{dt} \;=\; q\mathbf{E}\cdot\mathbf{v} \;=\; \mathbf{F}\cdot\mathbf{v}, \tag{119}$$

where $\mathbf{F}$ is exactly the same EM force which changes the momentum $\mathbf{p}$. Thus, eq. (119) is simply the power equation $P = \mathbf{F}\cdot\mathbf{v}$, without any relativistic corrections. (Although the particle's energy itself is the relativistic $E = \gamma mc^2$.)