Radiation by Magnetic Dipoles and Electric Quadrupoles

Summary of Radiation Basics

Last week we have learned that far away from an antenna carrying harmonic currents we have divergent spherical EM waves

$$\mathbf{H} \approx -ik\big(\mathbf{n}\times\mathbf{f}(\mathbf{n})\big)\,\frac{e^{ikr-i\omega t}}{r}\,, \tag{1}$$

$$\mathbf{E} \approx +ikZ_0\big(\mathbf{n}\times\big(\mathbf{n}\times\mathbf{f}(\mathbf{n})\big)\big)\,\frac{e^{ikr-i\omega t}}{r}\,, \tag{2}$$

$$\frac{dP}{d\Omega} = \frac{Z_0k^2}{2}\,\|\mathbf{n}\times\mathbf{f}(\mathbf{n})\|^2\,, \tag{3}$$

for

$$\mathbf{f}(\mathbf{n}) = \frac{1}{4\pi}\int d^3\mathbf{y}\,\mathbf{J}(\mathbf{y})\,\exp(-ik\mathbf{n}\cdot\mathbf{y}). \tag{4}$$

For compact antennas of size $\ll (1/k)$, this $\mathbf{f}$ can be expanded into electric and magnetic multipoles,

$$\mathbf{f}(\mathbf{n}) = \mathbf{f}_{Ed} + \big(\mathbf{f}_{Md}(\mathbf{n}) + \mathbf{f}_{Eq}(\mathbf{n})\big) + \big(\mathbf{f}_{Mq}(\mathbf{n}) + \mathbf{f}_{Eo}(\mathbf{n})\big) + \cdots, \tag{5}$$

where the leading term — the electric dipole — comes from approximating $\exp(-ik\mathbf{n}\cdot\mathbf{y}) \approx 1$ and hence

$$\mathbf{f} \approx \mathbf{f}_{Ed} = \frac{1}{4\pi}\int d^3\mathbf{y}\,\mathbf{J}(\mathbf{y}) = \frac{i\omega}{4\pi}\,\mathbf{d} \tag{6}$$

In the electric dipole approximation,

$$\mathbf{H} \approx \frac{\omega^2}{4\pi c}\,(\mathbf{n}\times\mathbf{d})\,\frac{e^{ikr-i\omega t}}{r}\,, \tag{7}$$

$$\mathbf{E} \approx -\frac{Z_0\,\omega^2}{4\pi c}\,(\mathbf{n}\times(\mathbf{n}\times\mathbf{d}))\,\frac{e^{ikr-i\omega t}}{r}\,, \tag{8}$$

$$\frac{dP}{d\Omega} \approx \frac{Z_0\,\omega^4}{32\pi^2c^2}\,\|\mathbf{n}\times\mathbf{d}\|^2, \tag{9}$$

and the net radiated power is

$$P_{\text{net}} \approx \frac{Z_0\,\omega^4}{12\pi c^2}\,\|\mathbf{d}\|^2. \tag{10}$$

Magnetic Dipole Radiation

While the electric dipole moment yields the leading contribution to the EM radiation by compact antennas, sometimes $\mathbf{d} = 0$ and the radiation is dominated by the first subleading terms in the expansion (5), namely the magnetic dipole and the electric quadrupole. In terms of eq. (4) for the $\mathbf{f}(\mathbf{n})$, the electric dipole terms comes from the leading term in the expansion

$$\exp(-ik\mathbf{n}\cdot\mathbf{y}) \;=\; 1 \;-\; ik(\mathbf{n}\cdot\mathbf{y}) \;+\; \cdots, \tag{11}$$

while the magnetic dipole and the electric quadrupole terms come from the first subleading term $-ik(\mathbf{n}\cdot\mathbf{y})$, thus

$$\mathbf{f}_{Md}(\mathbf{n}) \;+\; \mathbf{f}_{Eq}(\mathbf{n}) \;=\; \frac{-ik}{4\pi}\int\! d^3\mathbf{y}\,\mathbf{J}(\mathbf{y})(\mathbf{n}\cdot\mathbf{y}). \tag{12}$$

Specifically,

$$\mathbf{f}_{Md}(\mathbf{n}) \;=\; \frac{-ik}{8\pi}\int\! d^3\mathbf{y}\Big(\mathbf{J}(\mathbf{y}\cdot\mathbf{n}) \;-\; \mathbf{y}(\mathbf{J}\cdot\mathbf{n})\Big) \tag{13}$$

while

$$\mathbf{f}_{Eq}(\mathbf{n}) \;=\; \frac{-ik}{8\pi}\int\! d^3\mathbf{y}\Big(\mathbf{J}(\mathbf{y}\cdot\mathbf{n}) \;+\; \mathbf{y}(\mathbf{J}\cdot\mathbf{n})\Big). \tag{14}$$

To see the relation of the $\mathbf{f}_{Md}$ to the magnetic dipole moment, note that

$$\mathbf{J}(\mathbf{y}\cdot\mathbf{n}) \;-\; \mathbf{y}(\mathbf{J}\cdot\mathbf{n}) \;=\; \mathbf{n}\times(\mathbf{J}\times\mathbf{y}), \tag{15}$$

hence

$$\mathbf{f}_{Md}(\mathbf{n}) \;=\; \frac{-ik}{8\pi}\,\mathbf{n}\times\int\! d^3\mathbf{y}\,\mathbf{J}(\mathbf{y})\times\mathbf{y}, \tag{16}$$

where the integral is precisely $2\mathbf{m}$ — (twice) the magnetic dipole moment of the antenna, thus

$$\mathbf{f}_{Md}(\mathbf{n}) \;=\; \frac{-ik}{4\pi}\,(\mathbf{n}\times\mathbf{m}). \tag{17}$$

Consequently, in the radiation zone far away from the antenna,

$$\mathbf{H} \;\approx\; -\left(\frac{k^2}{4\pi} \;=\; \frac{\omega^2}{4\pi c^2}\right)\,\Big(\mathbf{n}\times(\mathbf{n}\times\mathbf{m})\Big)\,\frac{e^{ikr-i\omega t}}{r}\,, \tag{18}$$

$$\mathbf{E} \approx +\frac{Z_0\,\omega^2}{4\pi c^2}\big(\mathbf{n}\times(\mathbf{n}\times(\mathbf{n}\times\mathbf{m}))\big)\,\frac{e^{ikr-i\omega t}}{r}$$
$$= -\frac{Z_0\,\omega^2}{4\pi c^2}\,(\mathbf{n}\times\mathbf{m})\,\frac{e^{ikr-i\omega t}}{r}\,. \tag{19}$$

Note that the magnetic field (18) of the magnetic dipole behaves exactly like the electric field (8) of the electric dipole, while the electric field (19) of the magnetic dipole behaves exactly like the magnetic field (7) of the electric dipole.

Likewise, the power of the magnetic dipole radiation

$$\frac{dP}{d\Omega} = \frac{Z_0\,\omega^4}{32\pi^2 c^4}\,\big\|\mathbf{n}\times(\mathbf{n}\times\mathbf{m})\big\|^2 = \frac{Z_0\,\omega^4}{32\pi^2 c^4}\,\big\|\mathbf{n}\times\mathbf{m}\big\|^2 \tag{20}$$

has similar angular distribution and similar net power

$$P_{\text{net}} = \frac{Z_0\,\omega^4}{12\pi c^4}\,\|\mathbf{m}\|^2 \tag{21}$$

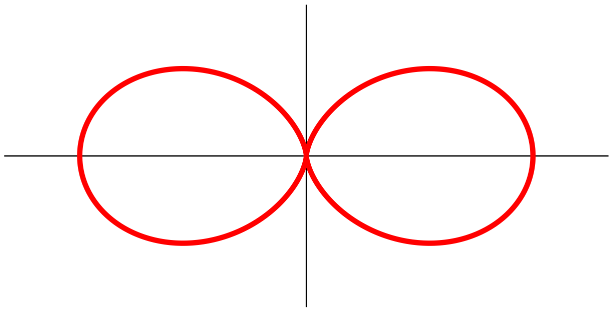
to the radiation of electric dipole

$$\mathbf{d} = \frac{\mathbf{m}}{c}\,. \tag{22}$$

As an example, consider a loop antenna — a flat loop of wire of area A. Or more often, a flat coil having N turns of area A. When this coil is fed a harmonic current of frequency ω and amplitude I_0, it has oscillating magnetic moment of amplitude $m = NAI_0$ in the direction of the coil's axis, *i.e.* $\perp$ to the plane of the flat coil. As a complex vector, $\mathbf{m}$ is real up to an overall phase, hence the angular distribution of the loop antenna's radiation is

$$\frac{dP}{d\Omega} \propto \sin^2\theta \tag{23}$$

where θ is the angle between the antenna's axis and the direction of the radiation. Thus, the radiation is strongest within the plane of the loop, while no power is emitted along the loop's axis. Graphically, the angular distribution (23) is illustrated by the *radiation power* *diagram*



The net power emitted by the loop antenna is

$$P_{\rm net} \;=\; \frac{Z_0\,\omega^4}{12\pi c^4}\,(NA\,|I_0|)^2. \tag{24}$$

From the point of view of the radio-frequency generator supplying this power — and the current of amplitude I_0 to the antenna,

$$P_{\rm net} \;=\; \frac{|I_0|^2}{2}\times{\rm Re}(Z) \tag{25}$$

where Z is the antenna's impedance. In light of eq. (24), the real (active) part of this impedance is

$$Z \;=\; \frac{Z_0}{6\pi}\times(\omega/c)^4\times(NA)^2 \;=\; \frac{Z_0}{6\pi}\times\left(NA\times\left(\frac{2\pi}{\lambda}\right)^2\right)^2. \tag{26}$$

For example, take antenna made of 10 turns of area $A = 1\ {\rm m}^2$, and let it radiate short-wave radio signal at wavelength $\lambda = 20$ m (frequency $\omega = 2\pi\times 15$ MHz). For this antenna, ${\rm Re}(Z)\approx 20\ \Omega$, so if we feed it with current of amplitude $I_0 = 10$ A, it would radiate 1 kW of net radio power.

Electric Quadrupole Radiation

The electric quadrupole contribution to the EM radiation stems from the

$$\mathbf{f}_{Eq}(\mathbf{n}) \;=\; \frac{-ik}{8\pi}\int\! d^3\mathbf{y}\Big(\mathbf{J}(\mathbf{y}\cdot\mathbf{n})\,+\,\mathbf{y}(\mathbf{J}\cdot\mathbf{n})\Big) \tag{14}$$

contribution to the first subleading term in the expansion of $\mathbf{f}$ into powers of $k\times(\text{size})$. To see the relation of this $\mathbf{f}_{Ed}$ to the electric quadrupole moment tensor

$$\mathcal{Q}_{ij} \;=\; \int\! d^3\mathbf{y}\,\rho(\mathbf{y})\Big(\tfrac{3}{2}y_iy_j\,-\,\tfrac{1}{2}\delta_{ij}\mathbf{y}^2\Big), \tag{27}$$

we use

$$\nabla\cdot\mathbf{J} \;=\; -\frac{\partial\rho}{\partial t} \;=\; +i\omega\rho. \tag{28}$$

Thus,

$$\begin{aligned} i\omega\mathcal{Q}_{ij} \;&=\; \int\! d^3\mathbf{y}\,(\nabla\cdot\mathbf{J})\Big(\tfrac{3}{2}y_iy_j\,-\,\tfrac{1}{2}\delta_{ij}\mathbf{y}^2\Big)\\ &\qquad\langle\!\langle\,\text{integrating by parts}\,\rangle\!\rangle\\ &=\; -\int\! d^3\mathbf{y}\,J_k\Big(\tfrac{3}{2}\delta_{ki}y_j\,+\,\tfrac{3}{2}y_i\delta_{kj}\,-\,\delta_{ij}y_k\Big)\\ &=\; -\int\! d^3\mathbf{y}\,\Big(\tfrac{3}{2}J_iy_j\,+\,\tfrac{3}{2}y_iJ_j\,-\,\delta_{ij}(\mathbf{y}\cdot\mathbf{J})\Big), \end{aligned} \tag{29}$$

and hence

$$\begin{aligned} i\omega\mathcal{Q}_{ij}n_j \;&=\; -\int\! d^3\mathbf{y}\,\Big(\tfrac{3}{2}J_i(\mathbf{y}\cdot\mathbf{n})\,+\,\tfrac{3}{2}y_i(\mathbf{J}\cdot\mathbf{n})\,-\,n_i(\mathbf{y}\cdot\mathbf{J})\Big)\\ &=\; -\frac{3}{2}\int\! d^3\mathbf{y}\,\Big(\mathbf{J}(\mathbf{y}\cdot\mathbf{n})\,+\,\mathbf{y}(\mathbf{J}\cdot\mathbf{n})\Big)_i\,+\,\mathbf{n}_i\int\! d^3\mathbf{y}\,(\mathbf{y}\cdot\mathbf{J}). \end{aligned} \tag{30}$$

The first term on the second line here is similar to eq. (14) for the $\mathbf{f}_{Eq}$ — except for the overall coefficient — while the second term has a form $\mathbf{n}$ times a scalar. Therefore,

$$\mathbf{f}_{Eq}(\mathbf{n}) \;=\; -\frac{\omega k}{12\pi}\,(\mathcal{Q}\circ\mathbf{n})\;+\;(\text{scalar})\,\mathbf{n} \tag{31}$$

where $(\mathcal{Q}\circ\mathbf{n})$ is a vector with components $(\mathcal{Q}\circ\mathbf{n})_i = \mathcal{Q}_{ij}n_j$.

Moreover, the second term (scalar)$\mathbf{n}$ in eq. (31) does not affect the EM fields $\mathbf{E}$ and $\mathbf{H}$ in the radiation zone or the power of the EM waves governed by the corresponding eqs. (1), (2), and (3) since in all these equations $\mathbf{f}$ appears only in the combination $\mathbf{n}\times\mathbf{f}(\mathbf{n})$. Consequently, from the radiation point of view,

$$\mathbf{f}(\mathbf{n}) \;+\; (\text{any scalar})\,\mathbf{n} \;\cong\; \mathbf{f}(\mathbf{n}). \tag{32}$$

In particular, for the electric quadrupole radiation,

$$\mathbf{f}_{Eq}(\mathbf{n}) \;\cong\; -\frac{\omega k}{12\pi}\,(\mathcal{Q}\circ\mathbf{n}). \tag{33}$$

In terms of the EM fields in the radiation zone, this means

$$\mathbf{H} \;\approx\; +i\frac{k^2\omega}{12\pi}\big(\mathbf{n}\times(\mathcal{Q}\circ\mathbf{n})\big)\,\frac{e^{ikr-i\omega t}}{r}\,, \tag{34}$$

$$\mathbf{E} \;\approx\; -i\frac{Z_0k^2\omega}{12\pi}\big(\mathbf{n}\times(\mathbf{n}\times(\mathcal{Q}\circ\mathbf{n}))\big)\,\frac{e^{ikr-i\omega t}}{r}\,, \tag{35}$$

while the EM power radiated per unit of solid angle is

$$\frac{dP}{d\Omega} \;=\; \frac{Z_0k^4\omega^2}{288\pi^2}\,\big\|\mathbf{n}\times(\mathcal{Q}\circ\mathbf{n})\big\|^2 \;=\; \frac{Z_0\,\omega^6}{288\pi^2c^4}\Big((\mathcal{Q}^*_{ij}n_j)(\mathcal{Q}_{ik}n_k) \;-\; |n_i\mathcal{Q}_{ij}n_j|^2\Big). \tag{36}$$

To calculate the net radiated power, we need to integrate eq. (36) over the 4π solid angle. In components,

$$\oiint d^2\Omega\Big((\mathcal{Q}^*\cdot\mathbf{n})\cdot(\mathcal{Q}\cdot\mathbf{n}) \;-\; \big|\mathbf{n}\cdot\mathcal{Q}\cdot\mathbf{n}\big|^2\Big) \;=\; Q^*_{ij}Q_{ik}\oiint d^2\Omega\,n_jn_k \;-\; Q^*_{ij}Q_{k\ell}\oiint d^2\Omega\,n_in_jn_kn_\ell \tag{37}$$

where the remaining integrals on the RHS must be rotationally invariant and also totally symmetric in the indices of all the $\mathbf{n}$ vectors. Thus

$$\oiint d^2\Omega\,n_jn_k \;=\; A_2\delta_{jk}\,, \quad \oiint d^2\Omega\,n_in_jn_kn_\ell \;=\; A_4\big(\delta_{ij}\delta_{k\ell} \;+\; \delta_{ik}\delta_{j\ell} \;+\; \delta_{i\ell}\delta_{jk}\big) \tag{38}$$

for some overall coefficients A_2 and A_4 which obtain by setting all the indices to 3 (*i.e.*, z):

$$A_2 \;=\; \oiint d^2\Omega\,\cos^2\theta \;=\; \frac{4\pi}{3}\,, \quad 3A_4 \;=\; \oiint d^2\Omega\,\cos^4\theta \;=\; \frac{4\pi}{5}\,. \tag{39}$$

Consequently,

$$\begin{aligned} &\oiint d^2\Omega\,\Big((\mathcal{Q}^*\circ\mathbf{n})\cdot(\mathcal{Q}\circ\mathbf{n}) \;-\; \big|\mathbf{n}\circ\mathcal{Q}\circ\mathbf{n}\big|^2\Big) \;= \\ &\quad=\; Q^*_{ij}Q_{ik}\times\frac{4\pi}{3}\,\delta_{jk} \;-\; Q^*_{ij}Q_{k\ell}\times\frac{4\pi}{15}\big(\delta_{ij}\delta_{k\ell}\,+\,\delta_{ik}\delta_{j\ell}\,+\,\delta_{i\ell}\delta_{jk}\big) \\ &\quad=\; \frac{4\pi}{3}\,Q^*_{ij}Q_{ij} \;-\; \frac{4\pi}{15}\Big(Q^*_{ii}Q_{kk}\,+\,Q^*_{ij}Q_{ji}\,+\,Q^*_{ij}Q_{ij}\Big) \\ &\qquad\langle\!\langle\text{ using symmetry and tracelessness of the quadrupole moment tensor }\rangle\!\rangle \\ &\quad=\; \frac{4\pi}{3}\,Q^*_{ij}Q_{ij} \;-\; \frac{4\pi}{15}\Big(0\,+\,2Q^*_{ij}Q_{ij}\Big) \\ &\quad=\; \frac{4\pi}{5}\,Q^*_{ij}Q_{ij} \;=\; \frac{4\pi}{5}\,\mathrm{tr}\big(\mathcal{Q}^\dagger\mathcal{Q}\big), \end{aligned} \tag{40}$$

and therefore the net radiated power is

$$P_{\mathrm{net}} \;=\; \frac{Z_0}{360\pi c^4}\,\omega^6\,\mathrm{tr}\big(\mathcal{Q}^\dagger\mathcal{Q}\big). \tag{41}$$

The angular distribution of the quadrupole radiation depends on the structure of the quadrupole moment tensor, which can range from a linear quadrupole (all charges arranged along a line) to planer quadrupole (all charges in the same plane) to complicated 3D setups where the charges move in different directions with different phases. For specific examples, let's consider the quadrupole moment tensors proportional to the spherical harmonics $Y_{\ell,m}$ with $\ell = 2$, namely

$$\begin{aligned} \mathcal{Q}^{(m=0)} \;&=\; \frac{Q}{\sqrt{3/2}}\begin{pmatrix} -\frac12 & 0 & 0 \\ 0 & -\frac12 & 0 \\ 0 & 0 & +1 \end{pmatrix}, \\ \mathcal{Q}^{(m=\pm1)} \;&=\; \frac{Q}{2}\begin{pmatrix} 0 & 0 & +1 \\ 0 & 0 & \pm i \\ +1 & \pm i & 0 \end{pmatrix}, \\ \mathcal{Q}^{(m=\pm2)} \;&=\; \frac{Q}{2}\begin{pmatrix} +1 & \pm i & 0 \\ \pm i & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \end{aligned} \tag{42}$$

- The $m = 0$ quadrupole mode included all linear quadrupoles as well as other configurations with similar symmetries. (An axial symmetry, or at least a symmetry of 90° rotations around the z axis.) For this mode

$$\mathcal{Q}\circ\mathbf{n} \;=\; \frac{Q}{\sqrt{3/2}}\begin{pmatrix} -\frac{1}{2}n_x \\ -\frac{1}{2}n_y \\ +n_z \end{pmatrix} \quad\Longrightarrow\quad \begin{aligned} \|\mathcal{Q}\circ\mathbf{n}\|^2 \;&=\; \frac{Q^2}{3/2}\left(\tfrac{1}{4}n_x^2+\tfrac{1}{4}n_y^2+n_z^2\right) \\ &=\; \frac{Q^2}{6}\Big(1+3n_z^2 \;=\; 1+3\cos^2\theta\Big) \end{aligned} \tag{43}$$

while

$$\mathbf{n}\circ\mathcal{Q}\circ\mathbf{n} \;=\; \frac{Q}{\sqrt{3/2}}\Big(-\tfrac{1}{2}n_x^2-\tfrac{1}{2}n_y^2+n_z^2 \;=\; \tfrac{3}{2}n_z^2-\tfrac{1}{2} \;=\; \tfrac{3}{2}\cos^2\theta-\tfrac{1}{2}\Big). \tag{44}$$

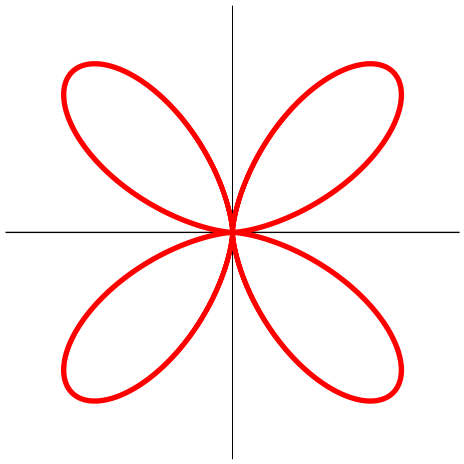
Consequently,

$$\begin{aligned} \|\mathcal{Q}\circ\mathbf{n}\|^2 \;-\; |\mathbf{n}\circ\mathcal{Q}\circ\mathbf{n}|^2 \;&=\; \frac{Q^2}{6}\left(1+3\cos^2\theta\right) \;-\; \frac{Q^2}{3/2}\left(\tfrac{3}{2}\cos^2\theta-\tfrac{1}{2}\right)^2 \\ &=\; \frac{Q^2}{6}\Big(1+3\cos^2\theta-1+6\cos^2\theta-9\cos^4\theta\Big) \\ &=\; \frac{Q^2}{6}\left(9\cos^2\theta-9\cos^4\theta\right) \\ &=\; \frac{3Q^2}{2}\times\cos^2\theta\sin^2\theta \end{aligned} \tag{45}$$

and therefore the angular distribution of the radiated power is

$$\frac{dP}{d\Omega} \;\propto\; \cos^2\theta\sin^2\theta. \tag{46}$$

Here is the radiation power diagram for this distribution:



- Next, consider the $m = \pm 1$ quadrupole modes, for which

$$\mathcal{Q} \circ \mathbf{n} \;=\; \frac{Q}{2} \begin{pmatrix} +n_z \\ \pm i n_z \\ n_x \pm i n_y \end{pmatrix} \quad \Longrightarrow \quad \begin{aligned} \|\mathcal{Q} \circ \mathbf{n}\|^2 \;&=\; \frac{Q^2}{4} \left(2n_z^2 + n_x^2 + n_y^2\right) \\ &=\; \frac{Q^2}{4} \left(1 + \cos^2\theta\right) \end{aligned} \tag{47}$$

while

$$\mathbf{n} \circ \mathcal{Q} \circ \mathbf{n} \;=\; \frac{Q}{2} \times 2n_z(n_x \pm i n_y) \;=\; Q \times \cos\theta \sin\theta \, e^{\pm i\phi}. \tag{48}$$

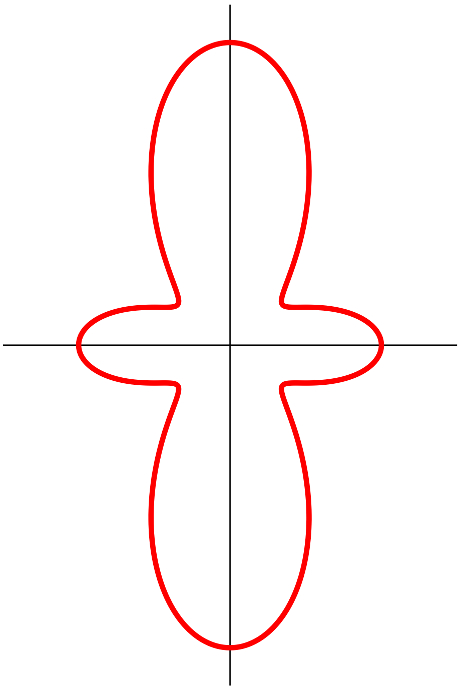
Consequently,

$$\begin{aligned} \|\mathcal{Q} \circ \mathbf{n}\|^2 \;-\; |\mathbf{n} \circ \mathcal{Q} \circ \mathbf{n}|^2 \;&=\; \frac{Q^2}{4} (1 + \cos^2\theta) \;-\; Q^2 \cos^2\theta \sin^2\theta \\ &=\; \frac{Q^2}{4} \times \left(1 - 3\cos^2\theta + 4\cos^4\theta\right) \end{aligned} \tag{49}$$

and therefore

$$\frac{dP}{d\Omega} \;\propto\; 1 \;-\; 3\cos^2\theta \;+\; 4\cos^4\theta. \tag{50}$$

The radiation power diagram for this distribution looks like



- Finally, the $m = \pm 2$ quadrupole modes, which include the planar quadrupoles. For these modes

$$\mathcal{Q} \circ \mathbf{n} \;=\; \frac{Q}{2}\begin{pmatrix} n_x \pm i n_y \\ \pm i n_x - n_y \\ 0 \end{pmatrix} \quad\Longrightarrow\quad \begin{aligned} \|\mathcal{Q} \circ \mathbf{n}\|^2 \;&=\; \frac{Q^2}{4} \times 2|n_x \pm i n_y|^2 \\ &=\; \frac{Q^2}{2}\,\sin^2\theta \end{aligned} \tag{51}$$

while

$$\mathbf{n} \circ \mathcal{Q} \circ \mathbf{n} \;=\; \frac{Q}{2} \times (n_x \pm i n_y)^2 = \frac{Q}{2} \times \sin^2\theta\, e^{\pm 2i\phi}. \tag{52}$$

Consequently,

$$
\begin{aligned}
\|\mathcal{Q}\circ\mathbf{n}\|^2 \;-\; |\mathbf{n}\circ\mathcal{Q}\circ\mathbf{n}|^2 \;&=\; \frac{Q^2}{2}\times\sin^2\theta \;-\; \frac{Q^2}{4}\times\sin^4\theta \\
&=\; \frac{Q^2}{4}\times(\sin^2 \;=\; 1-\cos^2\theta)\times(2-\sin^2\theta \;=\; 1+\cos^2\theta) \\
&=\; \frac{Q^2}{4}\times(1-\cos^4\theta)
\end{aligned}
\tag{53}
$$

and therefore

$$
\frac{dP}{d\Omega} \;\propto\; 1-\cos^4\theta. \tag{54}
$$

Here is the radiation power diagram for this angular distribution.