MAXWELL EQUATIONS

The Displacement Current

The magnetic field of a *steady* current obeys the Ampere's Law

$$\nabla \times \mathbf{H} \;=\; \mathbf{J}. \tag{1}$$

In the quasistatic approximation, we may apply this law to the fields of currents which are *slowly* changing with time, but only when such currents are divergence-less. Otherwise, the Ampere's law (1) would be mathematically inconsistent with $\nabla \cdot \mathbf{J} \neq 0$. But what if a time-dependent current density $\mathbf{J}(\mathbf{x}, t)$ is accompanied by a time-dependent change density $\rho(\mathbf{x}, t)$, in which case the continuity equation

$$\nabla \cdot \mathbf{J} \;+\; \frac{\partial \rho}{\partial t} \;=\; 0 \tag{2}$$

requires $\nabla \cdot \mathbf{J} \neq 0$? Obviously, the Ampere's law must be modified to handle such a current, but in such a way that for a steady current the modified law reverts to its original form (1).

In 1861, James Clerk Maxwell wrote down the modification now days called the Maxwell–Ampere law; in modern notations, it reads

$$\nabla \times \mathbf{H} \;=\; \mathbf{J} \;+\; \frac{\partial \mathbf{D}}{\partial t} \tag{3}$$

where $\mathbf{D}(\mathbf{x}, t)$ is the *electric displacement field,* whose time derivative

$$\mathbf{J}_d \;=\; \frac{\partial \mathbf{D}}{\partial t} \tag{4}$$

is often called the *displacement current,* even though it's not really a current. To make a distinction, the ordinary macroscopic current $\mathbf{J}$ is called the *conduction current.* Adding the displacement current to the conduction current makes for a combination whose divergence is identically zero, which makes the Maxwell–Ampere law mathematically consistent. Indeed, taking the time derivative of the Gauss law $\nabla \cdot \mathbf{D} = \rho$ we get

$$\nabla \cdot \mathbf{J}_d \;=\; \nabla \cdot \frac{\partial \mathbf{D}}{\partial t} \;=\; \frac{\partial}{\partial t}(\nabla \cdot \mathbf{D}) \;=\; \frac{\partial}{\partial t}\rho, \tag{5}$$

hence by the continuity equation (2)

$$\nabla \cdot \mathbf{J} + \; \nabla \cdot \mathbf{J}_d \;=\; \nabla \cdot \mathbf{J} + \frac{\partial \rho}{\partial t} \;=\; 0. \tag{6}$$

For alternating currents in wires and circuit elements, eq. (6) becomes the Kirchhoff law for the net conduction + displacement current. For example, consider a capacitor: the conduction current flows through the wires connected to the capacitor, while the displacement current flows between the capacitor's plates, but the net current flows un-interrupted through the whole circuit. For simplicity, take a parallel-plate capacitor with plates of area A; then given the charge $Q(t)$ on the plates at some moment of time, the electric displacement field between the plates at that time is uniform

$$D(t) \;=\; \frac{Q(t)}{A}\,. \tag{7}$$

Consequently, the displacement current density between the plates is

$$J_d(t) \;=\; \frac{\partial D}{\partial t} \;=\; \frac{1}{A}\frac{dQ}{dt}\,, \tag{8}$$

and the net displacement current between the plates is

$$I_d(t) \;=\; A \times J(t) \;=\; \frac{dQ}{dt}\,. \tag{9}$$

But by the charge conservation, the time dependence of the charges on the plates come solely due to the conduction current in the wires, thus

$$I_c(t) \;=\; \frac{dQ}{dt}\,, \tag{10}$$

hence at all times t,

$$I_c[\text{in the wires}](t) \;=\; I_d[\text{between the plates}](t). \tag{11}$$

Quod erat demonstrandum.

But the most profound consequence of the displacement current is that it allows for the *propagating electromagnetic waves.* We shall see how this works a couple of pages below.

Maxwell Equations in Hertz–Heaviside Form

And God said,

$$\nabla\cdot\mathbf{D} = \rho, \tag{12.a}$$

$$\nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t}, \tag{12.b}$$

$$\nabla\cdot\mathbf{B} = 0, \tag{12.c}$$

$$\nabla\times\mathbf{H} = \mathbf{J} + \frac{\partial\mathbf{D}}{\partial t}, \tag{12.d}$$

and there was light.

The set of four equations (12,a–d) was named *Maxwell equations* by Oliver Heaviside and Heinrich Hertz, who wrote these equations in the modern vector notations (Maxwell's own *Treatise on Electricity and Magnetism* from (1873) had a much larger set of equations in rather unwieldy notations.) Maxwell equations are the key to electrodynamics in general, and especially to the electromagnetic waves, including the light waves, hence the biblical quote above.

To see how the EM waves follow from the Maxwell equations, consider a uniform linear medium where $\mathbf{D}=\epsilon\epsilon_0\mathbf{E}$ and $\mathbf{B}=\mu\nu_0\mathbf{H}$, and without any macroscopic conduction currents and/or charges. In such a medium

$$\nabla\times\mathbf{H} = 0 + \frac{\partial\mathbf{D}}{\partial t} = \epsilon\epsilon_0\frac{\partial\mathbf{E}}{\partial t}, \tag{13}$$

$$\nabla\times\mathbf{B} = \mu\mu_0\nabla\times\mathbf{H} = (\mu\mu_0\epsilon\epsilon_0)\frac{\partial\mathbf{E}}{\partial t}, \tag{14}$$

$$\begin{aligned}\nabla\times(\nabla\times\mathbf{B}) &= (\mu\mu_0\epsilon\epsilon_0)\nabla\times\frac{\partial\mathbf{E}}{\partial t} = (\mu\mu_0\epsilon\epsilon_0)\frac{\partial}{\partial t}\left(\nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t}\right)\\ &= -(\mu\mu_0\epsilon\epsilon_0)\frac{\partial^2}{\partial t^2}\mathbf{B},\end{aligned} \tag{15}$$

$$\nabla^2\mathbf{B} = \nabla(\nabla\cdot\mathbf{B}) - \nabla\times(\nabla\times\mathbf{B}) = 0 + (\mu\mu_0\epsilon\epsilon_0)\frac{\partial^2}{\partial t^2}\mathbf{B}, \tag{16}$$

where the last line has the form of the wave equation

$$\left(\frac{1}{v^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right)\mathbf{B}(\mathbf{x},t) = 0 \tag{17}$$

for the wave speed

$$v = \frac{1}{\sqrt{\mu\mu_0\epsilon\epsilon_0}}. \tag{18}$$

The electric field also obeys the wave equation similar to eq. (17) for the same wave speed. Indeed, in the absence of macroscopic charges

$$\nabla\cdot\mathbf{E} = \frac{1}{\epsilon\epsilon_0}\nabla\cdot\mathbf{D} = \frac{\rho}{\epsilon\epsilon_0} = 0, \tag{19}$$

hence

$$\begin{aligned}
\nabla^2\mathbf{E} &= \nabla(\nabla\cdot\mathbf{E}) - \nabla\times(\nabla\times\mathbf{E}) \\
&= 0 + \nabla\times\left(\frac{\partial\mathbf{B}}{\partial t}\right) = \frac{\partial}{\partial t}(\nabla\times\mathbf{B}) \\
&\quad \langle\!\langle \text{ by eq. (14) } \rangle\!\rangle \\
&= \frac{\partial}{\partial t}\left((\mu\mu_0\epsilon\epsilon_0)\frac{\partial\mathbf{E}}{\partial t}\right) = (\mu\mu_0\epsilon\epsilon_0)\frac{\partial^2}{\partial t^2}\mathbf{E},
\end{aligned} \tag{20}$$

and thus the wave equation

$$\left(\frac{1}{v^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right)\mathbf{E}(\mathbf{x},t) = 0 \tag{21}$$

for the wave speed v exactly as in eq. (18). In the vacuum, this speed of the EM wave is the famous ‘speed of light’

$$c = \frac{1}{\sqrt{\mu_0\epsilon_0}} = 299\,792\,458 \text{ m/s}. \tag{22}$$

In other linear media, the EM waves propagate at slower speeds

$$V = \frac{c}{n} \quad \text{for} \quad n = \sqrt{\epsilon\mu} > 1; \tag{23}$$

the coefficient n here is called the *refraction index* since it governs the refraction of the EM waves as they cross a boundary between two different media.

Maxwell Equations for the Potentials

In light of the *homogeneous* Maxwell equations

$$\nabla\cdot\mathbf{B} \;=\; 0, \quad \nabla\times\mathbf{E} \,+\, \frac{\partial\mathbf{B}}{\partial t} \;=\; 0, \tag{24}$$

we may always express the electric and the magnetic fields in terms of the vector and scalar potentials as

$$\mathbf{B}(\mathbf{x},t) \;=\; \nabla\times\mathbf{A}(\mathbf{x},t), \tag{25}$$

$$\mathbf{E}(\mathbf{x},t) \;=\; -\frac{\partial}{\partial t}\mathbf{A}(\mathbf{x},t) \;-\; \nabla\Phi(\mathbf{x},t). \tag{26}$$

Indeed, at any moment of time the $\mathbf{B}$ field is divergenceless, so it's a curl of some vector potential $\mathbf{A}$, and if the $\mathbf{B}$ field is time dependent, then the vector potential should also be time-dependent, As to the electric field, by the induction law the combination $\mathbf{E}+\partial\mathbf{A}/\partial t$ has zero curl,

$$\nabla\times\left(\mathbf{E}\,+\,\frac{\partial\mathbf{A}}{\partial t}\right) \;=\; \nabla\times\mathbf{E} \,+\, \frac{\partial}{\partial t}(\nabla\times\mathbf{A} \;=\; \mathbf{B}) \;=\; 0, \tag{27}$$

so it's a gradient of some scalar field $-\Phi(\mathbf{x},t)$.

For any given time-dependent fields $\mathbf{E}(\mathbf{x},t)$ and $\mathbf{B}(\mathbf{x},t)$, the potentials $\mathbf{A}(\mathbf{x},t)$ and $\Phi(\mathbf{x},t)$ are far from unique: there is a whole family of such potentials related by time-dependent gauge transforms

$$\mathbf{A}'(\mathbf{x},t) \;=\; \mathbf{A}(\mathbf{x},t) \,+\, \nabla\Lambda(\mathbf{x},t), \quad \Phi'(\mathbf{x},t) \;=\; \Phi(\mathbf{x},t) \;-\; \frac{\partial}{\partial t}\Lambda(\mathbf{x},t) \tag{28}$$

for a completely general function $\Lambda(\mathbf{x},t)$. Indeed, adding a gradient to the vector potential would not affect its curl, *i.e.* the $\mathbf{B}$ field, while for the electric field, the effect of Λ cancels out from the transformed $\mathbf{A}'$ and Φ' potentials,

$$\begin{aligned}
\mathbf{E}' &= -\frac{\partial\mathbf{A}'}{\partial t} \;-\; \nabla\Phi' \\
&= -\frac{\partial\mathbf{A}}{\partial t} \;-\; \cancel{\frac{\partial}{\partial t}\nabla}\Lambda \;-\; \nabla\Phi \;+\; \cancel{\nabla\frac{\partial\Lambda}{\partial t}} \\
&= -\frac{\partial\mathbf{A}}{\partial t} \;-\nabla\Phi \;=\; \mathbf{E}.
\end{aligned} \tag{29}$$

Eqs. (25) and (26) automagically solve the two homogeneous Maxwell equations, while the two inhomogeneous equations become much messier second-order PDEs. For simplicity, let me write them down for the vacuum where $\mathbf{D}\equiv\epsilon_0\mathbf{E}$ and $\mathbf{B}\equiv\mu_0\mathbf{H}$:

$$\frac{\rho}{\epsilon_0} = \nabla\cdot\mathbf{E} = -\nabla\cdot\frac{\partial\mathbf{A}}{\partial t} - \nabla^2\Phi, \tag{30}$$

$$\begin{aligned}\mu_0\mathbf{J} &= \nabla\times\mathbf{B} - \mu_0\epsilon_0\frac{\partial\mathbf{E}}{\partial t} = \nabla\times(\nabla\times\mathbf{A}) - \frac{1}{c^2}\frac{\partial}{\partial t}\left(-\frac{\partial\mathbf{A}}{\partial t} - \nabla\Phi\right)\\ &= \nabla(\nabla\cdot\mathbf{A}) - \nabla^2\mathbf{A} + \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\mathbf{A} + \frac{1}{c^2}\nabla\left(\frac{\partial\Phi}{\partial t}\right).\end{aligned} \tag{31}$$

For every solution of these equations, there is a whole family of physically equivalent solutions related by the gauge transforms (28). To eliminate this redundancy, we should impose an additional linear condition — called the *gauge condition* — at every point $(\mathbf{x},t)$ of space-time. Two most commonly used gauge conditions are:

- the *transverse gauge* $\nabla\cdot\mathbf{A}\equiv 0$, also known as the *Coulomb gauge* or as the *radiation gauge*;
- the *Landau gauge*

$$\nabla\cdot\mathbf{A} + \frac{1}{c^2}\frac{\partial\Phi}{\partial t} = 0. \tag{32}$$

In the transverse gauge, eq. (30) reduces to the ordinary Poisson equation for the scalar potential,

$$\nabla^2\Phi = -\frac{\rho}{\epsilon_0}. \tag{33}$$

There are no time derivatives in this equation, only the space derivatives, so its solution is the *instantaneous* Coulomb potential of the charge density $\rho(\mathbf{x},t)$,

$$\Phi(\mathbf{x},t) = \iiint\frac{\rho(\mathbf{y},t)\,d^3\mathbf{y}}{4\pi\epsilon_0\,|\mathbf{x}-\mathbf{y}|}, \tag{34}$$

hence the name *Coulomb gauge.* Note that instantaneous propagation of the scalar potential does not mean instantaneous propagation of the electric field or any other physical quantity; instead, the electric field propagates only at the speed of light according to the wave equation (21). What happens is that any change of ρ at some point $\mathbf{y}$ leads to instantaneous changes of both Φ and $\mathbf{A}$ at other points $\mathbf{x}$, but these changes cancel out from the electric field $\mathbf{E}$. But in addition to the instantaneous $\delta\mathbf{A}$ there is also a delayed $\delta\mathbf{A}$ which propagates at the speed of light, and it's that delayed piece which changes the electric field. I'll come back to this issue later in these notes.

Meanwhile, consider the vector potential in the transverse gauge, In this gauge, eq. (31) becomes

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\,-\,\nabla^2\right)\mathbf{A}\;=\;\mu_0\mathbf{J}\;-\;\frac{1}{c^2}\nabla\left(\frac{\partial\Phi}{\partial t}\right), \tag{35}$$

where the second term on the RHS can be re-expressed in terms of the charge density ρ and hence of the current density $\mathbf{J}$. Indeed, in light of eq. (34),

$$\begin{aligned}\frac{1}{c^2}\nabla\left(\frac{\partial\Phi(\mathbf{x},t)}{\partial t}\right)\;&=\;\epsilon_0\mu_0\,\nabla_x\frac{\partial}{\partial t}\iiint\frac{\rho(\mathbf{y},t)\,d^3\mathbf{y}}{4\pi\epsilon_0\,|\mathbf{x}-\mathbf{y}|}\\&=\;\mu_0\,\nabla_x\iiint\frac{d^3\mathbf{y}}{4\pi|\mathbf{x}-\mathbf{y}|}\,\frac{\partial\rho(\mathbf{y},t)}{\partial t}\\&\qquad\langle\!\langle\text{ by the continuity equation }\rangle\!\rangle\\&=\;-\mu_0\,\nabla_x\iiint\frac{d^3\mathbf{y}}{4\pi|\mathbf{x}-\mathbf{y}|}\,(\nabla\cdot\mathbf{J})(\mathbf{y},t).\end{aligned} \tag{36}$$

Thus, eq. (35) becomes a forced wave equation for the vector potential,

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\,-\,\nabla^2\right)\mathbf{A}(\mathbf{x},t)\;=\;\mu_0\mathbf{J}_T(\mathbf{x},t), \tag{37}$$

where

$$\mathbf{J}_T(\mathbf{x},t)\;=\;\mathbf{J}(\mathbf{x},t)\;+\;\nabla_x\iiint\frac{d^3\mathbf{y}}{4\pi|\mathbf{x}-\mathbf{y}|}\,(\nabla\cdot\mathbf{J})(\mathbf{y},t) \tag{38}$$

is the transverse part of the current $\mathbf{J}(\mathbf{x},t)$ or simply the *transverse current*, hence the name the *transverse gauge*.

More generally, any vector field can be written as a sum of transverse field which has zero divergence and a longitudinal field which has zero curl,

$$\mathbf{f}(\mathbf{x}) \;=\; \mathbf{f}_T(\mathbf{x}) \;+\; \mathbf{f}_L(\mathbf{x}), \quad \nabla\cdot\mathbf{f}_T \;\equiv\; 0, \quad \nabla\times\mathbf{f}_L \;\equiv\; 0. \tag{39}$$

The simplest way to see how this works is via the Fourier transform

$$\mathbf{f}(\mathbf{x}) \;=\; \iiint \frac{d^3\mathbf{k}}{(2\pi)^3}\, e^{i\mathbf{k}\cdot\mathbf{x}}\, \mathbf{g}(\mathbf{k}), \quad \mathbf{g}(\mathbf{k}) \;=\; \iiint d^3\mathbf{x}\, e^{-i\mathbf{k}\cdot\mathbf{x}}\, \mathbf{f}(\mathbf{x}). \tag{40}$$

The Fourier transform of the curl and the divergence of $\mathbf{f}$ are simply $i\mathbf{k}\cdot\mathbf{g}(\mathbf{k})$ and $i\mathbf{k}\times\mathbf{g}(\mathbf{k})$, so the decomposition into the transverse and longitudinal pieces amounts to splitting the $\mathbf{g}(\mathbf{k})$ vector into the parts perpendicular and parallel to $\mathbf{k}$,

$$\mathbf{g}_L(\mathbf{k}) \;=\; \frac{\mathbf{k}\cdot\mathbf{g}(\mathbf{k})}{\mathbf{k}^2}\,\mathbf{k}, \quad \mathbf{g}_T(\mathbf{k}) \;=\; \mathbf{g}(\mathbf{k}) \;-\; \mathbf{g}_L(\mathbf{k}). \tag{41}$$

Fourier transforming back to the $\mathbf{f}(\mathbf{x})$, we find

$$\mathbf{f}_L(\mathbf{x}) \;=\; (i\nabla)\,\frac{-1}{\nabla^2}\,(i\nabla\cdot\mathbf{f}) \;=\; -\nabla_x \iiint \frac{d^3\mathbf{y}}{4\pi|\mathbf{x}-\mathbf{y}|}\,(\nabla\cdot\mathbf{f})(\mathbf{y}) \tag{42}$$

and hence

$$\mathbf{f}_T(\mathbf{x}) \;=\; \mathbf{f}(\mathbf{x}) \;+\nabla_x \iiint \frac{d^3\mathbf{y}}{4\pi|\mathbf{x}-\mathbf{y}|}\,(\nabla\cdot\mathbf{f})(\mathbf{y}). \tag{43}$$

Comparing this formula to eq (38), we see that $\mathbf{J}_T(\mathbf{x},t)$ is indeed the transverse part of the electric current density at time t.

Note that the transverse current at any point $\mathbf{x}$ and time t depends on $\mathbf{J}(\mathbf{y},t)$ at other points $\mathbf{y}$ but the same time t, so the vector potential $\mathbf{A}(\mathbf{x},t)$ which obeys the forced wave equation (37) contains an instantaneously propagating piece. Fortunately, this instantaneous piece is longitudinal, so it does not affect the magnetic field, while its contribution to the electric field cancels against the instantaneous scalar potential.

The transverse gauge is commonly used in Quantum Electrodynamics (QED), which has to be formulated in terms of the potentials to allow *local* interaction with the Dirac field of the electrons. It is particularly convenient for the EM radiation by the atoms, hence yet another name, the *radiation gauge.* Even classically, its convenient for the EM radiation by small antennas, since the gradient of the scalar potential (34) decreases with distance as $1/r^2$ or faster, so it affects the electric field only at the relatively short distances from the antenna. In the radiation zone further away from the antenna, all we need is the vector potential, thus

$$\mathbf{B} \;=\; \nabla\times\mathbf{A}, \quad \mathbf{E} \;\approx\; -\frac{\partial}{\partial t}\mathbf{A}. \tag{44}$$

Now consider the Landau gauge (32). Writing eq. (31) for the vector potential as

$$\mu_0\mathbf{J} \;=\; \left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right)\mathbf{A} \;+\; \nabla\left((\nabla\cdot\mathbf{A}) + \frac{1}{c^2}\frac{\partial\Phi}{\partial t}\right), \tag{45}$$

we see that in the Landau gauge it becomes simply

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right)\mathbf{A}(\mathbf{x},t) \;=\; \mu_0\mathbf{J}(\mathbf{x},t). \tag{46}$$

Likewise, eq. (30) for the scalar potential — which me may write as

$$\begin{aligned}\frac{\rho}{\epsilon_0} \;&=\; -\frac{\partial}{\partial t}(\nabla\cdot\mathbf{A}) \;-\; \nabla^2\phi \\ &=\; \left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right)\Phi \;-\; \frac{\partial}{\partial t}\left(\nabla\cdot\mathbf{A} + \frac{1}{c^2}\frac{\partial\Phi}{\partial t}\right),\end{aligned} \tag{47}$$

— in the Landau gauge becomes simply

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right)\Phi(\mathbf{x},t) \;=\; \frac{1}{\epsilon_0}\,\rho(\mathbf{x},t). \tag{48}$$

The Landau gauge is convenient for the manifestly relativistic treatment of electrodynamics. Indeed, the differential operator involved in both eqs. (46) and (48),

$$\Box \;=\; \frac{1}{c^2}\frac{\partial^2}{\partial t^2} \;-\; \nabla^2 \;=\; \frac{1}{c^2}\frac{\partial^2}{\partial t^2} \;-\; \frac{\partial^2}{\partial x_1^2} \;-\; \frac{\partial^2}{\partial x_2^2} \;-\; \frac{\partial^2}{\partial x_3^2} \tag{49}$$

— called the *d'Alembert operator* or simply the *d'Alembertian* — is invariant under Lorentz transformations of space and time coordinates between differently moving frames of reference.

Also, the similarity between the equations (46) and (48) allows combining the scalar and the vector potentials into a 4–vector potential, while the charge density and the current density are combined into another 4–vector.

The Landau gauge condition does not completely fix the gauge. Indeed, let the potentials $\mathbf{A}(\mathbf{x},t)$ and $\Phi(\mathbf{x},t)$ obey

$$\nabla\cdot\mathbf{A} + \frac{1}{c^2}\frac{\partial\Phi}{\partial t} = 0. \tag{32}$$

Then for any $\Lambda(\mathbf{x},t)$ which satisfy the free wave equation $\Box\Lambda(\mathbf{x},t) = 0$, the transformed potentials

$$\mathbf{A}' = \mathbf{A} + \nabla\Lambda, \quad \Phi' = \Phi - \frac{\partial\Lambda}{\partial t}, \tag{50}$$

also obey the Landau gauge condition,

$$\nabla\cdot\mathbf{A}' + \frac{1}{c^2}\frac{\partial\Phi'}{\partial t} = \nabla\cdot\mathbf{A} + \frac{1}{c^2}\frac{\partial\Phi}{\partial t} + \left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\Lambda = 0. \tag{51}$$

To eliminate this residual ambiguity of the potentials, we need an additional rule such as causality. Note that *causality* requires that physical quantities such as electric or magnetic fields $\mathbf{E}(\mathbf{x},t)$ and $\mathbf{B}(\mathbf{x},t)$ not be affected by the charges or currents at later times $t' > t$, — today's fields cannot be affected by tomorrow's charger and currents. In principle, the potentials $\mathbf{A}$ and Φ do not have to be causal, as long as the fields $\mathbf{E}$ and $\mathbf{B}$ are causal. However, we may require the potentials to be causal as an additional gauge-fixing constraint to eliminate the residual ambiguity of the Landau gauge.

Green's Functions of the D'Alembert Operator

Like any linear PDE with a source term, the forced wave equations

$$\Box\Phi(\mathbf{x},t) = \frac{1}{\epsilon_0}\rho(\mathbf{x},t), \quad \Box\mathbf{A}(\mathbf{x},t) = \frac{1}{\epsilon_0}\mathbf{J}(\mathbf{x},t), \tag{52}$$

for the potentials in the Landau gauge can be formally solved as

$$\begin{aligned}\Phi(\mathbf{x},t_x) &= \int dt_y \iiint d^3\mathbf{y}\, G(\mathbf{x},t_x;\mathbf{y},t_y)\,\rho(\mathbf{y},t_y),\\ \mathbf{A}(\mathbf{x},t_x) &= \int dt_y \iiint d^3\mathbf{y}\, G(\mathbf{x},t_x;\mathbf{y},t_y)\,\mathbf{J}(\mathbf{y},t_y),\end{aligned} \tag{53}$$

where G is the Green's function of the d'Alembert operator $\Box$. That is, G is the solution of

$$\Box_x G(\mathbf{x},t_x;\mathbf{y},t_y) \;=\; \delta^{(3)}(\mathbf{x}-\mathbf{y})\delta(t_x-t_y) \tag{54}$$

subject to the appropriate boundary and/or asymptotic conditions. Also, if we want the solutions (53) to be causal — that is, ρ and $\mathbf{A}$ at time t_x should not be affected by the ρ and $\mathbf{J}$ at a later time $t_y > t_z$), — then we need $G = 0$ for $t_y > t_x$.

Let's find the Green's function for the wave equation in the unlimited 3D space without any boundaries. Thanks to the translational symmetry of this space, the Green's function should depend on $\mathbf{x}$ and $\mathbf{y}$ only via their difference $\mathbf{x}-\mathbf{y}$. Likewise, there is a symmetry of translations in time, so G depends on the times t_x and t_y only via the difference $t_x - t_y$, thus

$$G(\mathbf{x},t_x;\mathbf{y},t_y) \;=\; G(\mathbf{x}-\mathbf{y},t_x-t_y), \tag{55}$$

where $G(\mathbf{x}_{\mathrm{rel}},t_{\mathrm{rel}})$ obeys

$$\Box G(\mathbf{x},t) \;=\; \delta^{(3)}(\mathbf{x})\delta(t). \tag{56}$$

To solve this equation, let's Fourier transform the time dependence of this Green's function,

$$\tilde{G}(\mathbf{x},\omega) \;=\; \int\limits_{-\infty}^{+\infty} dt\, e^{+i\omega t}\times G(\mathbf{x},t), \tag{57}$$

$$G(\mathbf{x},t) \;=\; \int\limits_{-\infty}^{+\infty} \frac{d\omega}{2\pi}\, e^{-i\omega t}\times \tilde{G}(\mathbf{x},\omega), \tag{58}$$

$$\frac{\partial^2}{\partial t^2}G(\mathbf{x},t) \;=\; \int\limits_{-\infty}^{+\infty} \frac{d\omega}{2\pi}\, e^{-i\omega t}\times (-\omega^2)\tilde{G}(\mathbf{x},\omega), \tag{59}$$

hence eq. (56) becomes

$$\begin{aligned}\left(-\frac{\omega^2}{c^2}-\nabla^2\right)\tilde{G}(\mathbf{x},\omega) \;&=\; \delta^{(3)}(\mathbf{x})\times\text{Fourier transform of }\delta(t)\\ &=\; \delta^{(3)}(\mathbf{x})\times 1 \quad \langle\!\langle\,\text{independent of }\omega\,\rangle\!\rangle.\end{aligned} \tag{60}$$

Next, let's solve this equation in 3D space for any fixed ω. Thanks to the rotational symmetry of the equation and of the space itself, we may take $\tilde{G}$ to be spherically symmetric, *i.e.*, depend only on the magnitude of $\mathbf{x}$ rather than its direction, thus $\tilde{G}(\mathbf{x},\omega) = \tilde{G}(r,\omega)$ for $r = |\mathbf{x}|$. For such spherically symmetric functions of r, the Laplacian is simply

$$\nabla^2 \tilde{G}(r,\omega) \;=\; \left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r}\right)\tilde{G}(r,\omega) \;=\; \frac{1}{r}\frac{\partial^2}{\partial r^2}\big(r\tilde{G}(r,\omega)\big), \tag{61}$$

so eq. (60) becomes

$$-\frac{\omega^2}{c^2} \times \tilde{G}(r,\omega) \;-\; \frac{1}{r}\frac{\partial^2}{\partial r^2}\big(r \times \tilde{G}(r,\omega)\big) \;=\; \delta^{(3)}(\mathbf{x}). \tag{62}$$

At $\mathbf{x} \neq 0$ we may further simplify this equation by multiplying both sides by $-r$, thus

$$\left(\frac{\omega^2}{c^2} + \frac{\partial^2}{\partial r^2}\right)\big(r\tilde{G}(r,\omega)\big) \;=\; 0, \tag{63}$$

and the general solution of this equation is

$$\tilde{G}(r,\omega) \;=\; \alpha(\omega) \times \frac{\exp(+i(\omega/c)r)}{4\pi r} \;+\; \beta(\omega) \times \frac{\exp(-i(\omega/c)r)}{4\pi r}\,. \tag{64}$$

At this point, α and β are arbitrary functions of the frequency ω, but there is a constraint stemming from the delta-function at $\mathbf{x} = 0$. Expanding $G(r,\omega)$ in powers of r for $r \to 0$, we have

$$G(\text{small}\, r;\omega) \;=\; \frac{\alpha+\beta}{4\pi r} \;+\; \frac{i(\alpha-\beta)\omega}{4\pi c} \;-\; \frac{(\alpha+\beta)\omega^2}{8\pi c^2} \times r \;+\; O(r^2), \tag{65}$$

hence

$$\nabla^2 G(r,\omega) \;=\; -(\alpha+\beta) \times \delta^{(3)}(\mathbf{x}) \;+\; 0 \;-\; \frac{(\alpha+\beta)\omega^2}{8\pi c^2} \times \frac{2}{r} + \text{ finite}, \tag{66}$$

where the $1/r$ term (as well as all the finite terms) cancel against the $(\omega/c)^2 G(r,\omega)$. Only the delta-function term remains un-canceled, and to get its coefficient right, we need

$$\alpha(\omega) \;+\; \beta(\omega) \;=\; 1 \quad \text{for all } \omega. \tag{67}$$

Despite this constraint, we still have an infinite family of solutions parametrized by the $\alpha(\omega)$, and their Fourier transforms gives us an infinite family of Green's functions. Of particular interest are:

- The *retarded* or *causal* Green's function, which obtains for $\alpha \equiv 1$ and $\beta \equiv 0$.
- The *advanced* or *anticausal* Green's function, which obtains for $\alpha \equiv 0$ and $\beta \equiv 1$.
- The *Feynman's propagator,* which obtains when

$$\text{for } \omega > 0, \quad \alpha = 1, \quad \beta = 0, \qquad \text{but for } \omega < 0, \quad \alpha = 0, \quad \beta = 1. \tag{68}$$

The Feynman's propagator plays a key role in Quantum Field Theory, but for our purposes let's focus on the other two Green's functions,

$$\tilde{G}_{\pm}(r, \omega) = \frac{\exp(\pm i\omega r/c)}{4\pi r}. \tag{69}$$

Fourier transforming them from functions of frequency to functions of time, we get

$$\begin{aligned} G_{\pm}(r, t) &= \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} \exp(-i\omega t) \times \frac{\exp(\pm i\omega r/c)}{4\pi r} \\ &= \frac{1}{4\pi r} \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} \exp(i\omega(t \mp (r/c))) \\ &= \frac{1}{4\pi r} \times \delta(t \mp (r/c)) \end{aligned} \tag{70}$$

and hence

$$G_{\pm}(\mathbf{x}, t_x; \mathbf{y}, t_y) = \frac{1}{4\pi|\mathbf{x} - \mathbf{y}|} \times \delta\left(t_x - t_y \mp \frac{|\mathbf{x} - \mathbf{y}|}{c}\right). \tag{71}$$

Note that the G_+ Green's function is causal: it vanishes unless t_x is later than t_y. Moreover, the t_x must be delayed relative to the t_y by precisely the time r/c needed by a signal moving at speed of light c to get from point $\mathbf{y}$ to point $\mathbf{x}$. Because of this specific delay or retardation, G_+ is called the *retarded Green's function.* On the other hand, the G_- Green's function is *anticausal,* it vanishes unless t_x is earlier than t_y. Moreover, the t_x must be advanced relative to the t_y by the light propagation time r/c, hence the name *advanced Green's function* for the G_-.

Now let's apply the Green's functions (71) to the EM potentials $\mathbf{A}$ and Φ. Allowing for free waves obeying $\Box\,\mathbf{A}_{\rm free}\equiv 0$ and $\Box\,\Phi_{\rm free}\equiv 0$, we have

$$\begin{aligned}
\mathbf{A}_\pm(\mathbf{x},t) &= \mathbf{A}_{\rm free}(\mathbf{x},t) + \frac{\mu_0}{4\pi}\iiint d^3\mathbf{y}\,\frac{\mathbf{J}(\mathbf{y},t\mp r/c)}{4\pi r}\,, & (72)\\
\Phi_\pm(\mathbf{x},t) &= \Phi_{\rm free}(\mathbf{x},t) + \frac{1}{4\pi\epsilon_0}\iiint d^3\mathbf{y}\,\frac{\rho(\mathbf{y},t\mp r/c)}{4\pi r}\,, & (73)
\end{aligned}$$

where $r\equiv|\mathbf{x}-\mathbf{y}|$. What is the physical meaning of these potentials and which solution should we use: the retarded potentials $\mathbf{A}_+$ and Φ_+, or the advanced potentials $\mathbf{A}_-$ and Φ_-? The answer depends on whether we know the past or the future of the system in question.

Suppose the charges and the currents exist only for a limited period of time. Before they have turned on, there was some pre-existing free wave $(\mathbf{A},\Phi)_{\rm in}$, and after the current and the changes have turned off, there remains some free wave $(\mathbf{A},\Phi)_{\rm out}$. If we know the pre-existing wave, then we should use the retarded Green's function, thus

$$\begin{aligned}
\mathbf{A}(\mathbf{x},t) &= \mathbf{A}_{\rm in}(\mathbf{x},t) + \frac{\mu_0}{4\pi}\iiint d^3\mathbf{y}\,\frac{\mathbf{J}(\mathbf{y},t-r/c)}{4\pi r}\,,\\
\Phi(\mathbf{x},t) &= \Phi_{\rm in}(\mathbf{x},t) + \frac{1}{4\pi\epsilon_0}\iiint d^3\mathbf{y}\,\frac{\rho(\mathbf{y},t-r/c)}{4\pi r}\,,
\end{aligned}\qquad(74)$$

Indeed, the integral terms here vanish at times t before the currents and the charges have turned on, so in the deep past, the potentials are precisely the known past potentials $(\mathbf{A},\Phi)_{\rm in}$. On the other hand, if we know the late future wave $(\mathbf{A},\Phi)_{\rm out}$ at times after all the currents and the charges have turned off, then we should use the advanced Green's function, thus

$$\begin{aligned}
\mathbf{A}(\mathbf{x},t) &= \mathbf{A}_{\rm out}(\mathbf{x},t) + \frac{\mu_0}{4\pi}\iiint d^3\mathbf{y}\,\frac{\mathbf{J}(\mathbf{y},t+r/c)}{4\pi r}\,,\\
\Phi(\mathbf{x},t) &= \Phi_{\rm out}(\mathbf{x},t) + \frac{1}{4\pi\epsilon_0}\iiint d^3\mathbf{y}\,\frac{\rho(\mathbf{y},t+r/c)}{4\pi r}\,,
\end{aligned}\qquad(75)$$

In practice, knowing the initial field is much more common then knowing the final fields, so the retarded Green's function and the retarded potentials (74) are used a lot more often than the advanced Green's function and the advanced potentials (75).

Efimenko Equations for the Retarded Fields **E** and **B**

Suppose far back in the past there were no electric or magnetic fields, and no charges or currents. Once the charges and the currents turn on, the potentials $\mathbf{A}$ and Φ also turn on and start spreading out at the speed of light according to eqs. (74) without the free terms. The $\mathbf{E}$ and $\mathbf{B}$ fields follow from these potentials by taking the appropriate derivatives,

$$\mathbf{B} = \nabla\times\mathbf{A}, \quad \mathbf{E} = -\nabla\Phi - \frac{\partial\mathbf{A}}{\partial t}, \tag{76}$$

but taking these derivatives requires extra care. Note that the current and charge density in eqs. (74) are evaluated at the *retarded time*

$$t_{\rm ret} = t - \frac{|\mathbf{x}-\mathbf{y}|}{c} \tag{77}$$

which depends on $\mathbf{x}$, hence taking an $\mathbf{x}$ derivative at fixed time t of a some function of $\mathbf{y}$ and the retired time yields

$$\frac{\partial}{\partial x_i} f(\mathbf{y}, t_{\rm ret}) = \left[\frac{\partial f}{\partial t}\right]_{y,\rm ret} \times \frac{\partial t_{\rm ret}}{\partial x_i} = \left[\frac{\partial f}{\partial t}\right]_{y,\rm ret} \times \frac{-n_i}{c} \tag{78}$$

where the subscript 'y,ret' indicates that the expression in brackets should be evaluated at point $\mathbf{y}$ and the retarded time $t_{\rm ret}$, and n_i is the $i^{\rm th}$ component of the unit vector $\mathbf{n}$ pointing from $\mathbf{y}$ to $\mathbf{x}$. In particular,

$$\frac{\partial}{\partial x_j}\left[\frac{J_k(\mathbf{y}, t_{\rm ret})}{r}\right] = \frac{\partial(1/r)}{\partial x_j}\times[J_k]_{y,\rm ret} + \frac{1}{r}\frac{\partial}{\partial x_j}[J_k]_{y,\rm ret} = -\frac{n_j}{r^2}\times[J_k]_{y,\rm ret} - \frac{n_j}{rc}\times\left[\frac{\partial J_k}{\partial t}\right]_{y,\rm ret}, \tag{79}$$

hence in vector notations

$$\nabla\times\left[\frac{\mathbf{J}}{r}\right]_{y,\rm ret} = +[\mathbf{J}]_{y,\rm ret}\times\frac{\mathbf{n}}{r^2} + \left[\frac{\partial\mathbf{J}}{\partial t}\right]_{y,\rm ret}\times\frac{\mathbf{n}}{rc}, \tag{80}$$

and therefore

$$\begin{aligned}\mathbf{B}(\mathbf{x},t) &= \nabla\times\left(\mathbf{A}(\mathbf{x},t) = \frac{\mu_0}{4\pi}\iiint d^3\mathbf{y}\left[\frac{\mathbf{J}}{r}\right]_{y,\rm ret}\right)\\ &= \frac{\mu_0}{4\pi}\iiint d^3\mathbf{y}\left([\mathbf{J}]_{y,\rm ret}\times\frac{\mathbf{n}}{r^2} + \left[\frac{\partial\mathbf{J}}{\partial t}\right]_{y,\rm ret}\times\frac{\mathbf{n}}{rc}\right).\end{aligned} \tag{81}$$

This is the *Efimenko equation* for the magnetic field of a general time-dependent current. For a steady current, this formula reduces to the Biot–Savart–Laplace formula, time-dependent currents the second term inside $(\cdots)$ provides a correction. For current varying on the time scale τ, the second term becomes larger than the Biot–Savart–Laplace first term at long distances $r \gtrsim c\tau$. At mush shorter distances $r \ll c\tau$, we may use the Biot–Savart–Laplace formula as a quasistatic approximation.

The Efimenko equation for the electric field obtains in a similar way,

$$\mathbf{E}(\mathbf{x},t) = -\nabla\Phi - \frac{\partial\mathbf{A}}{\partial t} = \frac{1}{4\pi\epsilon_0}\iiint d^3\mathbf{y}\left(-\nabla_x\left[\frac{\rho}{r}\right]_{y,\mathrm{ret}} - \mu_0\epsilon_0\frac{\partial}{\partial t}\left[\frac{\mathbf{J}}{r}\right]_{y,\mathrm{ret}}\right) \qquad (82)$$

where

$$\nabla_x\left[\frac{\rho}{r}\right]_{y,\mathrm{ret}} = -\frac{\mathbf{n}}{r^2}[\rho]_{y,\mathrm{ret}} - \frac{\mathbf{n}}{rc}\left[\frac{\partial\rho}{\partial t}\right]_{y,\mathrm{ret}} \qquad (83)$$

but

$$\frac{\partial}{\partial t}\left[\frac{\mathbf{J}}{r}\right]_{y,\mathrm{ret}} = \frac{1}{r}\left[\frac{\partial\mathbf{J}}{\partial t}\right]_{y,\mathrm{ret}} + 0. \qquad (84)$$

Also, $\mu_0\epsilon_0 = 1/c^2$. Thus altogether,

$$\mathbf{E}(\mathbf{x},t) = \frac{1}{4\pi\epsilon_0}\iiint d^3\mathbf{y}\left(\frac{\mathbf{n}}{r^2}[\rho]_{y,\mathrm{ret}} + \frac{\mathbf{n}}{rc}\left[\frac{\partial\rho}{\partial t}\right]_{y,\mathrm{ret}} - \frac{1}{rc^2}\left[\frac{\partial\mathbf{J}}{\partial t}\right]_{y,\mathrm{ret}}\right). \qquad (85)$$

For the static charges and steady currents, the first term inside the $(\cdots)$ here yields the good old Coulomb field while the other two terms vanish. But for the time-dependent fields and currents, all three terms can be important.